Language of Dynamical Systems

$(X, \mathcal{B}, \mu)$ - a probability space (X usually is a compact metric space)
$\mathcal{B}$ - σ-algebra of Borel sets
μ - a probability measure ($\mu(X) = 1$)
A discrete-time dynamical system is any measurable transformation $T : X \to X$.
Usually, T is a measurable automorphism, i.e. T is a bijection and T^{-1} is measurable.
$x \in X$ is interpreted as a state (position) of the dynamical system
$Tx \in X$ is interpreted as a state (position) after a discrete time unit.
The entire evolution of $x \in X$ is described by its orbit:

$$\dots, T^{-1}x, x, Tx, T^2x, T^3x, \dots$$

We can also observe the evolution of sets of objects $A \in \mathcal{B}$ (events):

$$\dots, T^{-1}A, A, TA, T^2A, T^3A, \dots$$

$$x \in T^{-n}A \iff T^n x \in A$$

$T^{-n}A$ is the set of points which after time n are in A
$\mu(A)$ - the probability of the event A
$\mu(T^{-n}A)$ - the probability of occurring the event A after time n

Definition 1. The map $T : X \to X$ preserves the measure μ if $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}$. Then $\mu(T^{-n}A) = \mu(A)$ for any $n \in \mathbb{N}$.

Definition 2. The quadruple $(X, \mathcal{B}, \mu, T)$ is called **a measure-preserving (discrete-time) dynamical system.**

Example 1. Rotation on the circle

$$X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

$\mu = \lambda$ the normalized Lebesgue measure
For every $\alpha \in [0, 1)$ denote by $T_\alpha : S^1 \to S^1$ the rotation $T_\alpha z = e^{2\pi i\alpha} z$
We can identify S^1 with the interval $[0, 1)$ or the quotient group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by

$$[0, 1) \ni x \mapsto e^{2\pi i x} \in S^1$$
$$\mathbb{R} \ni x \mapsto e^{2\pi i x} \in S^1$$
$$\mathbb{T} = \mathbb{R}/\mathbb{Z} \ni x \mapsto e^{2\pi i x} \in S^1$$

$$T_\alpha : \mathbb{T} \to \mathbb{T}, T_\alpha x = x + \alpha \mod 1$$

Example 2. Bernoulli shift
$X = \{0, 1\}^{\mathbb{Z}}$
$\mathcal{B}$ is the σ-algebra generated by the cylindrical sets
For every $x \in \{0, 1\}^{\mathbb{Z}}$ and integer $n \le m$ denote by $x[n, m]$ the word seen in x in places between n and m
For every word $a \in \{0, 1\}^k$ and $n \in \mathbb{Z}$ we define the cylinder

$$C_n(a) := \{x \in \{0, 1\}^{\mathbb{Z}} : x[n, n + k - 1] = a\}$$

μ is the Bernoulli measure, i.e. $\mu(C_n(a)) = \frac{1}{2^k}$

$$S : X \to X, \quad (Sx)_n = x_{n+1} \text{ for all } x \in \{0, 1\}^{\mathbb{Z}}, \quad n \in \mathbb{Z}$$

$$\dots x_{-2}x_{-1}.x_0x_1x_2 \dots \overset{S}{\longmapsto} \dots x_{-1}x_0.x_1x_2x_3 \dots$$

$$S^{-1}C_n(a) = C_{n+1}(a) \Longrightarrow \mu(S^{-1}C_n(a)) = \mu(C_{n+1}(a)) = \frac{1}{2^k} = \mu(C_n(a))$$
$$\Longrightarrow \mu \circ S^{-1} = \mu \Longrightarrow S \text{ preserves } \mu$$

Definition 3. A subset $A \subset X$ is T-invariant if $T^{-1}A = A \mod \mu$, i.e. $\mu(T^{-1}A \triangle A) = 0$.

$$\big(A \triangle B = (A \setminus B) \cup (B \setminus A)\big)$$

$$A' = \bigcap_{n\in\mathbb{Z}} T^n A, \quad T^{-1}A' = A', \quad A' = A \mod \mu$$

If $A \subset X$ is invariant then for every $x \in A$ its orbit $(T^n x)_{n\in\mathbb{Z}}$ is included in A.

If $\mu(A) > 0$ and $\mu(X \setminus A) > 0 (\Leftrightarrow \mu(A) < 1)$ then the dynamical system $(X, \mathcal{B}, \mu, T)$ splits into two subsystems.

Definition 4. A dynamical system $(X, \mathcal{B}, \mu, T)$ is **ergodic** if for every T-invariant set $A \in \mathcal{B}$ we have $\mu(A) = 0$ or $\mu(A) = 1$.

Remark. T is ergodic iff for every measurable $f : X \to \mathbb{C}$ with $f \circ T = f$ μ-a.e. the map f is constant (a.e.)

We can restrict the right hand side to L^1 or L^2 or bounded maps

For any $A \in \mathcal{B}$ let us consider $f = \chi_A$. Then $f \circ T = \chi_A \circ T = \chi_{T^{-1}A}$

$f : X \to \mathbb{C}$ is T-invariant iff $A \subset X$ is T-invariant

Theorem 1. *$T_\alpha : \mathbb{T} \to \mathbb{T}$ is ergodic iff $\alpha \in [0, 1)$ is irrational.*

Proof. If $\alpha = \frac{p}{q}$ is rational, then $\frac{1}{q}$-periodic set $A \subset [0, 1)$ is T_α-invariant.

If α is irrational, then let as consider any L^2-map $f : \mathbb{T} \to \mathbb{C}$ with $f \circ T_\alpha = f$. We use its Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}, \quad a_n := \int_0^1 f(x) e^{-2\pi i n x} dx \text{ for } n \in \mathbb{Z}$$

$$f(T_\alpha x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n(x+\alpha)} = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\alpha} e^{2\pi i n x}$$

Therefore $a_n e^{2\pi i n\alpha} = a_n$ for all $n \in \mathbb{Z}$. By the irrationality of α, $e^{2\pi i n\alpha} \neq 1$ if $n \neq 0$. It follows that $a_n = 0$ for all $n \neq 0$. Hence $f = a_0$. ☐

Theorem 2. *Bernoulli shift $S : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ is also ergodic.*

Theorem 3 (Birkhoff's Ergodic Theorem)**.** *For every ergodic dynamical system $(X, \mathcal{B}, \mu, T)$, for every L^1-function $f : X \to \mathbb{C}$ and for μ-a.e. $x \in X$ (from a subset of measure 1) we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \to \int_X f\, d\mu.$$

If $f = \chi_A$ then

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \frac{1}{n}\sum_{k=0}^{n-1} \chi_A(T^k x) = \frac{\#\{0 \le k < n : T^k x \in A\}}{n} \to \mu(A)$$

Definition 5. Let X is a compact metric space, $T : X \to X$ is a homeomorphism and μ is a T-invariant measure. The homeomorphism T is **uniquely ergodic** if

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \to \int_X f\, d\mu$$

for every continuous $f : X \to \mathbb{C}$ and every $x \in X$.

Then every semi-orbit of the dynamical system is **equidistributed** on X according to the measure μ.

Theorem 4. *If $\alpha \in [0,1)$ is irrational then $T_\alpha : \mathbb{T} \to \mathbb{T}$ is uniquely ergodic.*

Proof. Let us start from exponential maps $f(x) = e^{2\pi i m x}$. Then

$$\int_0^1 f(x)\,dx = \int_0^1 e^{2\pi i m x}\,dx = \frac{e^{2\pi i m} - 1}{2\pi i m} = 0 \text{ if } m \neq 0$$

$$\int_0^1 f(x)\,dx = \int_0^1 1\,dx = 1 \text{ if } m = 0.$$

If $m = 0$ then

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = 1 = \int_X f\,d\mu.$$

If $m \neq 0$ then

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i m(x+k\alpha)} = \frac{e^{2\pi i m x}}{n}\sum_{k=0}^{n-1}(e^{2\pi i m\alpha})^k = \frac{e^{2\pi i m x}}{n}\frac{e^{2\pi i m n\alpha} - 1}{e^{2\pi i m\alpha} - 1}$$

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)\right| \leq \frac{1}{n}\frac{2}{|e^{2\pi i m\alpha} - 1|} \to 0.$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \to 0 \int_X f\,d\mu.$$

The convergence can be easily extended to trigonometric polynomials $f(x) = \sum_{m=-N}^{N} a_m e^{2\pi i m x}$. Since trigonometric polynomials are dense in $C(\mathbb{T})$, we can extend it to all continuous maps. □

Remark. The Bernoulli shift is not uniquely ergodic. Take $f = \chi_{C_0(0)}$ and $x = \ldots 00.000\ldots$. Then $f(S^n x) = 1$, so

$$\frac{1}{n}\sum_{k=0}^{n-1} f(S^k x) = 1 \neq \frac{1}{2} = \mu(C_0(0)) = \int_X f\,d\mu.$$

Theorem 5. *A homeomorphism $T : X \to X$ is uniquely ergodic iff T has only one invariant probability measure.*

Definition 6. Two dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are **isomorphic** if there exists a measurable bijection $V : X \to Y$ such that

$$\nu \circ V^{-1} = \mu \quad \text{and} \quad V \circ T = S \circ V.$$

Continuous-time dynamical systems

Discrete-time dynamical system $(T^n)_{n\in\mathbb{Z}}$: $T^0 = Id$ and $T^{m+n} = T^m \circ T^n$ for all $m, n \in \mathbb{Z}$

Continuous-time dynamical system is a family $(T_t x)_{t\in\mathbb{R}}$, $T_t : X \to X$ measurable automorphism:

$$T_0 = Id \text{ and } T_{t+s} = T_t \circ T_s \text{ for all } t, s \in \mathbb{R}.$$

$x \in X$ is interpreted as a state (position) of the dynamical system
$T_t x \in X$ is interpreted as a state (position) after time $t \in \mathbb{R}$
The entire evolution of $x \in X$ is described by its orbit: $(T_t x)_{t\in\mathbb{R}}$
$(T_t)_{t\in\mathbb{R}}$ is called a **flow**. A flow $(T_t)_{t\in\mathbb{R}}$ preserves a probability measure μ on X if

$$\mu(T_t^{-1} A) = \mu(A) \text{ for all } A \in \mathcal{B} \text{ and } t \in \mathbb{R}.$$

Definition 7. The quadruple $(X, \mathcal{B}, \mu, (T_t)_{t\in\mathbb{R}})$ is called **a measure-preserving (continuous-time) dynamical system.**

Example 3. Rotation on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For any vector $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$ let

$$T_t(x,y) = (x + t\alpha, y + t\beta)$$

Let us consider the circle $I \subset \mathbb{T}^2$, $I = [0,1) \times \{0\}$. For any $(x,0) \in I$ we look for the first return time $t > 0$: $T_t(x,0) \in I$. Suppose that $\beta > 0$. Then

$$T_t(x,0) = (x + t\alpha, t\beta) \in I \iff t\beta = 1 \iff t = \frac{1}{\beta}$$

The first return time map

$$T_{1/\beta} : I \to I, \quad T_{1/\beta}(x,0) = (x + \alpha/\beta, 1) = (x + \alpha/\beta, 0)$$

is isomorphic to the rotation $T_{\alpha/\beta} : \mathbb{T} \to \mathbb{T}$.

Theorem 6. *The rotation flow $(T_t)_{t\in\mathbb{R}}$ is ergodic (uniquely ergodic) iff the map $T_{\alpha/\beta}$: $\mathbb{T} \to \mathbb{T}$ is ergodic (uniquely ergodic) iff α/β is irrational.*

Example 4. **Special flows.** Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving automorphism and $f : X \to \mathbb{R}_{>0}$ an integrable map. The special flow $(T_t^f)_{t\in\mathbb{R}}$ acts on

$$X^f := \{(x,s) : x \in X, 0 \le s < f(x)\}$$

$$T_t^f(x,s) = (x, s+t), \quad (x, f(x)) = (Tx, 0)$$

Remark. Every measure-preserving flow is isomorphic to a special flow.

Example 5. **Ordinary differential equations**. Let us consider any autonomous ordinary differential equation on $D \subset \mathbb{R}^N$ given by a C^2-vector field $X : D \to \mathbb{R}^N$: $x'(t) = X(x(t))$. Suppose that for every $x_0 \in D$ there is a solution $x : \mathbb{R} \to D$ of

$$\begin{cases} x'(t) = X(x(t)), & t \in \mathbb{R} \\ x(t_0) = x_0 \end{cases}$$

Then $(T_t)_{t\in\mathbb{R}}$ given by $T_t x_0 = x(t)$ is a flow.

$$(T_t)_{t\in\mathbb{R}} \text{ preserves the Lebesgue measure} \iff \operatorname{div} X = \sum_{i=1}^{N} \frac{\partial X_i}{x_i} = 0.$$

Remark. All the concepts defined so far for discrete time systems can be easily transferred to continuous time systems.

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \quad \longmapsto \quad \frac{1}{T}\int_0^T f(T_t x)\, dt$$

Billiards

Let us consider an area $\Omega \subset \mathbb{R}^2$ bounded by a finite chain of C^2-curves such that each curve is either convex or concave.

The phase space of the billiard flow $(b_t)_{t\in\mathbb{R}}$ is

$$X = S^1\Omega = \{(x,v) : x \in \Omega, v \in S^1 \text{ or } x \in \partial\Omega \text{ with } v \in S^1 \text{ inward}\}$$

The billiard flow $(b_t)_{t\in\mathbb{R}}$ preserves the product measure $\lambda_\Omega \times \lambda_{S^1}$.

Example 6. **Billiard on ellipses.**

Suppose that Ω is an ellipse. Every billiard orbit is tangent to a confocal ellipse or hyperbola.

Such curves are called **caustics.**

Let us denote by $(\Lambda_c)_c$ the family of confocal ellipses inside Ω.

Denote by $X_c^{\pm} \subset S^1\Omega$ the subset of all vectors whose orbits are tangent to Λ_c and which run clockwise $(+)$ and counterclockwise $(-)$.

The phase space $S^1\Omega$ splits into invariant sets $X_c^{\pm}$, so the billiard flow $(b_t)_{t\in\mathbb{R}}$ is highly non-ergodic.

The set X_c^+ can be identified with the union of two copies of the ring $\Omega \setminus \Lambda_c$.

After gluing the boundaries of the two copies we obtain a topological torus. Moreover, the billiard flow $(b_t)_{t\in\mathbb{R}}$ is isomorphic to a rotation flow on $\mathbb{T}^2$ along a vector $(\alpha(c), \beta(c))$.

If the ratio $\frac{\alpha(c)}{\beta(c)}$ is irrational then every billiard orbit tangent to Λ_c is equidistributed in the ring $\Omega \setminus \Lambda_c$.

If the ratio $\frac{\alpha(c)}{\beta(c)}$ is rational then every billiard orbit tangent to Λ_c is periodic

Remark. The behavior of billiard orbits on general strictly convex tables is much more complicated. The are results about the existence of a single caustic. This implies the absence of unique ergodicity.

Billiards on concave tables.

Here we assume that at least one part of the boundary is strictly concave.

The billiard flow on a concave table is ergodic and or even isomorphic (in a sense) to Bernoulli shift. Concave billiard are very chaotic.

Billiards on polygons.

Theorem 7 (Kerkhoff-Masur-Smillie, 1986)**.** *For every $n \geq 3$ for a typical n-gon P (from a dense G_δ-set of n-gon; a countable intersection of dense and open sets) the billiard flow on P is ergodic.*

Problem 1. It is very difficult to find concrete polygons for which $(b_t)_{t\in\mathbb{R}}$ is ergodic.

Even in the class of right triangles, any characterization of ergodicity is not known.

Problem 2. **The existence of periodic orbits** is not know for obtuse triangles.

Billiards on rational polygons.

A polygon P is called **rational** if all its angles $\in \mathbb{Q}\pi$.

Billiard on a rectangle.

All possible directions: θ, $\pi - \theta$, $\pi + \theta$, $-\theta$.

$$\{\theta, \pi - \theta, \pi + \theta, -\theta\} = \Gamma\theta$$

Γ is a group of isometries generated by the vertical reflection: $\theta \mapsto -\theta$, and the horizontal reflection: $\theta \mapsto \pi - \theta$.

For every $\theta \in \mathbb{T}$ the set $P_\theta = P \times \{\theta, \pi - \theta, \pi + \theta, -\theta\} \subset S^1P$ is invariant, the flow $(b_t)_{t\in\mathbb{R}}$ is not ergodic.

The billiard flow $(b_t)_{t\in\mathbb{R}}$ is isomorphic to the rotation flow on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ in the direction of the vector $(\frac{\cos\theta}{a}, \frac{\sin\theta}{b})$. Its ergodicity depends on the irrationality of $\tan\theta\frac{b}{a}$.

Unfolding procedure: for any rational polygon P. Denote by Γ a group of isometries (fixing zero) generated by reflections across lines parallel to all sides of P. Since the angles between the sides $\in \frac{\pi}{N}\mathbb{N}$, the group Γ is finite - the dihedral group D_N.

The set $P_\theta = P \times \Gamma\theta \subset S^1 P$ is invariant. Moreover,

$$P_\theta \cong \bigcup_{\gamma\in\Gamma} \gamma^{-1} P$$

The resulting object is a compact surface $M(P)$ and the billiard flow $(b_t)_{t\in\mathbb{R}}$ is isomorphic to the transflation in direction θ on $M(P)$.

Example 7. **Billiard on the right triangle with the acute angle $\pi/8$ (and $3\pi/8$).**
The resulting object $M(P)$ a surface of genus 2. All vertices of the octagon are glued in $M(P)$ one point such that the total angle around it is 6π. Such points are called **singular** in $M(P)$.
If the total angle around a singular point is $2\pi(k+1)$, then $k \in \mathbb{N}$ is called its index.

Definition 8. A **translation surface** is a topological compact connected surface M with a finite subset $\Sigma \subset M$ of singular points and an atlas of charts $\omega = \{\xi_\alpha : U_\alpha \to \mathbb{C}\}_{\alpha\in\mathcal{A}}$ on $M \setminus \Sigma$ such that any transition map $\xi_\beta \circ \xi_\alpha^{-1} : \xi_\alpha(U_\alpha \cap U_\beta) \to \xi_\beta(U_\alpha \cap U_\beta)$ is a translation.

Definition 9. For every $\theta \in [0,1)$ and any regular point $x \in M \setminus \Sigma$ **the translation flow** $(\varphi_t^\theta)_{t\in\mathbb{R}}$ **in direction** θ is defined by $\varphi_t^\theta x = x + te^{2\pi i\theta}$ (in local coordinates).

Remark. The billiard flow $(b_t)_{t\in\mathbb{R}}$ on P_θ is isomorphic with the translation flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ in direction θ on $M(P)$.

Theorem 8 (Kerkhoff-Masur-Smillie, 1986)**.** *For every translation surface (M,ω) for a.e. direction θ the translation flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ is uniquely ergodic.*

Remark. There is a dense set of directions θ for which the translation flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ has periodic orbits.

Corollary 9. *For every rational polygon P for almost every direction θ all billiard orbits are equidistributed in P. For a dense set of directions θ the billiard flow allows periodic orbits.*

Moduli space of translation surfaces

Remark 1. Every translation surface can be represented as a polygon for which sides are partitioned into pairs of parallel sides of the same length.

Remark 2. Polygonal representations are not unique. We can cut-and-paste some triangles.

Remark 3. One polygonal representation can be obtained from another representation after a finite chain of cut-and-paste operations.

Definition 10. The moduli space of translation surfaces of genus g

$$\mathcal{M}_g^1 = \{\text{the set polygons with area 1 leading to genus } g \text{ surfaces}\}/\sim$$

$\sim$ the relation described in Remark 3.

$\mathcal{M}_g^1$ has a natural manifold structure given by the coordinates of vertices.
$\mathcal{M}_g^1$ is not compact and not connected (except $g = 1$).
$\mathcal{M}_1^1 \simeq SL_2(\mathbb{R})/SL_2(\mathbb{Z})$.
On $\mathcal{M}_g^1$ we deal with two important flows: the Teichmüller $(g_t)_{t\in\mathbb{R}}$ and the rotation $(r_\theta)_{\theta\in\mathbb{T}}$ given by linear transformation determined by

$$g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \text{ and } r_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Theorem 10 (Masur, 1991)**.** *For every translation surface (M,ω) if its forward Teichmüller orbit $(g_t(M,\omega))_{t>0}$ is* ***recurrent*** *(there exists a compact subset $C \subset \mathcal{M}_g^1$ and $t_n \to +\infty$ such that $g_{t_n}(M,\omega) \in C$) then the vertical translation flow $(\varphi_t^v)_{t\in\mathbb{R}}$ on (M,ω) is uniquely ergodic.*

Theorem 11 (Kerkhoff-Masur-Smillie, 1986)**.** *For every translation surface (M,ω) for a.e. direction θ the rotated surface $r_\theta(M,\omega)$ has a recurrent forward Teichmüller orbit.*

Since the translation flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ on (M,ω) coincides with $(\varphi_t^v)_{t\in\mathbb{R}}$ on $r_{\frac{\pi}{2}-\theta}(M,\omega)$, both give the unique ergodicity of $(\varphi_t^\theta)_{t\in\mathbb{R}}$ on (M,ω) for a.e. direction θ.