On the extension of bi-Lipschitz mappings

L. Birbrair, A. Fernandes, Z. Jelonek

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1. In modern Lipschitz Geometry of singularities there are two natural metric and three natural classification problems.

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2. The relation between the inner and the outer metrics are very well investigated and the theory of LNE (Lipschitz Normal embedding) is on the rapid development (see, for example the works of Misef and Pichon and Kerner, Pedersen and Ruas.

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2. That is why it makes sense to study the relation of outer and ambient equivalence, but it makes sense only in the case, when one has no topological obstructions.

1. One of very basic examples in the paper show that two compact curves in the plane can be Outer Lipschitz equivalent, ambient topologically equivalent, but not Ambient Lipschitz equivalent.

1. The examples of Birbrair and Gabrielov show that for germs of real surfaces the outer Lipschitz equivalence, considered together with ambient topological equivalence, do not imply ambient Lipschitz equivalences.

1. Given a semialgebraic set $X \subset \mathbb{R}^n$ of dimension k. How one can estimate in terms of k the dimension of the space $\mathbb{R}^m$, such that there exists a bi-Lipschitz embedding of X into $\mathbb{R}^m$? We prove :

2. **Theorem 3.3.**(*Whitney type theorem for bi-Lipschitz category*)
Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set of dimension k with the induced metric. Then there is a bi-Lipschitz and semi-algebraic embedding $f : X \to \mathbb{R}^{2k+1}$. Moreover, if X is an algebraic variety, then we can assume that $f(X)$ is also an algebraic variety.

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1. We give also a complex version of this theorem. Another result of this type is:

2. **Theorem 3.5.** *Any compact semialgebraic set X of dimension k is bi-Lipschitz equivalent, with respect to the inner metric, to a Lipschitz normally embedded semialgebraic set $Y \subset \mathbb{R}^{2k+1}$.*

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1. When the Outer Lipschitz equivalence implies the ambient Lipschitz equivalence. In general two bi-Lipschitz embeddings into $\mathbb{R}^{2k+1}$ are not Ambient equivalent, i.e., the embedding is not unique.

2. We say that a semialgebraic set X admits a unique embedding to $\mathbb{R}^n$, if for any two bi-Lipschitz embeddings $f, g : X \to \mathbb{R}^n$, where $f(X), g(X)$ are semialgebraic sets, there is a bi-Lipschitz homeomorphism Φ of $\mathbb{R}^n$, such that $g = \Phi \circ f$.

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1. In other words the subset $X \subset \mathbb{R}^n$ has a unique embedding into $\mathbb{R}^n$, if any other semialgebraic subset $Y \subset \mathbb{R}^n$, which is equivalent to X with respect to the outer metric, is Ambient Lipschitz equivalent to X.
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1. We say that a bi-Lipschitz homeomorphism is said to be *tame* if it is a composition of bi-Lipschitz homeomorphisms of the form

$$\Phi : \mathbb{R}^n \ni (x_1, ..., x_n) \to (x_1, ..., x_{n-1}, x_n + p(x_1, ..., x_{n-1})) \in \mathbb{R}^n$$

and linear mappings with determinant one.

2. **Note that such a mapping has almost everywhere jacobian equal to one, hence it preserves the $n-$dimensional volume of $\mathbb{R}^n$.!!**

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1. **Theorem 4.5.** *Let X be a closed semialgebraic subset of $\mathbb{R}^n$ of dimension k. Let $f : X \to \mathbb{R}^n$ be a semialgebraic and bi-Lipschitz embedding. If $n \geq 2k+2$, then there exists a tame bi-Lipchitz and semialgebraic homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$F|_X = f.$$

Moreover, there is a continues semialgebraic family of tame bi-Lipchitz semialgebraic homeomorphisms $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, such that $F_0 = identity\ and\ F_1|_X = f$.

1. **Corollary 4.6.** *Let X be a closed semialgebraic set of dimension k. If $n \geq 2k+2$, then X has a unique semialgebraic and bi-Lipchitz embedding into $\mathbb{R}^n$ (up to a semialgebraic and bi-Lipschitz tame homeomorphism of $\mathbb{R}^n$).*

1. Using slightly more subtle methods in the local case, we obtain better local results. The analog of Theorem 3.3 is the following:

2. **Theorem 5.4.** *Let* $(X,0)$ *be a germ of a closed semialgebraic set of dimension* k*. Then there is a bi-Lipschitz semi-algebraic embedding of* $(X,0)$ *into* $(\mathbb{R}^{2k},0)$.

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1. We give also a complex version of this theorem:
2. **Theorem 5.5.** *Let $(X,0)$ be a germ of closed complex algebraic affine variety of dimension k. Then there is a bi-Lipschitz (and regular) embedding of $f:(X,0)\to(\mathbb{C}^{2k},0)$, such that $(f(X),0)$ is a germ of complex algebraic variety.*
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1. **Theorem 5.9.** *Let* $(X,0)$ *be a germ of a closed semialgebraic subset of* $(\mathbb{R}^n,0)$ *of dimension* k. *Let* $f:(X,0)\to(\mathbb{R}^n,0)$ *be a semialgebraic and bi-Lipschitz embedding. If* $n\geq 2k+1$, *then there exists a germ of tame bi-Lipchitz and semialgebraic homeomorphism* $F:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ *such that*

$$F|_X = f.$$

Moreover, there exists a continues semialgebraic family of germs of tame bi-Lipchitz and there exists a semialgebraic homeomorphisms $F_t:(\mathbb{R}^n,0)\times\mathbb{R}\to(\mathbb{R}^n,0)$, *such that* $F_0 = identity \text{ and } F_1|_X = f$.

1. **Corollary 5.10.** *Let* $(X,0)$ *be a germ of a closed semialgebraic set of dimension* k*. If* $n \geq 2k+1$*, then* X *has a unique semialgebraic and bi-Lipchitz embedding into* $(\mathbb{R}^n,0)$ *(up to a germ of semialgebraic and bi-Lipschitz tame homeomorphism of* $(\mathbb{R}^n,0)$*).*

2. **Corollary 5.18.** *Let* $(X,0)$ *and* $(Y,0)$ *be germs of one-dimensional semialgebraic subsets of* $\mathbb{R}^2$*. If* $(X,0)$ *and* $(Y,0)$ *are outer bi-Lipschitz equivalent, then they are ambient bi-Lipschitz equivalent.*

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2. **Corollary 5.18.** *Let $(X, 0)$ and $(Y, 0)$ be germs of one-dimensional semialgebraic subsets of $\mathbb{R}^2$. If $(X, 0)$ and $(Y, 0)$ are outer bi-Lipschitz equivalent, then they are ambient bi-Lipschitz equivalent.*

1. However for surfaces the situation is more complicated. There are germ of surfaces $(X,0)$ and $(Y,0)$ in $\mathbb{R}^n$ (where $n=3$ or $n=4$) and a semialgebraic bi-Lipschitz mapping $f:(X,0)\to(Y,0)$ which can be extended to homeomorphism of $(\mathbb{R}^n,0)$ but it can not be extended to a bi-Lipschitz homeomorphism of $(\mathbb{R}^n,0)$.

1. In this paper we consider the Lipschitz category $\mathcal{L}$. Its objects are closed subsets of $\mathbb{R}^n$ (mainly semialgebraic) equipped with the outer metric and morphisms are Lipschitz mappings with respect to the outer metric.

2. We also consider the Lipchitz semialgebraic category $\mathcal{LS}$. Its objects are closed semialgebraic subsets of $\mathbb{R}^n$ equipped with the outer metric and morphisms are Lipschitz and semialgebraic mappings with respect to the outer metric. We start with the following basic definition which is valid for both categories (for the category $\mathcal{LS}$ we use brackets):

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1. **Definition** Let X be closed (semialgebraic) subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^n$ be a Lipschitz mapping. We say that f is a bi-Lipschitz (semialgebraic) embedding if the mapping $f : X \to Y$ is a bi-Lipschitz (semialgebraic) isomorphism, i.e., f and f^{-1} are Lipschitz (semialgebraic) mappings.

1. **Remark** Note that the bi-Lipschitz embedding is a closed mapping, in particular the set $f(X) = Y$ is closed and the mapping $f : X \to Y$ is a homeomorphism. Indeed, if $Y \ni y_k \to y \in \overline{Y}$, then $x_k = f^{-1}(y_k)$ is a Cauchy sequence in X, so it has a limit $x \in X$. By continuity of f, $y = f(x) \in f(X)$. In particular a bi-Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. Indeed, f is open (invariance of domain) and closed, hence $f(\mathbb{R}^n) = \mathbb{R}^n$.

1. Let X be a closed subset of $\mathbb{R}^n$. We will denote by $\mathbf{V}(X)$ the vector space of all Lipschitz functions on X. If $f : X \to Y$ is a Lipschitz mapping of closed sets, then we have the natural homomorphism $\mathbf{V}(f) : \mathbf{V}(Y) \ni h \to h \circ f \in \mathbf{V}(X)$.

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1. Since every Lipschitz function on X is the restriction of some Lipschitz function on $\mathbb{R}^n$, we have that the mapping $\iota^* : \mathbf{V}(\mathbb{R}^n) \ni h \mapsto h \circ \iota \in \mathbf{V}(X)$ is an epimorphism, where the mapping ι is the inclusion $\iota : X \to \mathbb{R}^n$. We have the following more general fact:
2. **Proposition** Let $X \subset \mathbb{R}^m$ be a closed subset and $f : X \to \mathbb{R}^n$ be a Lipschitz mapping. The following conditions are equivalent:
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1. *Proof:*

1. The same result holds in the category $\mathcal{LS}$ (with the same proof as above):
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1. **Lemma** Let X be a closed subset of $\mathbb{R}^n$. Assume that the projection $\pi : X \ni (x_1, ..., x_n) \to (x_1, ..., x_l, 0, ..., 0) \in \mathbb{R}^l \times \{0\}$ is a bi-Lipschitz embedding. Then, there exists a tame bi-Lipschitz homeomorphism $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Pi|_X = \pi$.

1. *Proof:*

1. The next Lemma is a version of the Whitney Embedding Theorem:
2. **Lemma** Let X be a closed semialgebraic set of $\mathbb{R}^n$ of dimension k. If $n > 2k+1$, then there exists a system of coordinates $(x_1, ..., x_n)$ such that the projection $\pi : X \ni (x_1, ..., x_n) \to (x_1, ..., x_{2k+1}, 0, ..., 0) \in \mathbb{R}^{2k+1} \times \{0\}$ is a bi-Lipschitz embedding. Moreover, if X is an algebraic variety, then we can assume that $\pi(X)$ is also an algebraic variety.

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1. A direct consequence of "Whitney" Lemma is:
2. **Theorem** Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set of dimension k with the induced metric. Then there is a bi-Lipschitz and semi-algebraic embedding $f : X \to \mathbb{R}^{2k+1}$. Moreover, if X is an algebraic variety, then we can assume that $f(X)$ is also an algebraic variety.

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1. This result is sharp. In general we cannot embed a semi-algebraic curve in a bi-Lipschitz way into $\mathbb{R}^2$, even if there is no topological obstruction to do it. Indeed, we have the example given by Edson Sampaio.

1. **Theorem** Any compact semialgebraic set X of dimension k is bi-Lipschitz equivalent, with respect to the inner metric, to a Lipschitz normally embedded semialgebraic set $Y \subset \mathbb{R}^{2k+1}$.
2. Proof: By Birbrair-Mostowski there exists $N > 0$, such that X is bi-Lipschitz equivalent, with respect to the inner metric, to a Lipschitz normally embedded semialgebraic set $Y \subset \mathbb{R}^N$. By previuos Lemmas there is a bi-Lipchitz homeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\Phi(Y) \subset \mathbb{R}^{2k+1} \times \{0\}$. But Φ preserves the equivalence of the inner metric and the outer metric.

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1. **Lemma** Let $X, Y \subset \mathbb{R}^n$ be closed subsets. Assume that there are linear subspaces L^s and H^{n-s}, of dimensions s and $n-s$ respectively, such that $X \subset L^s$ and $Y \subset H^{n-s}$. If $f : X \to Y$ is a bi-Lipschitz isomorphism, then there is a tame bi-Lipschitz homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$F|_X = f.$$

1. *Proof:*

1. **Theorem** Let $X, Y \subset \mathbb{R}^n$ be closed semialgebraic sets of dimension k. Let $f : X \to Y$ be a bi-Lipschitz mapping. If $n \geq 4k + 2$, then f can be extended to a tame bi-Lipschitz mapping $F : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, there is a continuous family of tame bi-Lipschitz mappings $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, such that $F_0 = identity \text{ and } F_1|_X = f$.

1. *Proof:*

1. **Corollary** Let X be a closed semialgebraic set of dimension k. If $n \geq 4k + 2$, then f has a unique bi-Lipschitz embedding into $\mathbb{R}^n$ as a semialgebraic set (up to a bi-Lipschitz mapping of $\mathbb{R}^n$).

1. **Definition** Let L^s, H^{n-s-1} be two disjoint linear subspaces of $\mathbb{P}^n$. Let π_∞ be a hyperplane (a hyperplane at infinty) and assume that $L^s \subset \pi_\infty$. By a projection π_L with center L^s we mean the mapping:

$$\pi_L : \mathbb{R}^n = \mathbb{P}^n \backslash \pi_\infty \ni x \mapsto < L^s, x > \cap H^{n-s-1} \in H^{n-s-1} \backslash \pi_\infty = \mathbb{R}^{n-s-1}.$$

Here by $< L, x >$ we mean a linear subspace spanned by L and $\{x\}$.

1. **Lemma** Let X be a closed subset of $\mathbb{R}^n$. Denote by $\Lambda \subset \pi_\infty$ the set of directions of all secants of X and let $\Sigma = cl(\Lambda)$. where π_∞ is a hyperplane at infinity and we consider the projective closure. Let $\pi_L : \mathbb{R}^n \to \mathbb{R}^l$ be a projection with center L. Then $\pi_L|_X$ is a bi-Lipschitz embedding if and only if $L \cap \Sigma = \emptyset$.

1. *Proof:*

1. **Lemma** Let $X \subset \mathbb{R}^n$ be a closed set and let $f : X \to \mathbb{R}^m$ be a Lipschitz mapping. Let $Y := graph(f) \subset \mathbb{R}^n \times \mathbb{R}^m$. Then the mapping $\phi : X \ni x \mapsto (x, f(x)) \in Y$ is a bi-Lipschitz embedding.

1. *Proof*: Since f is Lipschitz, there is a constant C such that

$$||f(x) - f(y)|| < C||x - y||.$$

We have

$$||\phi(x) - \phi(y)|| = ||(x - y, f(x) - f(y))|| \leq$$

$$\leq ||x-y||+||f(x)-f(y)|| \leq ||x-y||+C||x-y|| \leq (1+C)||x-y||.$$

Moreover

$$||x - y|| \leq ||\phi(x) - \phi(y)||.$$

Hence

$$||x - y|| \leq ||\phi(x) - \phi(y)|| \leq (1 + C)||x - y||.$$

Lemma Let $X \subset \mathbb{R}^{2n}$ be a closed semialgebraic set of dimension k and $n \geq 2k+1$. Assume that mappings

$$\pi_1 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (x_1, ..., x_n) \in \mathbb{R}^n$$

and

$$\pi_2 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (y_1, ..., y_n) \in \mathbb{R}^n$$

are bi-Lipschitz embeddings. Then there exist linear isomorphisms S, T of $\mathbb{R}^n$ with determinant equals to 1, such that if we change coordinates in $\mathbb{R}^n \times \{0\}$ by $(z_1, ..., z_n) = S(x_1, ..., x_n)$ and we change coordinates in $\{0\} \times \mathbb{R}^n$ by $(w_1, ..., w_n) = T(y_1, ..., y_n)$, then all the projections:

$$q_r : X \ni (z_1, ..., z_n, w_1, ..., w_n) \to (z_1, ..., z_r, w_{r+1}, ..., w_n) \in \mathbb{R}^n,$$

$$r = 0, ..., n$$

are bi-Lipschitz embeddings.

1. *Proof:*

Theorem Let X be a closed semialgebraic subset of $\mathbb{R}^n$ of dimension k. Let $f : X \to \mathbb{R}^n$ be a semialgebraic bi-Lipschitz embedding. If $n \geq 2k + 2$, then there exists a tame bi-Lipchitz semialgebraic mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F|_X = f.$$

Moreover, there is a continues semialgebraic family of tame bi-Lipchitz semialgebraic mappings $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, such that $F_0 = \mathit{identity\ and}\ F_1|_X = f$.

1. *Proof:*

Corollary Let X be a closed semialgebraic set of dimension k. In $n \geq 2k + 2$, then X has a unique semialgebraic bi-Lipchitz embedding into $\mathbb{R}^n$ (up to a semialgebraic and bi-Lipschitz mapping of $\mathbb{R}^n$).

Example Let $X \subset \mathbb{R}^3$ be semi-algebraic curves as on the picture (a) and $Y \subset \mathbb{R}^3$ as on the picture (b). Hence X, Y are bi-Lipschitz outer equivalent, however from topological reasons they are not ambient bi-Lipschitz equivalent in $\mathbb{R}^3$. Hence our Theorem does not work for $n = 3$ and $k = 1$.

It also does not work for $n = 2$ and $k = 1$. Indeed, in $\mathbb{R}^2$ there are compact semialgebraic curves which are bi-Lipschitz outer equivalent and topologically ambient equivalent but not ambient bi-Lipchitz equivalent. Indeed let $X \subset \mathbb{R}^2$ be semi-algebraic curves as on the picture (a) below and $Y \subset \mathbb{R}^2$ as on the picture (b) below. Here red curves have order of contact different than blue curves. Hence X, Y are bi-Lipschitz outer equivalent and topologically ambient equivalent, however they are not ambient bi-Lipschitz equivalent in $\mathbb{R}^2$.

1. Hence our Theorem is sharp for $k = 1$. Moreover, there are germ of surfaces $(X,0)$ and $(Y,0)$ in $\mathbb{R}^n$ (where $n = 3$ or $n = 4$) and a semialgebraic bi-Lipschitz mapping $f : (X,0) \to (Y,0)$ which can be extended to homeomorphism of $(\mathbb{R}^n,0)$ but it can not be extended to a bi-Lipschitz homeomorphism of $(\mathbb{R}^n,0)$. Hence our theorem is nearly sharp also for $k = 2$. Let us note that we can obtain in a similar way a complex version of our Theorems:
2. **Lemma** Let X be a closed affine variety of $\mathbb{C}^n$ of dimension k. If $n > 2k+1$, then there exists a system of complex coordinates $(x_1,...,x_n)$ such that the projection $\pi : X \ni (x_1,...,x_n) \to (x_1,...,x_{2k+1},0,...,0) \in \mathbb{C}^{2k+1} \times \{0\}$ is a bi-Lipschitz embedding.

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2. **Lemma** Let X be a closed affine variety of $\mathbb{C}^n$ of dimension k. If $n > 2k + 1$, then there exists a system of complex coordinates $(x_1, ..., x_n)$ such that the projection $\pi : X \ni (x_1, ..., x_n) \to (x_1, ..., x_{2k+1}, 0, ..., 0) \in \mathbb{C}^{2k+1} \times \{0\}$ is a bi-Lipschitz embedding.

Theorem Let X be a complex affine algebraic variety of dimension k. Then there is a bi-Lipschitz (and regular) embedding $f : X \to \mathbb{C}^{2k+1}$ such that the set $f(X)$ is an affine algebraic variety. Moreover, this embedding is unique up to a bi-Lipchitz semialgebraic homeomorphism of $\mathbb{C}^{2k+1}$.

1. **Example** This Theorem is sharp. Indeed we show that there exists an affine curve Γ without self-intersections, which cannot be embedded in a bi-Lipschitz way into $\mathbb{C}^2$ as an algebraic curve. Note that such a curve in an obvious way has a continuous embedding into $\mathbb{C}^2$. Let Γ' be a rational curve with nine cusps (such a curve can be constructed by gluing a punctured cusp $\{x^2 = y^3\}$ with a punctured $\mathbb{P}^1$ several times). It is a projective curve. Let $O \in \Gamma'$ be a smooth point and take $\Gamma = \Gamma' \setminus O$.

1. Fix an embedding $\Gamma \subset \mathbb{C}^N$ and consider Γ with the induced metric structure from $\mathbb{C}^N$. Hence Γ is an affine curve with nine cusps. Assume that there is a bi-Lipschitz embedding $\phi : \Gamma \to \mathbb{C}^2$ and $\phi(\Gamma) = \Lambda$ is an algebraic curve. The curve Λ has also nine cusps, hence the projective closure of Λ in $\mathbb{P}^2$ is a rational cuspidal curve with at least nine cusps- this is a contradiction.

Definition Let $(X, 0)$ be closed (semialgebraic) subset of $(\mathbb{R}^n, 0)$ and let $f : (X, 0) \to (\mathbb{R}^n, 0)$ be a Lipschitz mapping. We say that f is a bi-Lipschitz (semialgebraic) embedding if

1. $f((X, 0)) = (Y, 0)$ is a germ of closed semialgebraic subset of $\mathbb{R}^n$,
2. the mapping $f : (X, 0) \to (Y, 0)$ is a bi-Lipschitz and semialgebraic isomorphism, i.e., f and f^{-1} are Lipschitz and semialgebraic mappings.

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1. Let X be a closed subset of $\mathbb{R}^n$. We will denote by $\mathbf{VS}_0(X)$ the vector space of all germs of Lipschitz functions on $(X,0)$. If $f:(X,0)\to(Y,0)$ is a Lipschitz mapping of closed sets, then we have the natural homomorphism $\mathbf{VS}_0(f):\mathbf{VS}_0(Y)\ni h\to h\circ f\in\mathbf{VS}_0(X)$.
2. If $(X,0)$ is a germ of closed semialgebraic subset of $\mathbb{R}^n$, then by $\mathbf{VS}_0(X)$ we will denote the vector space of all germs of Lipschitz and semialgebraic functions on $(X,0)$.

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2. If $(X,0)$ is a germ of closed semialgebraic subset of $\mathbb{R}^n$, then by $\mathbf{VS}_0(X)$ we will denote the vector space of all germs of Lipschitz and semialgebraic functions on $(X,0)$.

1. **Proposition** Let $(X,0) \subset (\mathbb{R}^m, 0)$ be a closed semialgebraic subset and $f : (X,0) \to (\mathbb{R}^n, 0)$ be a Lipschitz semialgebraic mapping. The following conditions are equivalent:
2. f is a bi-Lipschitz semialgebraic embedding,
3. the induced mapping $\mathbf{VS}_0(f) : \mathbf{VS}_0(\mathbb{R}^n) \to \mathbf{VS}_0(X)$ is an epimorphism.

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1. **Lemma** Let $(X, 0)$ be a closed semialgebraic set of $(\mathbb{R}^n, 0)$ of dimension k. If $n > 2k$, then there exists a system of coordinates $(x_1, ..., x_n)$ such that the projection $\pi : X \ni (x_1, ..., x_n) \to (x_1, ..., x_{2k}, 0, ..., 0) \in \mathbb{R}^{2k} \times \{0\}$ gives a bi-Lipschitz embedding of $(X, 0)$ into $(\mathbb{R}^{2k}, 0)$.

1. **Theorem** Let $(X, 0)$ be a germ of closed semialgebraic set of dimension k. Then there is a bi-Lipschitz (and semialgebraic) embedding of $(X, 0)$ into $(\mathbb{R}^{2k}, 0)$.

1. Theorem Let $(X, 0)$ be a germ of closed complex algebraic affine variety of dimension k. Then there is a bi-Lipschitz (and regular) embedding of $f : (X, 0) \to (\mathbb{C}^{2k}, 0)$, such that $(f(X), 0)$ is a germ of complex algebraic variety.

1. **Theorem** Let $(X,0)$ be a germ of a closed semialgebraic subset of $(\mathbb{R}^n,0)$ of dimension k. Let $f:(X,0)\to(\mathbb{R}^n,0)$ be a bi-Lipschitz embedding. If $n\geq 4k$, then there exists a germ of tame bi-Lipchitz homeomorphism $F:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ such that

$$F|_X = f.$$

Moreover, there is a continuos semialgebraic family of germs of tame bi-Lipchitz and semialgebraic homeomorphisms $F_t:(\mathbb{R}^n,0)\times\mathbb{R}\to(\mathbb{R}^n,0)$, such that $F_0 = \textit{identity and } F_1|_X = f$.

1. **Corollary** Let $(X, 0)$ be a germ of a closed semialgebraic set of dimension k. If $n \geq 4k$, then f has a unique bi-Lipschitz embedding into $\mathbb{R}^n$ as semialgebraic set (up to a germ of a bi-Lipschitz mapping of $\mathbb{R}^n$).

Lemma Let $(X,0) \subset (\mathbb{R}^{2n},0)$ be a germ of a closed semialgebraic set of dimension k and $n \geq 2k$. Assume that mappings

$$\pi_1 : (X,0) \ni (x_1, ..., x_n, y_1, ..., y_n) \to (x_1, ..., x_n) \in (\mathbb{R}^n, 0)$$

and

$$\pi_2 : (X,0) \ni (x_1, ..., x_n, y_1, ..., y_n) \to (y_1, ..., y_n) \in (\mathbb{R}^n, 0)$$

are bi-Lipschitz embeddings. Then there exist linear isomorphisms S, T of $\mathbb{R}^n$ with determinant 1, such that if we change coordinates in $\mathbb{R}^n \times \{0\}$ by $(z_1, ..., z_n) = S(x_1, ..., x_n)$ and we change coordinates in $\{0\} \times \mathbb{R}^n$ by $(w_1, ..., w_n) = T(y_1, ..., y_n)$, then all projections

$$q_r : (X,0) \ni (z_1, ..., z_n, w_1, ..., w_n) \to (z_1, ..., z_r, w_{r+1}, ..., w_n) \in (\mathbb{R}^n, 0),$$

$$r = 0, ..., n$$

are bi-Lipschitz embeddings.

1. **Theorem** Let $(X,0)$ be a germ of a closed semialgebraic subset of $\mathbb{R}^n$ of dimension k. Let $f:(X,0)\to(\mathbb{R}^n,0)$ be a semialgebraic and bi-Lipschitz embedding. If $n\geq 2k+1$, then there exists a germ of tame bi-Lipchitz and semialgebraic homeomorphism $F:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ such that
$$F|_X = f.$$
Moreover, there is a continuos and semialgebraic family of germs of tame bi-Lipchitz and semialgebraic homeomorphisms $F_t:(\mathbb{R}^n,0)\times\mathbb{R}\to(\mathbb{R}^n,0)$, such that $F_0 = \textit{identity and } F_1|_X = f.$

1. **Corollary** Let $(X, 0)$ be a germ of a closed semialgebraic set of dimension k. If $n \geq 2k + 1$, then $(X, 0)$ has a unique semialgebraic and bi-Lipchitz embedding into $(\mathbb{R}^n, 0)$ (up to a germ of a semialgebraic and bi-Lipschitz mapping of $(\mathbb{R}^n, 0)$).

THANK YOU FOR ATTENTION!