# On some application of topological dynamics and model theory

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# Basic notions in topological dynamics

Let $G$ be a topological (often discrete) group.

1. A *$G$-flow* is a pair $(G, X)$, where $X \neq \emptyset$ is compact and $G$ acts continuously on $X$.
2. $(G, X)$ is *point-transitive* if $X$ contains a dense $G$-orbit.
3. A *$G$-ambit* is a $G$-flow $(G, X, x_0)$, where $x_0 \in X$ with $Gx_0$ dense in $X$.
4. A $G$-flow is *minimal* if it contains no proper subflows.

**Fact**

Each flow contains a minimal subflow.

**Example: Bernoulli shift**

$G := \mathbb{Z}$ acts on $X := 2^{\mathbb{Z}}$ via $(k * f)(n) := f(n-k)$. If $f_0 \in X$ contains all finite 0-1 sequences as subsequences of consecutive elements, then $G * f_0$ is dense, so $(G, X, f_0)$ is an ambit. If $f \in X$ is periodic, then $G * f$ is finite (so closed), so $G * f$ is a minimal subflow. There are other minimal subflows of $(G, X)$.

# Some goals of abstract topological dynamics

### Fact

1. For every group $G$ there is a unique (up to $\cong$) universal $G$-ambit. If $G$ is discrete, it is $(G, \beta G, \mathcal{U}_e)$.
2. For every group $G$ there is a unique (up to $\cong$) universal minimal $G$-flow.

### General goals of abstract topological dynamics

For a given $G$:

- classify minimal $G$-flows: deep structural theory with lots of tools and special classes of flows (Auslander, Ellis, Furstenberg, Glasner, and others).
- describe the universal minimal $G$-flow, especially when $G$ is the group of automorphisms of a Fraïssé structure (Kechris, Pestov, Todorčević, and others).

In the mid 2000's Newelski came up with an idea of applying topological dynamics framework and tools to model theory. The point is that various type spaces can naturally be treated as flows; some fundamental results of stable group theory are in fact top. dyn. statements, and there is hope to generalize various ideas to an unstable context via this top. dyn. point of view.

My personal look at the subject is that various type spaces are naturally flows and topological dynamics yields new notions and methods to study [strong] types, definable sets, and other model-theoretic objects, and prove new results about them. On the other hand, model theory gives new descriptions of various objects in topological dynamics, yields new objects, and may lead to new results or new proofs.

# Topological dynamics in model theory

- The theory developed by Newelski, including applications to generating in finitely many steps in groups covered by countably many type-definable sets and *Newelski's conjecture.*
- Solutions to Newelski's conjecture (Chernikov, Gismatullin, Jagiella, Penazzi, Pillay, Simon, Yao, and others).
- Definably amenable groups and theories (Chernikov, Hrushovski, Krupiński, Pillay, Simon, and others).
- Applications of topological dynamics to quotients of definable groups by model-theoretic connected components and to the spaces of strong types (Krupiński, Pillay, Rzepecki). The *Ellis group* of a theory (Krupiński, Newelski, Simon).
- Variants of the last item (the "core" of a theory and its automorphism group) with applications to approximate subgroups (Hrushovski).

# Topological dynamics in model theory — cont.

- Kechris, Pestov, Todorčević theory: correspondences between various Ramsey and dynamical properties of Fraïssé structures; describing universal minimal flows of groups of automorphisms (many people involved in this program).
- Definable counterparts of Kechris, Pestov, Todorčević theory: correspondences between some "definable" Ramsey properties and dynamical properties of theories; criteria for [pro]finiteness of the Ellis group of the theory (Krupiński, Lee, Moconja).
- Correspondences between model-theoretic notions and dynamical notions. e.g. "stable=WAP", "NIP=tame" (Ben-Yaacov, Ibarlucía, and others).
- Applications of model theory to the theory of algebras of functions and compactifications of groups (Ben-Yaacov, Ibarlucía, Tsankov,...).
- Some recent developments on Keisler measures (Chernikov, Conant, Gannon, Hanson).
- ...

# Large sets and points in flows

Let $(G, X)$ be a flow.

## Definition

1. $U \subseteq X$ is *generic (syndetic)* if $AU = X$ for some finite $A \subseteq G$.
2. $U \subseteq X$ is *weak generic* if $U \cup U'$ is generic for some non-generic $U' \subseteq X$.
3. $p \in X$ is *[weak] generic* if every neighborhood $U$ of $p$ is [weak] generic.
4. $p \in X$ is *almost periodic* if it belongs to a minimal subflow.

Let $Gen(X) := \{p \in X : p \text{ generic}\}$. Similarly, we define $WGen(X)$ and $APer(X)$.

## Fact (Newelski)

1. $WGen(X) = \text{cl}(APer(X)) \neq \emptyset$.
2. If $Gen(X) \neq \emptyset$, then $Gen(X) = WGen(X) = APer(X)$ is a unique minimal subflow of $X$.

# Large sets and points — cont.

## Example

In $(\mathbb{Z}, 2^{\mathbb{Z}})$, any $f_0$ with $\mathbb{Z} * f_0$ dense is not almost periodic. Since $Per(2^{\mathbb{Z}})$ is dense in $2^{\mathbb{Z}}$, we get $WGen(2^{\mathbb{Z}}) = \mathrm{cl}(APer(2^{\mathbb{Z}})) \supseteq \mathrm{cl}(Per(2^{\mathbb{Z}})) = 2^{\mathbb{Z}}$. So $Gen(2^{\mathbb{Z}}) = \emptyset$.

## Example

Consider the flow $(\mathbb{Z}, S^1)$, where $S^1$ is the unit circle and $\mathbb{Z}$ acts on $S^1$ by: $nz := \alpha^n z$, where $\alpha \in S^1$ is not a root of unity. This flow is clearly minimal. Thus, $Gen(S^1) = S^1$.

## Proof of $Gen(S^1) = S^1$

By compactness of $S^1$, it is enough to show that for every nonempty open $U \subseteq S^1$ finitely many translates of $U$ by elements from $\{\alpha^n : n \in \mathbb{Z}\}$ cover $S^1$. This in turn follows from the fact that $S^1$ is a topological group and $\{\alpha^n : n \in \mathbb{Z}\}$ is dense in it.

- $\mathcal{L} = \{f_i, P_j, c_k : i \in I, j \in J, k \in K\}$: a *language* (or *signature*), i.e. the $f_i$'s are function symbols, $P_j$'s relational symbols, $c_k$'s constant symbols. For example, $\mathcal{L}_{gr} = \{\cdot, e\}$ is the language of group theory.
- Using symbols of $\mathcal{L}$, $=$, variables, logical connectives, and quantifiers, one builds *$\mathcal{L}$-formulas*. For example, $\varphi(x) := (\exists y)(\neg(x \cdot y = y \cdot x))$ is an $\mathcal{L}_{gr}$-formula with the free variable $x$.
- An *$\mathcal{L}$-theory* is a set of $\mathcal{L}$-sentences (i.e. $\mathcal{L}$-formulas without free variables).
- An *$\mathcal{L}$-structure* is a set $M$ together with interpretations of all symbols of the language; e.g. if $f_i$ is a binary function symbol, then its interpretation is a function $f_i^M : M^2 \to M$.
- An $\mathcal{L}$-structure $M$ is a *model* of a theory $T$ (symbolically, $M \models T$) if $M$ satisfies all sentences from $T$. For example, if $T$ is group theory (i.e. the set of the well-known three axioms), then $M \models T$ iff $(M, \cdot, e)$ is a group.

- Theorem (Gödel). Let $T$ be a theory.
  - A theory $T$ is *consistent* (i.e. does not prove a contradiction in a syntactic sense) iff $T$ has a model.
  - (Completeness) For any formula $\varphi$, $T \vdash \varphi$ iff $T \models \varphi$, where $T \vdash \varphi$ means that $T$ proves $\varphi$, and $T \models \varphi$ means that $\varphi$ holds in every model of $T$.
  - (Compactness) A theory $T$ has a model iff every finite subset of $T$ has a model.
- For an $\mathcal{L}$-structure $M$ the *theory of $M$* is $\mathrm{Th}(M) := \{\varphi \text{ a sentence in } \mathcal{L} : M \models \varphi\}$.
- A consistent theory $T$ is *complete* if for every sentence $\varphi$ we have $T \models \varphi$ or $T \models \neg\varphi$.
- $T = \mathrm{Th}(M)$ is complete for any $\mathcal{L}$-structure $M$.
- Let $M \subseteq N$ be $\mathcal{L}$-structures. We say that $M$ is an *elementary substructure* of $N$ (in symbols $M \prec N$), if for every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $\bar{a} = (a_1, \dots, a_n) \in M^n$, $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a})$.

- Theorem (Löwenheim-Skolem). Let $M$ be an infinite $\mathcal{L}$-structure and $\kappa \geq |\mathcal{L}| + \aleph_0$. Then,
  - if $\kappa \geq |M|$, there is $N \succ M$ with $|N| = \kappa$,
  - if $\kappa < |M|$, there is $N \prec M$ with $|N| = \kappa$.
- A $\emptyset$-*definable set* in an $\mathcal{L}$-structure $M$ is a set of the form $\varphi(M) := \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}$ for an $\mathcal{L}$-formula $\varphi(\bar{x})$. One can also allow parameters from $M$ in $\varphi(\bar{x})$, and then the resulting set is called *definable* (over the parameters involved in $\varphi(\bar{x})$).
- For example, if $(M, \cdot, e)$ is a group and $g \in M$, then the centralizer $C(g)$ is the definable set $\varphi(M)$ for $\varphi(x) := (x \cdot g = g \cdot x)$.

# Types

$T$ a complete $\mathcal{L}$-theory; $n \in \mathbb{N}_{>0}$; $\bar{x} = (x_1, \dots, x_n)$ variables.

## Definition

An *n-type* $\pi(\bar{x})$ is a consistent (with respect to $T$) collection of $\mathcal{L}$-formulas in free variables $\bar{x}$, where "consistent" means that any finitely many formulas from this collection have a realization in every model of $T$. The type $\pi(\bar{x})$ is *complete* if for every formula $\varphi(\bar{x})$ either $\varphi(\bar{x}) \in \pi(\bar{x})$ or $\neg\varphi(\bar{x}) \in \pi(\bar{x})$.

## Example

If $a \in M \models T$, $tp^M(a) := \{\varphi(x) : M \models \varphi(a)\}$ is a complete type.

Formulas $\varphi(\bar{x})$, $\psi(\bar{x})$ are *equivalent* if $T \models (\forall \bar{x})(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

## Remark

If $p(\bar{x})$ is a complete type, and $\varphi(\bar{x})$ and $\psi(\bar{x})$ are equivalent, then $\varphi(\bar{x}) \in p(\bar{x}) \iff \psi(\bar{x}) \in p(\bar{x})$.

# Types — cont.

**Definition**

The *Lindenbaum-Tarski algebra* $L_n(\emptyset)$ of $T$ in $n$-variables is the quotient of the set of all formulas in $n$-variables by the above equivalence relation (say denoted by $\sim$), with Boolean algebra operations induced by $\wedge, \vee, \neg$; $0 := [\varphi \wedge \neg\varphi]_\sim$; $1 := [\varphi \vee \neg\varphi]_\sim$.

- By the last remark, complete types can be identified with ultrafilters in $L_n(\emptyset)$.
- It is also convenient to use another description. Take any $M \models T$. Then two formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are equivalent iff $\varphi(M) = \psi(M)$. Therefore, $L_n(\emptyset)$ is isomorphic to the Boolean algebra of $\emptyset$-definable subsets of $M^n$. And complete types can be seen as ultrafilters in this algebra.

# Space of types

- The set of all complete types in $n$-variables is denoted by $S_n(\emptyset)$. Viewing it as the set of ultrafilters (as above), it can be equipped with the Stone space topology. Explicitly, a basis of open sets consists of the sets of the form

$$[\varphi(\bar{x})] := \{p(\bar{x}) \in S_n(\bar{x}) : \varphi(\bar{x}) \in p(\bar{x})\},$$

where $\varphi(\bar{x})$ is a formula. These sets are then clopen, and $S_n(\emptyset)$ is a compact, totally disconnected space, which plays a key role in model theory.
- Let $M \models T$ and $A \subseteq M$. Let us add new constant symbols corresponding to all elements of $A$, and denote the obtained language by $L(A)$. Then $T(A) := \mathrm{Th}((M, a)_{a \in A})$ is an extension of $T$ to a complete $L(A)$-theory. Then we can talk about types over $A$ and the space $S_n(A)$ with respect $T$, which are formally types over $\emptyset$ and the space $S_n(\emptyset)$ in the sense of $T(A)$.

# Realizing types and saturation

Let $M \models T$ and $A \subseteq M$. By compactness theorem:

**Fact**

Each type $p \in S_n(A)$ is realized in some $N \succ M$, i.e. there is $a \in N$ with $\mathrm{tp}(a/A) = p$.

Using this fact, by a recursive construction, we get

**Fact**

Let $\kappa$ be a cardinal number. Every model can be extended to a *$\kappa$-saturated model* $\mathfrak{C}$, i.e. a model such that for every $B \subseteq \mathfrak{C}$ of cardinality $< \kappa$ every type in $S_n(B)$ has a realization in $\mathfrak{C}$.

One can also construct $\mathfrak{C}$ as above which is additionally *strongly $\kappa$-homogeneous* (each partial elementary map between subsets of cardinality $< \kappa$ extends to an automorphism of $\mathfrak{C}$). For most of the purposes, it is enough to work in such a $\mathfrak{C}$ (which is called a *monster model* of $T$) in place of the class of all models of $T$.

# Space of types — Example

Let $T := \mathrm{Th}(\mathbb{R}, +, \cdot, \leq)$. We will describe the space $S_1(\mathbb{R})$. $T$ has *quantifier elimination*, that is every formula is equivalent to a formula without quantifiers. In particular, each $L(\mathbb{R})$-formula in one variable is equivalent to a finite disjunction of formulas defining intervals (proper or not). So $S_1(\mathbb{R}) = \{p_a^-, p_a, p_a^+ : a \in \mathbb{R}\} \cup \{p_{-\infty}, p_\infty\}$, where:

- $p_a = \mathrm{tp}(a/\mathbb{R})$ (a type whose unique realization is $a$),
- $p_a^-$ is the type determined by $\{x > b : b < a\} \cup \{x < a\}$,
- $p_a^+$ is the type determined by $\{x > a\} \cup \{x < b : b > a\}$,
- $p_{-\infty}$ is the type determined by $\{x < a : a \in \mathbb{R}\}$,
- $p_{+\infty}$ is the type determined by $\{x > a : a \in \mathbb{R}\}$.

A basis of the topology on $S_1(\mathbb{R})$ consists of the sets determined by the intervals (closed, open, etc.) on the real line. For example, the sets $\{p_a\}$, $a \in \mathbb{R}$, are all open, whereas all other singletons are not.

# Type-definable sets

For a type $\pi(\bar{x})$ and $\bar{a} \in \mathfrak{C}^n$ we write $\bar{a} \models \pi$ when $\bar{a}$ satisfies all formulas in $\pi(\bar{x})$.

**Definition**

A *type-definable set* in $\mathfrak{C}$ is a set $X$ of the form $\{\bar{a} \in \mathfrak{C}^n : \bar{a} \models \pi(\bar{x})\}$ for some type $\pi(\bar{x})$ over a small (i.e. of cardinality $< \kappa$) set $A$ of parameters in $\mathfrak{C}$. More precisely, we say that $X$ is *$A$-type-definable*.

**Remark**

$\emptyset$-type-definable sets are invariant under the natural action of $\mathrm{Aut}(\mathfrak{C})$.

**Definition**

A group $G$ is *definable* in a model $M$ if both $G$ and the group law are definable in $M$.

# General goals of model theory

- Describe the structure of models of theories satisfying some general assumptions. Compute the number of models (up to isomorphism) of a given cardinality of such theories.
- Describe the structure (combinatorial, algebraic, ...) of definable (or, more generally, type-definable, invariant, etc.) sets of models of a given theory.
- Prove structural results about algebraic structures satisfying some general model-theoretic assumptions.
- Apply model theory of concrete algebraic structures (or theories) to obtain purely algebraic or geometric results. There are deep applications from the 90's to diophantine geometry, e.g. to the proofs of Mordell-Lang conjecture and Manin-Mumford conjecture.
- In recent years, an important part of model theory is the study of interactions with many other branches of mathematics, e.g. combinatorics, topological dynamics, non-archimedean geometry.

### $S_n(M)$ as an $\mathrm{Aut}(M)$-flow

Let $M$ be a structure, and $\mathrm{Aut}(M)$ the group of automorphisms of $M$. Then $S_n(M)$ is naturally an $\mathrm{Aut}(M)$-flow: $gp := \{gD : D \in p\}$.

### $S_G(M)$ as a $G$-flow

Let $G$ be a group $\emptyset$-definable in a model $M$. $S_G(M)$ denotes the space of complete types over $M$ concentrated on $G$, i.e. containing the formula defining $G$; equivalently, the space of ultrafilters of definable subsets of $G$. $S_G(M)$ is naturally a $G$-flow: $gp := \{gD : D \in p\} = \mathrm{tp}(ga/M)$, where $a \models p$ (here $a \in G(\mathfrak{C})$, where $\mathfrak{C} \succ M$ is a monster model).

Another important flow is $S_{G,ext}(M)$, i.e. the space of all external complete types over $M$ concentrated on $G$, i.e. ultrafilters of *externally definable* subsets of $G$ (i.e. subsets of the form $\varphi(M, \bar{a}) \cap G$, where $\bar{a} \in \mathfrak{C}^n$).

**Example**

$M := (\mathbb{R}, +, \cdot)$, $G := S^1 \subseteq \mathbb{R}^2$. Then $S_G(M) = \{p_a : a \in G\} \cup \{p_a^- : a \in G\} \cup \{p_a^+ : a \in G\}$, where $p_a := \text{tp}(a/\mathbb{R})$, $p_a^-$ is the left cut at $a$, $p_a^+$ is the right cut at $a$. Here, $Gen(S_G(M)) = \{p_a^- : a \in G\} \cup \{p_a^+ : a \in G\} \neq \emptyset$, so $Gen(S_G(M))$ is the unique minimal subflow.

**Theorem (Newelski, Petrykowski)**

Let $G$ be $\emptyset$-definable in an $\aleph_0$-saturated model $M$. Assume $G = \bigcup_{n<\omega} X_n$, where all $X_n$'s are $\emptyset$-type-definable [or just Borel over $\emptyset$]. Then for some finite $A \subseteq G$ and $n < \omega$, $G = AX_nX_n^{-1}$.

# Sketch of the proof

Let $Y_n := \{\text{tp}(a/M) : a \in X_n\}$: a closed [or Borel] set. Let $S \subseteq S_G(M)$ be a minimal subflow. By BCT, there is $n < \omega$ such that $Y_n \cap S$ is non-meager in $S$. Since it is also Borel, using minimality of $S$ and basic descriptive set theory, one can find a finite $A \subseteq G$ for which $A(Y_n \cap S)$ is comeager in $S$.

**Claim.** $G = AX_nX_n^{-1}$.

## Proof of claim.

For any $g \in G$, $g(Y_n \cap S)$ is non-meager in $S$, so $g(Y_n \cap S) \cap A(Y_n \cap S) \neq \emptyset$. Hence, there are $p, q \in Y_n$ and $a \in A$ with $gp = aq$, i.e. $g \in aq(\mathfrak{C})p(\mathfrak{C})^{-1}$. In particular, $g \in aq'(\mathfrak{C})p'(\mathfrak{C})^{-1}$ for $p' := p|_\emptyset$, $q' := q|_\emptyset$. So, by $\aleph_0$-saturation of $M$, $g \in aq'(M)p'(M)^{-1} \subseteq aX_nX_n^{-1}$. □

# Definable maps

Let $X$ be a definable set in a model $M$; $X^* := X(\mathfrak{C})$.

**Definition**

A function $f\colon X \to C$ , where $C$ is a compact (Hausdorff) space, is *definable* if preimages of any disjoint closed subsets of $C$ can be separated by a definable set.

**Definition**

A *definable compactification* of $X$ is a definable map $f\colon X \to C$ with dense image, where $C$ is compact. If $X$ is equipped with the full structure (or just all subsets of $X$ are definable), this is the usual compactification of the discrete space $X$.

# Topological restatement of the previous N-P theorem

**Fact**

The map $t\colon X \to S_X(M)$ given by $t(a) := \mathrm{tp}(a/M)$ is the unique (up to $\cong$) universal definable compactification of $X$.

**Restatement of N-P theorem**

Let $G$ be a group definable in $M$ and $f\colon G \to C$ a definable compactification. Assume $C = \bigcup_{n<\omega} X_n$, where all $X_n$ are closed. Then for some finite $A \subseteq G$ and $n < \omega$, for every open $U \supseteq X_n$, we have $G = Af^{-1}[U]f^{-1}[U]^{-1}$.

**Remark**

Taking the full structure on $M$, we obtain a model-theory free version of this result by removing the word "definable" in the above statement.

Let $(G, X)$ be a flow. For every $g \in G$ we have $\pi_g : X \to X$ given by $\pi_g(x) := gx$. (One often "identifies" $g$ with $\pi_g$, although this "identification" need not be injective.)

## Definition/Fact

$E(X)$ is defined as the closure of $\{\pi_g : g \in G\} \subseteq X^X$ in the pointwise convergence topology on $X^X$. Then $E(X)$ with $\circ$ (i.e. composition) is a left topological semigroup which is compact. It is called the *Ellis semigroup* of the flow $(G, X)$.

## Comment

Various dynamical properties of the flow $(G, X)$ can be expressed in terms of some algebraic or topological properties of $E(X)$, but we will not go into this.

# Ellis theorem

## Ellis Theorem

Suppose $S$ is a compact (Hausdorff) left topological semigroup. Then there exists a minimal left ideal $\mathcal{M}$ in $S$ (i.e. a minimal non-empty subset for which $S\mathcal{M} \subseteq \mathcal{M}$). And every such $\mathcal{M}$ satisfies the following properties.

1. $\mathcal{M}$ is closed, and $\mathcal{M} = Ss$ for all $s \in \mathcal{M}$.
2. If $u \in J(\mathcal{M}) := \{u \in \mathcal{M} : u^2 = u\}$, then $u\mathcal{M}$ is a group with the neutral element $u$.
3. $\mathcal{M} = \bigsqcup_{u \in J(\mathcal{M})} u\mathcal{M}$; in particular, $J(\mathcal{M}) \neq \emptyset$.
4. For every $u \in J(\mathcal{M})$ and $s \in \mathcal{M}$, we have $su = s$.
5. For every minimal left ideal $\mathcal{N}$ of $S$ (e.g. $\mathcal{N} = \mathcal{M}$), $u \in J(\mathcal{M})$ and $v \in J(\mathcal{N})$, we have $u\mathcal{M} \cong v\mathcal{N}$.

# Ellis group of a flow

Let $(G, X)$ be a flow. Ellis theorem applies to the Ellis semigroup $E(X)$.

**Definition**

The *Ellis group* of the flow $(G, X)$ is the isomorphism type of the isomorphic groups of the form $u\mathcal{M}$ for any minimal left ideal $\mathcal{M}$ of $E(X)$ and $u \in J(\mathcal{M})$. Any group $u\mathcal{M}$ as above will also be called the Ellis group of $X$.

**Warning**

In topological dynamics, something else (but strongly related) is called the Ellis group of a flow, but in model theory we use the above terminology.

# Ellis group of a flow — cont.

- The Ellis group of a flow is equipped with so-called $\tau$-topology which makes it a quasi-compact, $T_1$ semitopological group (i.e. group operation is separately continuous) with continuous inversion. One can then take the largest Hausdorff quotient to obtain a compact topological group.
- The Ellis group of a flow plays an important role in abstarct topological dynamics in the proofs of structural theorems.
- In model theory, it was an essential tool to describe the complexity in various senses of the Lascar Galois group of a theory (a generalization of the absolute Galois group of rationals) as well as spaces of strong types (by Krupiński, Pillay, Rzepecki). In fact, one defines the *Ellis group of a theory* (which does not depend on the choice of the monster model of the theory) which is an interesting new invariant (this was shown by Krupiński, Newelski, and Simon). Very recently, Hrsuhovski developed a variant of this notion and applied it to approximate subgroups.