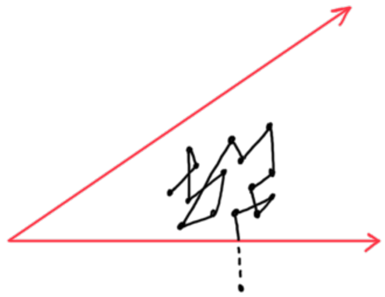
Talk Bedlewo Kilian Raschel baby steps

Enumeration of walks in cones

at the crossroad of several mathematical fields

Kilian Raschel (CNRS, Univ. Tours)



Main definitions

domain: cone C in $\mathbb{Z}^d$

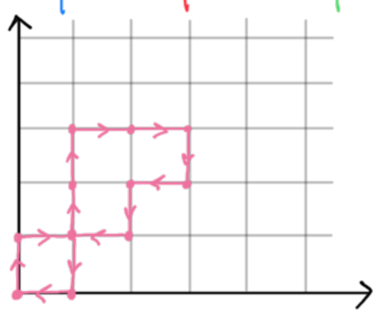
moves: step set $\mathcal{S} \subset \mathbb{Z}^d$

walk / excursion: succession of points $P_0, P_1, \dots, P_n$ in C with $P_{i+1} - P_i \in \mathcal{S}$

number of walks: $\#\{P \rightsquigarrow Q$ in n steps staying in C with step set $\mathcal{S}\}$

$\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$

example simple walk quarter pl



$\#\{(0,0) \rightsquigarrow (0,0)$ in n steps in the quarter plane for the simple walk$\}$

Questions

number of walks $\# \{ P \rightsquigarrow Q$ in n steps staying in C with step set $\mathcal{S} \}$

$\# \{ (0,0) \rightsquigarrow (0,0)$ in n steps in the quarter plane for the simple walk $\}$

Q1 Compute these numbers in closed form $n!$ $\binom{n}{k}$

Q2 Compute their generating function

Q3 Complexity of the generating function

Q4 Asymptotics $n\infty$

Motivations

Walks in cones are in bijection with many discrete combinatorial objects

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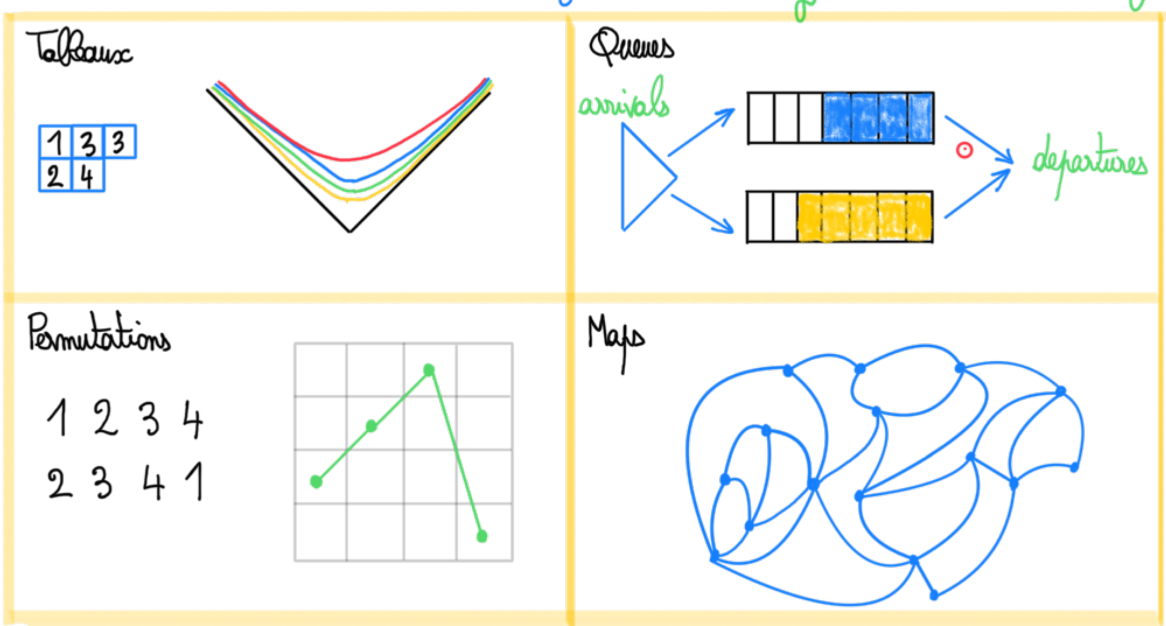

Main tool generating function

Example $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$ n^{th} Catalan number

$$\sum_{n\geqslant 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

$F_0 = F_1 = 1$

$F_{n+2} = F_{n+1} + F_n$

$$\sum_{n\geqslant 0} F_n x^n = \frac{1}{1+x+x^2}$$

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$$\sum_{n\geqslant 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

$y'' + y - 2y^3 = 0$

Hierarchy

rational $\subset$ algebraic $\subset$ differentially finite $\subset$ differentially algebraic

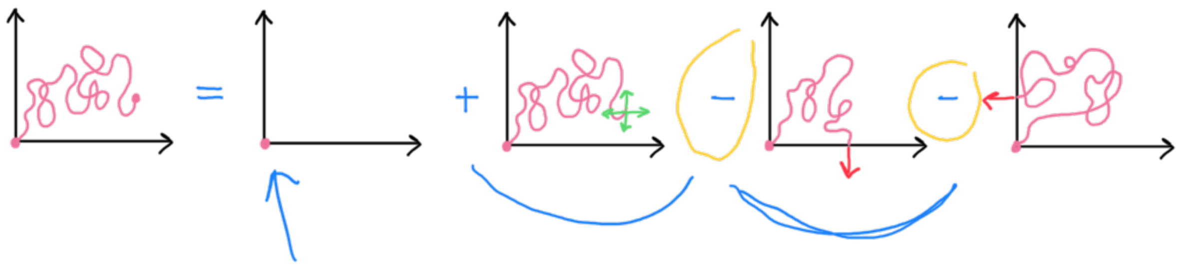
Hypertranscendental $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$

① A combinatorial approach   the kernel method

Generating function in the quarter plane

$$Q(x,y,t)=\sum_{n\geq 0}\sum_{(i,j)\in\mathbb{N}^2} \#\{(0,0)\overset{n}{\rightsquigarrow}(i,j)\}\, x^i y^j t^n$$

Functional equation



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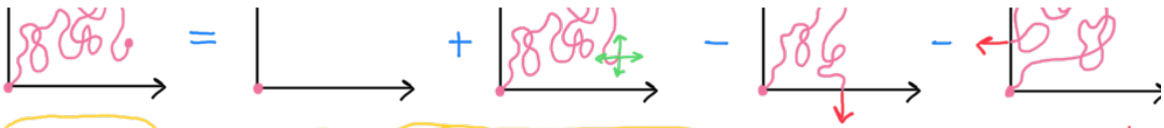
Functional equation

$x^{i\geq 0}\, y^{j\geq 0}$



$$Q(x,y,t) = 1 + tQ(x,y,t)\left(x+\frac{1}{x}+y+\frac{1}{y}\right) - tQ(x,0,t)\frac{1}{y} - tQ(0,y,t)\frac{1}{x}$$

$$\boxed{\left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right)xyQ(x,y,t) = xy - txQ(x,0,t) - tyQ(0,y,t)}$$

Applying the kernel method

$$\left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right)xyQ(x,y,t) = xy - txQ(x,0,t) - tyQ(0,y,t)$$

kernel, symmetries $(x,y)\longmapsto\left(x,\frac{1}{y}\right)$ and $(x,y)\longmapsto\left(\frac{1}{x},y\right)$

Applying the kernel method

$$\oplus \quad \underbrace{\left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right)}_{\text{kernel}} xy\,Q(x,y,t) = xy - tx\,Q(x,0,t) - ty\,Q(0,y,t$$

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$$\ominus \quad \left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right) x\frac{1}{y}\,Q\left(x,\frac{1}{y},t\right) = x\frac{1}{y} - tx\,Q(x,0,t) - t\frac{1}{y}\,Q\left(0,\frac{1}{y},t\right.$$

$$\oplus \quad \left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right) \frac{1}{x}\frac{1}{y}\,Q\left(\frac{1}{x},\frac{1}{y},t\right) = \frac{1}{x}\frac{1}{y} - t\frac{1}{x}\,Q\left(\frac{1}{x},0,t\right) - \cancel{t\frac{1}{y}\,Q\left(0,\frac{1}{y},t\right.}$$

$$\ominus \quad \left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right) \frac{1}{x}y\,Q\left(\frac{1}{x},y,t\right) = \frac{1}{x}y - t\frac{1}{x}\,Q\left(\frac{1}{x},0,t\right) - ty\,Q(0,y,t$$

$x^i y^j$

Taking alternating sum

$$\left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right)\left(xy\,Q(x,y,t) - \frac{x}{y}\,Q\left(x,\frac{1}{y},t\right) + \frac{1}{xy}\,Q\left(\frac{1}{x},\frac{1}{y},t\right) - \frac{y}{x}\,Q\left(\frac{1}{x},y,t\right.\right.$$

$$= xy - \frac{x}{y} + \frac{1}{xy} - \frac{y}{x}$$

$$\boxed{xy\,Q(x,y,t) = [x^{>}y^{>}]\,\frac{xy - \frac{x}{y} + \frac{1}{xy} - \frac{y}{x}}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}}$$

Conclusion and perspectives of the kernel method

$$xy\,Q(x,y,t) = [x^{>}y^{>}]\,\frac{xy - \frac{x}{y} + \frac{1}{xy} - \frac{y}{x}}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}$$

- Nice, combinatorial and explicit solution
- Allows to answer Q1, Q2, Q3, Q4
- Relation with reflection principle
- Symmetry group finite ... very strong condition
- Larger jumps? Higher dimension?

② Complex analysis

Random walks in the quarter plane
(queuing networks)

Malyshev '70
Fayolle '80
Iasnogorodski
Cohen
Boxma '90
...

Boundary value problems

② Complex analysis

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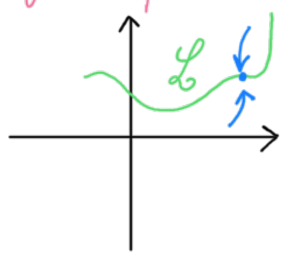
Boundary value problem



Given $a(x)$ and $b(x)$, find f analytic in $\mathbb{C} \setminus \mathcal{L}$ such that on $\mathcal{L}$

Boundary value problems

$$\lim_{z\downarrow x} f(z) - a(x)\lim_{z\uparrow x} f(z) = b(x$$

(Sochocki-Plemelj formulas)

From the functional equation to boundary value problems

$$\cancel{\left(1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right)xyQ(x,y,t)} = xy - txQ(x,0,t) - tyQ(0,y,t)$$

$\boxed{\text{Step 1:}}$ take the kernel $K(x,y,t) = 1 - t\left(x+\frac{1}{x}+y+\frac{1}{y}\right) = 0$

$$txQ(x,0,t) + tyQ(0,y,t) = xy$$

$\xrightarrow[(1,0)]{}$ $x^1 y^0 = x$

$\downarrow \qquad \frac{1}{y}$

From the functional equation to boundary value problems

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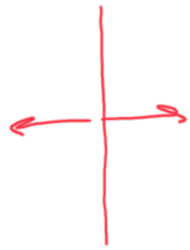
$$\left(tx\,Q(x,0,t)+\cancel{ty\,Q(0,y,t)}=xy\right.$$



$\boxed{\text{Step 2:}}$ if $K(x,y,t)=0$ then $K\left(\frac{1}{x},y,t\right)=0$

$$\left(\frac{t}{x}\,Q\left(\frac{1}{x},0,t\right)+\cancel{ty\,Q(0,y,t)}=\frac{y}{x}\right.$$

$\boxed{\text{Step 3:}}$ substraction

$$tx\,Q(x,0,t)-\frac{t}{x}\,Q\left(\frac{1}{x},0,t\right)=\left(x-\frac{1}{x}\right)y$$

Reformulation as a true boundary value problem with shift

The equation $\left(tx\,Q(x,0,t)-\frac{t}{x}\,Q\left(\frac{1}{x},0,t\right)=\left(x-\frac{1}{x}\right)y\right.$ on the unit circle:

$$tx\,Q(x,0,t)-t\bar{x}\,Q(\bar{x},0,t)=(x-\bar{x})y \quad \text{for } |x|=1$$

Reformulation as a true boundary value problem with shift

The equation $tx\,Q(x,0,t) - \frac{t}{x}\,Q\left(\frac{1}{x},0,t\right) = \left(x-\frac{1}{x}\right)y$ on the unit circle

$$tx\,Q(x,0,t) - t\bar{x}\,Q(\bar{x},0,t) = (x-\bar{x})y \quad \text{for } |x|=1$$

$$\lim_{z\downarrow x} f(z) - a(x)\lim_{z\uparrow x} f(z) = b(x$$

Solution
$$\boxed{tx\,Q(x,0,t) = \frac{1}{2i\pi}\int_{|u|=1} \frac{(u-\bar{u})y}{u-x}\,du}$$

Conclusion and perspectives of the boundary value problem approach

- Integral expression for the generating function (Sochocki-Plemelj formulas)
- Works for a large class of models (even without symmetries)
  [see two technical aspects in the next slide]
- Weights, other (more general) boundary conditions
- Larger jumps? Higher dimension?

Two interesting features: conformal gluing and Riemann surfaces

Kernel $K(x,y,t) = 1 - t \sum_{(i,j)\in \mathcal{S}} x^i y^j$

(see functional equation)

$\{(x,y)\in \mathbb{C}^2 : K(x,y,t) = 0\}$
is a Riemann surface of genus

- 0 or 1 for small steps
- arbitrary large in general

Two interesting features : conformal gluing and Riemann surfaces

Kernel $K(x,y,t) = 1 - t \sum_{(i,j)\in\mathcal{S}} x^i y^j$

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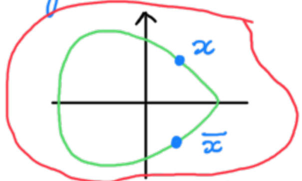
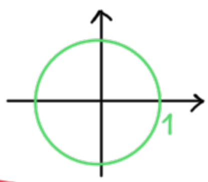
Given a symmetric curve



find f such that $f(x) = f(\bar{x})$

Example : the unit circle



$f(x) = x + \frac{1}{x}$

$\bar{x} = \frac{1}{x}$

③ Galois theory of difference equations

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Q3 Complexity of the generating function

Existence of an algebraic differential relation ?

Examples: $y = \frac{1}{\cos}$ satisfies $y'' + y - 2y^3 = 0$

$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt$ does not satisfy any diff. eq. (Hölder 1887)

How does this apply to enumeration problems ?

$$\boxed{\Gamma(x+1) = x\,\Gamma(x)}$$

$f(s_0) = \infty$

$D \longrightarrow qD$

Ishizaki theorem ('98)

- $a(s)$, $b(s)$ given rational functions
- $q \in \mathbb{C}$, $|q| \neq 0, 1$
- f unknown mero. function satisfying $\boxed{f(qs) - a(s)f(s) = b(s)}$

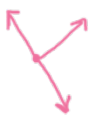
Conclusion : dichotomy

- f is rational
- f is hypertranscendental

An example : the walk

$$\left(1 - t\left(\frac{y}{x} + xy + \frac{x}{y}\right)\right) xy\, Q(x,y,t) = xy - tx^2 Q(x,0,t) - ty^2 Q(0,y,t)$$

An example : the walk



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Step 1: parametrization of the kernel curve

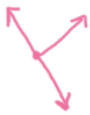
Example $\{(x,y) \in \mathbb{C}^2 : x^2 + y^2 = 1\} = \{(\cos(u), \sin(u)) : u \in \mathbb{C}\}$

$$= \left\{\left(\frac{s^2-1}{s^2+1}, \frac{2s}{s^2+1}\right) : s \in \mathbb{C}\right\}$$

$$\left\{(x,y) \in \mathbb{C}^2 : 1 - t\left(\frac{y}{x} + xy + \frac{x}{y}\right) = 0\right\} = \left\{\left(\frac{q-1}{\sqrt{q}} \frac{s}{s^2+1}, (q-1)\frac{s}{s^2+q}\right) : s \in \mathbb{C}\right\}$$

where $t = \frac{\sqrt{q}}{q+}$

$x(s) = x\left(\frac{1}{s}\right)$ $y(s) = y\left(\frac{q}{s}\right)$

An example : the walk 

$$\left(1 - t\left(\frac{y}{x} + xy + \frac{x}{y}\right)\right) xy\, Q(x,y,t) = xy - tx^2 Q(x,0,t) - ty^2 Q(0,y,t)$$

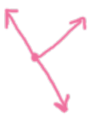
Step 2: evaluate the functional eq. at $(x(s), y(s))$

$$\underbrace{tx^2(s)\,Q(x(s),0,t)}_{f(s)} + \cancel{ty^2(s)\,Q(0,y(s),t)} = x(s)\,y(s) \qquad (*)$$

Step 2': evaluate the functional eq. at $\left(x\left(\frac{q}{s}\right), y\left(\frac{q}{s}\right)\right)$

$$f\left(\frac{q}{s}\right) + \cancel{ty^2(s)\,Q(0,y(s),t)} = x\left(\frac{q}{s}\right) y(s) \qquad (*)_{q/s}$$

Step 3: substraction $\boxed{f(s) - f\left(\frac{q}{s}\right) = \left(x(s) - x\left(\frac{q}{s}\right)\right) y(s)}$

An example : the walk 

The equation $f(s) - f\left(\frac{q}{s}\right) = \left(x(s) - x\left(\frac{q}{s}\right)\right) y(s)$ + invariance property $f(s) = f\left(\frac{1}{s}\right)$

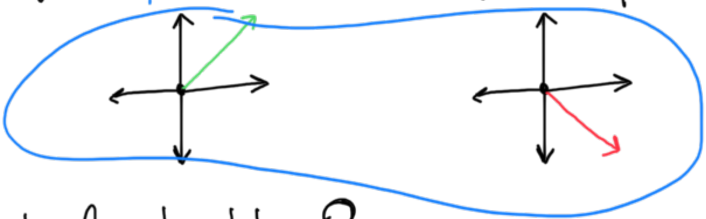
gives Ishizaki's eq.

$$\underbrace{f(qs) - \underbrace{a(s)}_{1} f(s) = b(s)}$$

$$f(s) - f\left(\frac{q}{s}\right) = \left(x(s) - x\left(\frac{q}{s}\right)\right) y($$

Conclusion and perspectives of this approach

- Small steps $\longleftrightarrow$ genus 0 or 1 are generically hypertranscendental
- Interesting exceptional cases with differential equation



- Combinatorial interpretation ?
- Larger jumps ? Higher dimension ?

④ A probabilistic approach: Brownian motion in cones

Classical approach in probability theory:

analysis

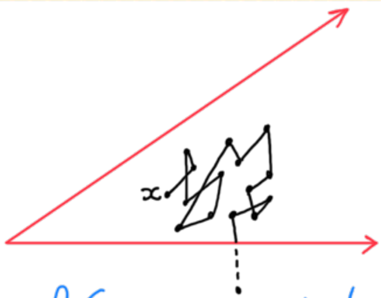
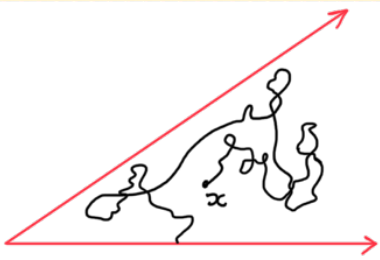
approximate discrete structures by scaling limits

④ A probabilistic approach: Brownian motion in cones

analysis

Classical approach in probability theory:

approximate discrete structures by scaling limits

Discrete	vs	Continuous
		
$\tau = \inf\{n \geq 0 : x + S(n) \notin \text{cone}\}$		$\tau = \inf\{t > 0 : x + B(t) \notin \text{cone}\}$

4 A probabilistic approach: Brownian motion in cones

Classical approach in probability theory:

approximate discrete structures by scaling limits

analysis

Random walk

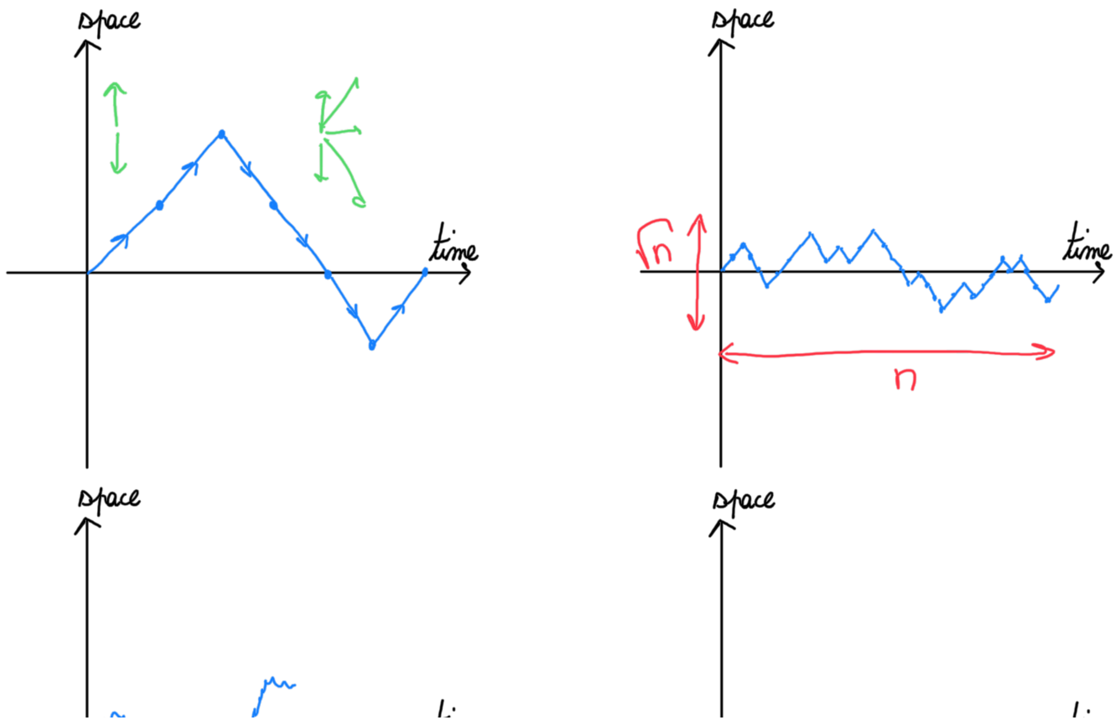
$$\#\{x \overset{n}{\rightsquigarrow} y \text{ in a cone } C\} = \mathbb{P}(x + S(n) = y,\ \tau^{RW} > n) \times |\mathcal{S}|^n$$

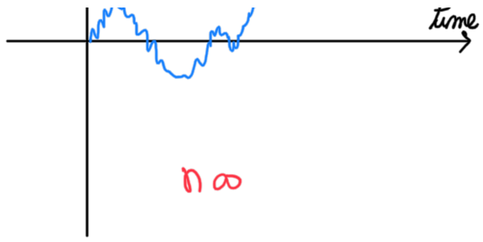
$$\approx \mathbb{P}(x + B(n) = y,\ \tau^{BM} > n) \times |\mathcal{S}|^n$$

Brownian motion

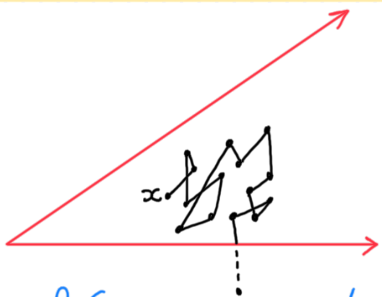
$$S(n) = X(1) + X(2) + \dots + X(n)$$

Brownian motion in dimension 1

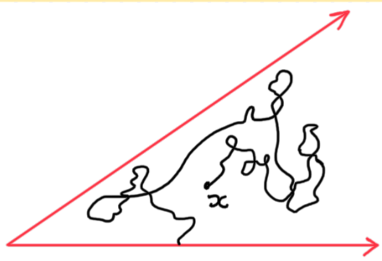




Discrete vs Continuous



$\tau = \inf \{ n \geq 0 : x + S(n) \notin \text{cone} \}$



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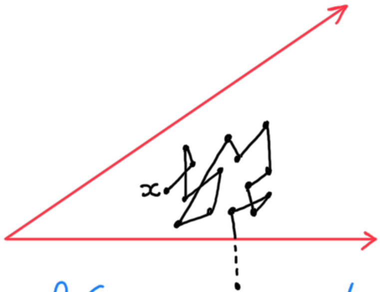
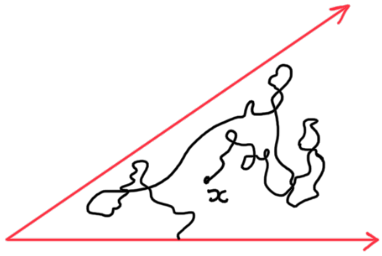
Discrete	vs	Continuous
		
$\tau = \inf\{n \geq 0 : x + S(n) \notin \text{cone}\}$		$\tau = \inf\{t > 0 : x + B(t) \notin \text{cone}\}$
Number of excursions $\#\{x \overset{n}{\rightsquigarrow} y, \text{ confined to the cone}\}$		Heat kernel $\mathbb{P}_x(B_t = y, \tau > t)$
Total number of walks $\#\{x \overset{n}{\rightsquigarrow} \text{cone}, \text{ confined to the cone}\}$		Survival probability $\mathbb{P}_x(\tau > t)$

Computation of the heat kernel

$\mathbb{P}_x(\tau > t)$ and $\mathbb{P}_x(\tau > t, B_t = y)$ satisfy PDE

Example: $\underbrace{\frac{\partial}{\partial t} - \frac{1}{2}\Delta = 0}$ [the heat equation]

Analysis on manifolds

Conclusion and perspectives of the probabilistic approach

(Q4) Asymptotics $n\infty$

$$\#\{x \overset{n}{\rightsquigarrow} y, \text{ confined to the cone}\} \underset{n\infty}{\sim} c \cdot h(x,y) \cdot \rho^n \cdot n$$

α

eigenvalues for some Dirichlet problem

Conclusion and perspectives of the probabilistic approach

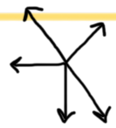
(Q4) Asymptotics $n\infty$

$$\#\{x \overset{n}{\rightsquigarrow} y, \text{ confined to the cone}\} \underset{n\infty}{\sim} c \cdot h(x,y) \cdot \rho^n \cdot n$$

α

eigenvalues for some Dirichlet problem

(Q3) Complexity of the generating function

Example 

$$\alpha = \frac{\pi}{\arccos(1/4)} \notin \mathbb{Q}$$

arithmetic properties of ρ and α

$$\begin{cases} \sum a_n x^n \text{ linear diff eq ?} \\ a_n \sim \rho^n n^{\alpha} \end{cases}$$

General question:
when $\alpha \in \mathbb{Q}$?

⑤ Spectral theory

Q4 Asymptotics $n\infty$

$$\#\{x \overset{n}{\rightsquigarrow} y, \text{ confined to the cone}\} \underset{n\infty}{\sim} c \cdot h(x,y) \cdot \rho^n \cdot n^{\alpha}$$

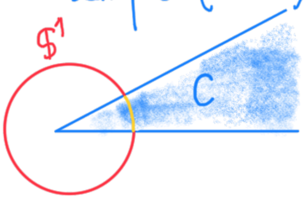
α — eigenvalues for some Dirichlet problem

Goal: understand / compute $\alpha = \frac{d}{2} - 1 + \sqrt{\left(\frac{d}{2}-1\right)^2 + \lambda_1}$

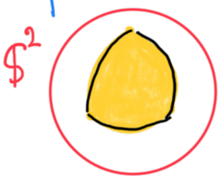
A formula for λ_1

Step 1: $\mathcal{D} = C \cap \mathbb{S}^{d-1}$

Example $(d=2)$



Example $(d=3)$

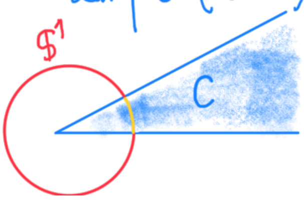


L $\mathbb{R}^3_+$

A formula for λ_1

Step 1: $\mathcal{D} = C \cap \mathbb{S}^{d-1}$

Example $(d=2)$



Step 2: a Dirichlet problem

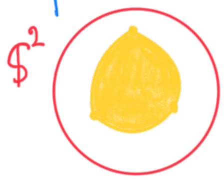
$$\boxed{\begin{cases} \Delta u = -\lambda u & \text{on } \mathcal{D} \\ u = 0 & \text{on } \partial\mathcal{D} \end{cases}}$$

Step 3: smallest eigen

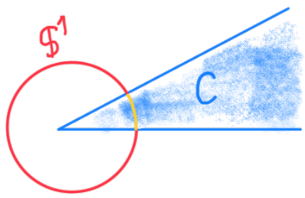
Discrete spectrum

$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$

Example $(d=3)$

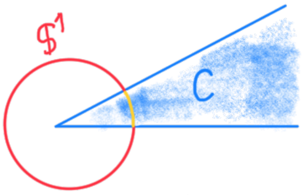


Dimension 2



$$\begin{cases} \Delta u = -\lambda u & \text{on } D \\ u = 0 & \text{on } \partial D \end{cases} \iff \begin{cases} u''(\theta) = -\lambda u(\theta) & \theta \in [0,\beta] \\ u(0) = u(\beta) = 0 \end{cases}$$

Dimension 2



$$\begin{cases} \Delta u = -\lambda u & \text{on } D \\ u = 0 & \text{on } \partial D \end{cases} \iff \begin{cases} u''(\theta) = -\lambda u(\theta) & \theta \in [0,\beta] \\ u(0) = u(\beta) = 0 \end{cases}$$

$$\iff u(\theta) = \sin\left(\left(k\frac{\pi}{\beta}\right)^2 \theta\right)$$

$$\boxed{\text{Conclusion: } \lambda_1 = \left(\frac{\pi}{\beta}\right)^2 \text{ and } \alpha = \frac{\pi}{\beta}}$$

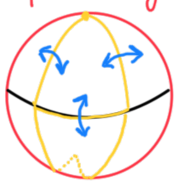
Exponent $\in \mathbb{Q} \iff$ angle of cone commensurable with π

Dimension 3

$$\begin{cases} \Delta u = -\lambda u & \text{on } D \\ u = 0 & \text{on } \partial D \end{cases}$$

in general not solvable

Reflection group of a triangle



reflection group finite $\longleftrightarrow$ tilings of the sphere with spherical triangles

Special families of triangles with explicit spectrum

$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ tetrahedral group

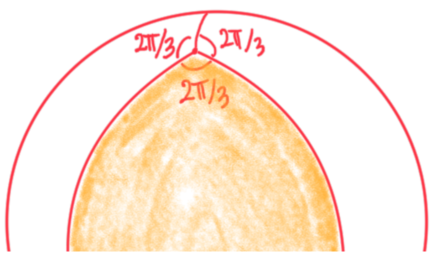
$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$ octahedral group

$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}\right)$ icosahedral group

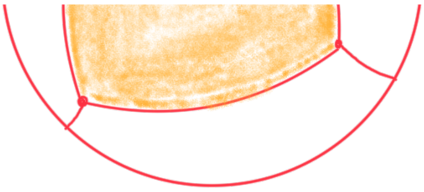
$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{n}\right)$ dihedral group

An open question

the equilateral triangle $\frac{2\pi}{3}$ tiling with 4 triangl



$\lambda_1 \approx 5,1591456\ldots$



$\in \mathbb{Q}$?

$(-1,-1,-1)$
$(1,0,0)$
$(0,1,0)$
$(0,0,1)$

Conclusion and perspectives

- very well suited for numerical analysis
- works for many cones and dimensions (dimension 4 ?)
- combinatorics / spectral theory / geometry

merci

$$\rho = \frac{1}{\min\limits_{\mathbb{R}_+^2} \sum\limits_{(i,j)\in S} x^i y^j}$$