DIMENSION-FREE ESTIMATES IN ANALYSIS: HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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L^p SPACES AND (SUB)-LINEAR OPERATORS

- $L^p = \{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} |f(x)|^p dx < \infty \}$, $\|f\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}$,

  Here $1 \leq p \leq \infty$. For us usually $1 < p < \infty$ COMMENT $\rightarrow$

- dx above denotes Lebesgue measure. By $|A|$ we mean Lebesgue measure of A

- An operator T on L^p is a mapping $T : L^p \to L^p$ COMMENT $\rightarrow$

- T is linear if $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$

- T is sublinear if

$$T(\alpha f) = \alpha T(f) \quad \text{and} \quad |T(f+g)| \leq |T(f)| + |T(g)|$$

GENERAL SETTING:

- d – dimension
- we have a family of (sub)-linear operators T_d such that T_d acts on $L^p(\mathbb{R}^d)$ (later also $L^p(\mathbb{Z}^d)$),
- the family $\{T_d\}$ has dimension-free estimates on L^p, $1 \le p \le \infty$, if

$$\boxed{\sup_d \|T_d f\|_{L^p(\mathbb{R}^d)} \le C(p) \|f\|_{L^p(\mathbb{R}^d)}, \qquad f \in L^p(\mathbb{R}^d)}$$

- the constant $C(p)$ depends only on p

EXAMPLES: (LINEAR OPERATORS)

1. Fourier transform

$$\mathcal{F}_d(f)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx$$

Then $\|\mathcal{F}_d(f)\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$, i.e. $C(2)=1$

2. Gaussian kernel

Let $h_d(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$, $x\in\mathbb{R}^d$, and $T_d(f) = f * h_d$

Then

$$\|T_d f\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)},$$ i.e. $C(p)=1$, $1\leq p\leq\infty$

EXAMPLES (SUB-LINEAR OPERATORS - MAXIMAL FUNCTIONS)

1. L^∞ estimates (supremum norm $\|f\|_{L^\infty(\mathbb{R}^d)} = \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|$)

Let g be such that $\int_{\mathbb{R}^d} |g(x)|\,dx = 1$ and let $g_t(x) = t^{-d} g(\frac{x}{t})$

Then $\left\| \sup_{t>0} |g_t * f| \right\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\mathbb{R}^d)}$, i.e. $C(\infty) = 1$

2. Gaussian maximal function

Let $h_t(x) = t^{-d} h(\frac{x}{t}) = (4\pi t^2)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t^2}}$

Then
$$\boxed{\left\| \sup_{t>0} |h_t * f| \right\|_{L^p(\mathbb{R}^d)} \leq C(p) \|f\|_{L^p(\mathbb{R}^d)}}$$

NOT SO EASY $\to$ SEMIGROUP THEORY

Here $C(p)$ is a constant independent of d, $1 < p < \infty$

I. HARDY-LITTLEWOOD MAXIMAL FUNCTIONS – CONTINUOUS SETTING

HARDY-LITTLEWOOD MAXIMAL FUNCTION OVER THE EUCLIDEAN BALLS

Let $|x| = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$ and $B = \{x \in \mathbb{R}^d : |x| \leq 1\}$

Denote $tB = \{tx : x \in B\}$ and $M_t^B f(x) = \frac{1}{|tB|} \int_{tB} f(x-y)\,dy$

Write $M_*^B f(x) = \sup_{t>0} |M_t^B f(x)|$.

THEOREM, [E.M. STEIN, 82] For each $p \in (1,\infty)$ there is $C(p)$ such that

$$\| M_*^B f \|_{L^p(\mathbb{R}^d)} \leq C(p) \|f\|_{L^p(\mathbb{R}^d)} \quad \text{uniformly in } d$$

Before it was only known that $\| M_*^B f \|_{L^p(\mathbb{R}^d)} \leq 3^d \tilde{C}(p) \|f\|_{L^p(\mathbb{R}^d)}$.

- Replacing the ball B^2 by some other sets

<u>Question</u>: What happens for the cube $Q=[-1,1]^d$?

→ If $x \in Q$, then $\left(\sum_{j=1}^{d} x_j^2\right)^{\frac{1}{2}} \leq \sqrt{d} \Rightarrow x \in \sqrt{d}\, B$

→ If $x \in B$, then $\max_j |x_j| \leq 1 \Rightarrow x \in Q$

→ Similarly $tB \subseteq tQ \subseteq t\sqrt{d}\, B$

→ From this, $|tA| = t^d |A|$, and Stein's result we may deduce

$$\left\| \sup_{t>0} \frac{1}{|tQ|} \left| \int_{tQ} f(x-y)\, dy \right| \right\|_{L^p(\mathbb{R}^d)} \leq d^{\frac{d}{2}}\, C(p) \|f\|_{L^p(\mathbb{R}^d)}$$

Note that $d^{\frac{d}{2}}$ grows very fast → no transfer of dimension-free estimates

HARDY-LITTLEWOOD MAXIMAL OPERATORS OVER SYMMETRIC CONVEX BODIES

DEFINITION: We say that $G \subseteq \mathbb{R}^d$ is a symmetric convex body if G is a compact set with non-empty interior such that $x \in G \iff -x \in G$

For a symmetric convex body G we define $tG = \{tx : x \in G\}$,

$$\boxed{M_t^G f(x) = \frac{1}{|tG|} \int_{tG} f(x-y)\,dy}$$

HL averages over G

and

$$\boxed{M_*^G f(x) = \sup_{t>0} \left| M_t^G f(x) \right|}$$

HL maximal operator over G

Since, for some constants $a = a(G)$, $b = b(G)$, $\quad aB \subseteq G \subseteq bB$

$$\| M_*^G f \|_{L^p(\mathbb{R}^d)} \leq \left(C(a,b) \right)^d \, C(p) \, \| f \|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty,$$

by comparison with Stein's result. For the cube $C(a,b) = \sqrt{d}$

$\longrightarrow$ Simple comparison gives boundedness but not dimension-free

$\longrightarrow$ The argument used by Stein works only for balls

It relies on rotation invariance

$\longrightarrow$ Here enters Bourgain....

THEOREM, [J. BOURGAIN, 85] There is a constant C such that

$$\| M_G^* f \|_{L^2(\mathbb{R}^d)} \leq C \| f \|_{L^2(\mathbb{R}^d)}, \qquad f \in L^2(\mathbb{R}^d)$$

uniformly, for all symmetric convex bodies G in all dimensions d

→ The proof relies on estimates for the Fourier transform of $1_{\tilde{G}}$, where $\tilde{G} = A(G)$ for some invertible linear transformation A

→ Replacing G by $A(G)$ does not change the task

e.g. the estimate for B is the same as for the ellipsoid

$$E = \{ x \in \mathbb{R}^d : \sum_{i=1}^{d} (1 + \tfrac{1}{i})^2 x_i^2 \leq 1 \}$$

What is the appropriate choice of A and $\tilde{G}$?

DEFINITION: We say that a symmetric convex body G is in the isotropic position if $|G|=1$ and

$$\int_G \left(\sum_{i=1}^d x_i \xi_i\right)^2 dx = L(G)|\xi|^2, \quad \xi \in \mathbb{R}^d$$

Then $L(G) = \frac{1}{d}\int_G |x|^2 dx$ is called the isotropic constant of G

REMARK: A major conjecture in ASYMPTOTIC CONVEX GEOMETRY asks whether $L(G) \approx 1$ uniformly in all symmetric convex bodies G.

This is known as the ISOTROPIC CONSTANT CONJECTURE and last year Y. CHEN made a significant progress on it

For a function $f : \mathbb{R}^d \to \mathbb{R}$ we write $\mathcal{F}(f)$ for its Fourier transform

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^d$$

Note that in this case Plancherel's theorem states that

$$\int_{\mathbb{R}^d} |\mathcal{F}(f)(\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |f(x)|^2\, dx$$

In particular for $f = 1_G$, $\xi \in \mathbb{R}^d$

$$\boxed{m^G(\xi) := \mathcal{F}(1_G)(\xi) = \int_G e^{-2\pi i x\cdot\xi}\, dx}$$

The crucial ingredient for proving Bourgain's maximal theorem is

PROPOSITION: (CONTINUOUS FT ESTIMATES), [BOURGAIN 86]

Let G be a symmetric convex body which is in the isotropic position. Then, for $\xi \in \mathbb{R}^d$

estimates

1) $|m^G(\xi) - 1| \leq 150 \cdot L(G)|\xi|$ (at 0)

2) $|m^G(\xi)| \leq \dfrac{150}{L(G)|\xi|}$ (at ∞)

3) $|\langle \xi, \nabla m^G(\xi) \rangle| \leq 150$ (differential)

The Proposition above allows us to prove the (dimension-free) L^2 maximal theorem of Bourgain even though we do not know if $L(G) \lesssim 1$.

A flavour of the reason why, here $a = L(G)|\xi|$

$$\sum_{j\in\mathbb{Z}} \min\left(2^j a, (2^j a)^{-1}\right) = \sum_{j: 2^j a \le 1} -\text{"}- + \sum_{j: 2^j a > 1} -\text{"}-$$

$$= \sum_{j \le \log_2 \frac{1}{a}} 2^j a + \sum_{j > \log_2 \frac{1}{a}} 2^{-j} a^{-1} \approx$$

„sum of a geometric series is approximately its largest term"

$$\approx 2^{\log_2 \frac{1}{a}} \cdot a + 2^{-\log_2 \frac{1}{a}} \cdot a^{-1} = 2$$

regardless of a

Interlude: Why symmetric convex sets are important?

DEFINITION: Let G be a symmetric convex body in $\mathbb{R}^d$. The function

$$\rho_G(x) := \inf\{t>0 : tx \in G\}$$

is called the Minkowski norm of G.

It turns out that:

LEMMA: If G is a symmetric convex body in $\mathbb{R}^d$ the ρ_G is a norm.

Conversely if ρ is a norm on $\mathbb{R}^d$ then there is a symmetric convex body G such that $\rho = \rho_G$

Shortly after Bourgain's L^2 result

THEOREM [J. BOURGAIN, A. CARBERY 86]

For each $p > \frac{3}{2}$ there is a constant $C(p)$ such that

$$\| M_*^G f \|_{L^p(\mathbb{R}^d)} \leq C(p) \, \|f\|_{L^p(\mathbb{R}^d)}, \qquad f \in L^p(\mathbb{R}^d)$$

uniformly in all dimensions d and all symmetric convex bodies G.

→ Since 86' there was progress for specific convex bodies by [D. Müller, 90] and [J. BOURGAIN, 14]

→ No progress (lowering p) for general convex bodies

CONJECTURE: BOURGAIN MAXIMAL FUNCTION CONJECTURE ≈ 86

Is it true that for each $1<p<\infty$ there exists a $C(p)$ s.t.

$$\|M_*^G f\|_{L^p(\mathbb{R}^d)} \leq C(p) \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d),$$

uniformly in dimensions d and symmetric convex bodies G ?

Three years ago we were able to reduce the question to a difference estimate

THEOREM [J. BOURGAIN, M. MIREK, E.M. STEIN, B. WRÓBEL, 18]

Assume that for all $1<p<\infty$ and $0<\alpha<1$ we have

$$\|M_{1+h}^G f - M_1^G f\|_{L^p(\mathbb{R}^d)} \leq C(\alpha,p)\, h^\alpha \|f\|_{L^p(\mathbb{R}^d)}, \quad 0<h<1, \quad f \in L^p(\mathbb{R}^d),$$

where $C(\alpha,p)$ depends only on α and p. Then Bourgain's conjecture holds

5

→ Recall that $M_t^G f(x) = \frac{1}{|tG|} \int_{tG} f(x-y)\,dy = \int_G f(x-ty)\,dy$

→ From PROPOSITION (CONTINUOUS FT ESTIMATES) we can deduce

$C(1,2) \leq 150$

→ Since M_t^G are averaging operators we also see that

$C(0,1) \leq 2$ and $C(0,\infty) \leq 2$ COMMENT →

→ No better estimates are known for general G

II. HARDY-LITTLEWOOD MAXIMAL FUNCTIONS - DISCRETE SETTING

DISCRETE SETTING

For a symmetric convex body G and $t>0$ we denote

- $$\boxed{tG\cap\mathbb{Z}^d=\{x\in tG : x\in\mathbb{Z}^d\},\quad |tG\cap\mathbb{Z}^d|=\#\{tG\cap\mathbb{Z}^d\}}$$

- $$\boxed{\mathcal{M}_t^G f(x)=\frac{1}{|tG\cap\mathbb{Z}^d|}\sum_{y\in tG\cap\mathbb{Z}^d} f(x-y)\quad \text{DISCRETE H-L AVERAGE}}$$

- $$\boxed{\mathcal{M}_*^G f(x)=\sup_{t>0}\left|\mathcal{M}_t^G f(x)\right|\quad \text{DISCRETE H-L MAXIMAL FUNCTION}}$$

Difficulties compared with the continuous case

$\longrightarrow$ No scaling, cannot replace G by $A(G)$

$\longrightarrow$ Need to count lattice points $|tG \cap \mathbb{Z}^d|$ in a dimension-free way

e.g. in $\{tB^2 \cap \mathbb{Z}^d\} = \{x \in \mathbb{Z}^d : \sum_{i=1}^{d} x_i^2 = t^2\}$

These are <u>real problems</u>. For $f : \mathbb{Z}^d \to \mathbb{R}$, denote

$$\boxed{\|f\|_{\ell^p(\mathbb{Z}^d)} = \Big(\sum_{x \in \mathbb{Z}^d} |f(x)|^p\Big)^{\frac{1}{p}}}$$

and

$$\ell^p(\mathbb{Z}^d) = \{f : \mathbb{Z}^d \to \mathbb{R} : \|f\|_{\ell^p(\mathbb{Z}^d)} < \infty\}$$

THEOREM [J. BOURGAIN, M. MIREK, E.M. STEIN, B. WRÓBEL 18]

For each $d \in \mathbb{N}$ there exists an ellipsoid E_d such that for all $1 < p < \infty$ there is a function $f \in \ell^p(\mathbb{Z}^d)$ for which

$$\| \mathcal{M}_*^{E_d} f \|_{\ell^p(\mathbb{Z}^d)} \geq C(p) (\log d)^{\frac{1}{p}} \|f\|_{\ell^p(\mathbb{Z}^d)}$$

$\longrightarrow$ This is in sharp contrast with the continuous case where $E_d = A_d(B)$ for some invertible A_d

$\longrightarrow$ No hope for a discrete result for all G

$\longrightarrow$ The problem appears for small values of $1 \leq t \leq 2$ in the supremum

For $1<p<\infty$ we let $C(p,G)$ be the optimal constant in the continuous inequality

$$\boxed{\| M_*^G f\|_{L^p(\mathbb{R}^d)} \leq C(p,G)\, \|f\|_{L^p(\mathbb{R}^d)}}$$

and denote by $\mathcal{C}(p,G)$ the optimal constant in discrete inequality

$$\boxed{\| \mathcal{M}_*^G f\|_{\ell^p(\mathbb{Z}^d)} \leq \mathcal{C}(p,G)\, \|f\|_{\ell^p(\mathbb{Z}^d)}}$$

COMPARISON PRINCIPLE 1 [D. KOSZ, M. MIREK, P. PLEWA, B. WRÓBEL 20]

For all $1<p<\infty$ we have

$$C(p,G) \leq \mathcal{C}(p,G)$$

In short: <u>The discrete question is harder</u>

Are there any positive results?

For a symmetric convex body $G \subseteq \mathbb{R}^d$ we let $Q_{\frac{1}{2}} = [-\frac{1}{2}, \frac{1}{2}]^d$ and

$$\boxed{c(G) = \inf\{t > 0 : Q_{\frac{1}{2}} \subseteq t\, G\}}$$

COMPARISON PRINCIPLE 2 [BOURGAIN, MIREK, STEIN, WRÓBEL 19]

Let $C(p, G)$ be the optimal constant in the continuous inequality

Then, for $1 < p < \infty$ and $f \in \ell^p(\mathbb{Z}^d)$

$$\Big\| \sup_{t > c(G) d} |M_t^G f| \Big\|_{\ell^p(\mathbb{Z}^d)} \leq 10^4\, C(p, G)\, \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Thus, the discrete dimension-free question is <u>harder</u> only for $t \leq c(G)\,d$.

What is $c(G)$ for various G?

$\longrightarrow$ For $G = Q = [-1,1]^d$, $c(Q) = \frac{1}{2}$. By [BOURGAIN 14] (continuous cube)

$C(p, Q) \leq C_p$, $1 < p < \infty$. Hence, the discrete dimension-free estimate holds for $\boxed{t > \frac{d}{2}}$

$\longrightarrow$ For $G = B = \{x \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 \leq 1\}$, $c(B) = \frac{\sqrt{d}}{2}$. By [STEIN 82]

$C(p, B) \leq C_p$, $1 < p < \infty$. Hence the discrete dimension-free estimate holds for $\boxed{t > \frac{1}{2} d^{\frac{3}{2}}}$

We shall now compare the difficulties for $G = Q$ vs $G = B$.

We focus on discrete Fourier transforms

$$m_t^{Q}(\xi) = \frac{1}{|tQ \cap \mathbb{Z}^d|} \sum_{x \in tQ \cap \mathbb{Z}^d} e^{-2\pi i x \cdot \xi}$$

$$m_t^{B}(\xi) = \frac{1}{|tB \cap \mathbb{Z}^d|} \sum_{x \in tB \cap \mathbb{Z}^d} e^{-2\pi i x \cdot \xi}$$

Here $\xi \in \mathbb{T}^d = [-\frac{1}{2}, \frac{1}{2})^d$, $t > 0$.

1. THE DISCRETE CUBE CASE $G = Q$

This case is simpler because of explicit formulas

- If $\lfloor t \rfloor = \lfloor s \rfloor$ then $tQ \cap \mathbb{Z}^d = sQ \cap \mathbb{Z}^d$ COMMENT→
- $|tQ \cap \mathbb{Z}^d| = (2\lfloor t \rfloor + 1)^d$
- $t \to M_t^Q f(x)$ changes only at $t \in \mathbb{N}$. WLOG we may restrict to such t.
- $$m_t^Q(\xi) = \frac{1}{(2t+1)^d} \prod_{k=1}^{d} \frac{\sin((2t+1)\pi \xi_k)}{(2t+1)\sin \pi \xi_k}$$

We can prove estimates analogous to the continuous case

PROPOSITION (DISCRETE FT ESTIMATES FOR Q), [BOURGAIN, MIREK, STEIN, WRÓBEL 19]

There is a universal constant $C>0$ such that

estimates

1) $|m_t^{Q}(\xi)-1| \leq C\,t|\xi|,$ (at 0)

2) $|m_t^{Q}(\xi)| \leq (t|\xi|)^{-1}$ (at ∞)

3) $|m_{t_1}^{Q}(\xi)-m_{t_2}^{Q}(\xi)| \leq C\,|t_1-t_2|\max(t_1^{-1},t_2^{-1}),$

where $\xi\in\mathbb{T}^d$, $t_1,t_2,t\in\mathbb{N}\setminus\{0\}$ (difference)

<u>Remark:</u> The above are much similar to general estimates in the continuous case when specified to $G = Q_{\frac{1}{2}} = [-\frac{1}{2}, \frac{1}{2}]^d$

For $t \geq 0$, $\xi \in \mathbb{R}^d$ let
$$\boxed{\nu_t(\xi) = \frac{1}{|(2t+1)Q_{\frac{1}{2}}|} \int_{(2t+1)Q_{\frac{1}{2}}} e^{-2\pi i x\cdot\xi}\, dx}.$$

Then $\nu_t(\xi) = \nu_0((2t+1)\xi)$ and $\nu_0(\xi) = \mathcal{F}(1_{Q_{\frac{1}{2}}})(\xi)$

$Q_{\frac{1}{2}}$ is in the isotropic position with $L(Q_{\frac{1}{2}}) = \frac{1}{12}$ COMMENT →

Thus from Bourgain FT estimates

$$\boxed{\begin{aligned} &1)\ |\nu_t(\xi) - 1| \leq 150\,((2t+1)|\xi|) && (\text{at } 0)\\ &2)\ |\nu_t(\xi)| \leq 150\,((2t+1)|\xi|)^{-1} && (\text{at } \infty)\end{aligned}}$$

Moreover by the mean value theorem

$$\left|\nu_{t_1}(\xi)-\nu_{t_2}(\xi)\right| = \left|\nu_0((2t_1+1)\xi)-\nu_0((2t_2+1)\xi)\right| \leq 2|t_1-t_2| \sup_{s\in[t_1,t_2]} \left|\frac{\partial}{\partial s}\nu_0((2s+1)\xi)\right|$$

Using the chain rule $\frac{\partial}{\partial s}\nu_0(s\xi) = 2\cdot\frac{1}{2s+1}\langle(2s+1)\xi, \nabla\nu_0((2s+1)\xi)\rangle$

$$3)\quad \left|\nu_{t_1}(\xi)-\nu_{t_2}(\xi)\right| \leq |t_1-t_2|\max(t_1^{-1},t_2^{-1})\sup_{\xi\in\mathbb{R}^d}|\langle\xi,\nabla\nu_0(\xi)\rangle| \leq$$

$$\leq 150\,|t_1-t_2|\max(t_1^{-1},t_2^{-1})$$ (difference)

The last inequality above comes from Bourgain's continuous estimate for $G=Q_{\frac{1}{2}}$.

PROOF (SKETCH) OF THE ESTIMATE FOR $m^Q_{t_1} - m^Q_{t_2}$

It suffices to take $t_1 \neq t_2$. In the sketch we take $|m^Q_t(\xi)| \le C(t|\xi|)^{-1}$ for granted.

Idea: compare m^Q_t with ν_t.

$$m^Q_{t_1}(\xi) - m^Q_{t_2}(\xi) = m^Q_{t_1}(\xi) - \nu_{t_1}(\xi) + \nu_{t_2}(\xi) - m^Q_{t_2}(\xi) + \nu_{t_1}(\xi) - \nu_{t_2}(\xi)$$

Recall that the goal is $\boxed{|m^Q_{t_1}(\xi) - m^Q_{t_2}(\xi)| \le C|t_1 - t_2| \max(t_1^{-1}, t_2^{-1})}$

The term $\nu_{t_1}(\xi) - \nu_{t_2}(\xi)$ is controlled by 3) (difference)

Using the explicit formula $\nu_t(\xi) = \frac{1}{(2t+1)^d} \prod_{k=1}^{d} \frac{\sin \pi (2t+1)\xi_k}{\pi (2t+1)\xi_k}$ we write

$$\boxed{m^Q_t(\xi) - \nu_t(\xi)} = \left(1 - \prod_{k=1}^{d} \frac{(2t+1)\sin \pi \xi_k}{\pi (2t+1)\xi_k}\right) \cdot \prod_{k=1}^{d} \frac{\sin \pi (2t+1)\xi_k}{(2t+1)\sin \pi \xi_k} = \boxed{(1 - \nu_0(\xi))\, m^Q_t(\xi)}$$

Now, using the estimate 1) (at 0) for ν_t and 2) (at ∞) for m_t^Q we reach

$$\left| m_t^Q(\xi) - \nu_t(\xi) \right| \leq C\, |\xi| \cdot \frac{1}{t|\xi|} \leq C\, t^{-1}$$ and thus

$$\boxed{\left| m_{t_1}^Q(\xi) - m_{t_2}^Q(\xi) \right|} = \left| m_{t_1}^Q(\xi) - \nu_{t_1}(\xi) + \nu_{t_2}(\xi) - m_{t_2}^Q(\xi) + \nu_{t_1}(\xi) - \nu_{t_2}(\xi) \right| \leq$$

$$\leq C \quad t_1^{-1} + t_2^{-1} + |t_1 - t_2| \max(t_1^{-1}, t_2^{-1}) \leq \boxed{C\, |t_1 - t_2| \max(t_1^{-1}, t_2^{-1})},$$

since we assumed that $t_1 \neq t_2$, $t_1, t_2 \in \mathbb{N} \setminus \{0\}$.

This completes the sketch. $\square$

<u>REMARK:</u> This works because ν_t is a very good approximation of m_t^Q.

Using the PROPOSITION on discrete FT estimates for m_t^Q we were able to justify:

THEOREM, [BMSW, 19] (DIM-FREE BOUNDS FOR $\mathcal{M}_*^Q$)

For each $\frac{3}{2} < p < \infty$ there exists a universal constant $C(p)$ such that

$$\| \mathcal{M}_*^Q f \|_{\ell^p(\mathbb{Z}^d)} \leq C(p) \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d)$$

<u>REMARK:</u> We did not cover $1 < p < \frac{3}{2}$. This seems to require new ideas not used in the continuous setting.

2. THE DISCRETE BALL CASE $G=B$

This case is much more difficult – no explicit formulas

- $\sum_{i=1}^{d} x_i^2 \leq t^2 \iff \sum_{i=1}^{d} x_i^2 \leq \lfloor t^2 \rfloor \Rightarrow$ only integer values of t^2 matter $\Rightarrow t \in \sqrt{\mathbb{N}}$

- It is not hard to see that

$$|tB \cap \mathbb{Z}^d| \approx |tB|, \qquad \text{for } t > \tfrac{1}{2} d^{\frac{3}{2}}$$

- However by COMPARISON PRINCIPLE 2 we are interested in $t < \frac{1}{2} d^{\frac{3}{2}}$

- for small $t < \frac{1}{2} d^{\frac{3}{2}}$ the number of lattice points in $tB \cap \mathbb{Z}^d$ is of a different nature e.g. if $t < 1$, then $|tB \cap \mathbb{Z}^d| = 1$, but $|tB| \xrightarrow{t \to 0} 0$

Comment →

REMARK: It is easy to see that $|tB \cap \mathbb{Z}^d| = t^d |B| + R_d(t)$, where the term $R_d(t)$ is of smaller order than t^d. The question on the optimal bound for $d=2$, i.e. for $R_2(t)$ is a major open question of ANALYTIC NUMBER THEORY labelled GAUSS CIRCLE PROBLEM. It is easy to see that $R_2(t) \leq ct$ and it is conjectured that the optimal bound is

$$R_2(t) \leq C t^{\frac{1}{2}+\varepsilon}, \quad t \to \infty$$

Even more problematic is the estimation of

$$m_t^B(\xi) = \frac{1}{|tB \cap \mathbb{Z}^d|} \sum_{x \in tB \cap \mathbb{Z}^d} e^{2\pi i x \cdot \xi}, \quad \xi \in \mathbb{T}^d$$

Something however is possible COMMENT →

LEMMA: For all $t > 0$ and $\xi \in \mathbb{T}^d$ we have

1) $$\left| m_t^B(\xi) - 1 \right| \leq C \min\left(1, \frac{t^2}{d} |\xi|^2\right) \leq C \frac{t}{\sqrt{d}} |\xi|,$$

where C is a universal constant (at 0)

→ Recall 1) for the discrete cube

$$\left| m_t^Q(\xi) - 1 \right| \leq C t |\xi|$$

→ The change from t to $\frac{t}{\sqrt{d}}$ is due to the fact that $\frac{1}{2}Q$ is in the isotropic position, whereas to scale the ball B to the isotropic position we need to consider $\approx \sqrt{d}\, B$

→ Compare $|Q| = 2^d$ vs. $|B| \approx \frac{1}{\sqrt{d\pi}} (2\pi e)^{\frac{d}{2}} d^{-\frac{d}{2}}$

The LEMMA suggest to define

$$\boxed{K = \frac{t}{\sqrt{d}}}$$

In place of 2) for the cube $\left| m_t^{\alpha}(\xi) \right| \leq C\big(t|\xi|\big)^{-1}$ we would like e.g.

$$\left| m_t^{\beta}(\xi) \right| \leq \frac{C}{K|\xi|} = \frac{C\sqrt{d}}{t|\xi|}$$

However this is much harder and not known

What is known in place of 2) for m_t^β ?

PROPOSITION [BMSW, 19] (FT ESTIMATES FOR m_t^β AT ∞)

There is a universal constant such that

2) for $\sqrt{d} \leq t \iff K \geq 1$

$$|m_t^\beta(\xi)| \leq C(K|\xi|)^{-1} + CK^{-\frac{1}{7}},$$

2') for $t < \sqrt{d} \iff K < 1$

$$|m_t^\beta(\xi)| \leq C\exp\left(-\frac{K}{400}|\xi|^2\right) + C\exp\left(-\frac{K}{400}\left|\xi+\tfrac{1}{2}\right|^2\right)$$

where $\xi + \frac{1}{2} = (\xi_1 + \frac{1}{2}, \dots, \xi_d + \frac{1}{2})$

REMARKS:

→ The proofs of 2) and 2') are involved. We shall give a rough sketch of 2') later, if time permits

→ Still we have no reasonable estimate for the difference

$$\left| m_{t_1}^{B}(\xi) - m_{t_2}^{B}(\xi) \right|$$

This seems <u>hard</u>

→ This is why we could only prove

THEOREM [BMSW19] (DIM-FREE BOUNDS ON ℓ^2 FOR DYADIC $\mathcal{M}_*^B$)

There is a universal constant such that,

$$\Big\| \sup_{n\in\mathbb{N}} \big| \mathcal{M}_{2^n}^B f(x) \big| \Big\|_{\ell^p(\mathbb{Z}^d)} \leq C \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f\in \ell^p(\mathbb{Z}^d)$$

for all dimensions $d\in\mathbb{N}$ and $p\in[2,\infty]$.

REMARK:

$\longrightarrow$ It is easy to see that $\|\mathcal{M}_*^B f\|_{\ell^\infty} \leq \|f\|_{\ell^\infty}$. The difficult part of the above theorem is $p=2$.

The rest follows by „interpolation".

A conjecture that is open for ≈ 25 years is

CONJECTURE: STEIN DISCRETE MAXIMAL FUNCTION CONJECTURE ≈ 95

Is there a universal constant $C>0$ such that

$$\|\mathcal{M}_*^B f\|_{\ell^2(\mathbb{Z}^d)} \leq C \|f\|_{\ell^2(\mathbb{Z}^d)}, \qquad f \in \ell^2(\mathbb{Z}^d),$$

uniformly in dimension $d \in \mathbb{N}$?

REMARK:

$\longrightarrow$ Our dyadic maximal theorem may be regarded a first step towards a solution of Stein's conjecture

VERY ROUGH SKETCH OF THE PROOF OF 2)'

The goal is, for $\xi \in \mathbb{T}^d$, $\kappa < 1$ (i.e. $t < \sqrt{d}$)

$$\boxed{\left| m_t^B(\xi) \right| \leq C \exp\left(-\frac{\kappa}{400} |\xi|^2\right) + C \exp\left(-\frac{\kappa}{400} \left|\xi + \frac{1}{2}\right|^2\right)}$$

IDEA 1):

If $x \in tB \cap \mathbb{Z}^d$ then $\tau \circ x = (x_{\tau(1)}, \dots, x_{\tau(d)}) \in tB \cap \mathbb{Z}^d$

for any permutation $\tau : \{1, 2, \dots, d\} \to \{1, 2, \dots, d\}$.

In fact $x \in tB \cap \mathbb{Z}^d \Longleftrightarrow (\tau \circ x) \in tB \cap \mathbb{Z}^d$

Permutations form the permutation group $Sym(d)$, $|Sym(d)| = d!$

We have

$$m_t^B(\xi) = \frac{1}{|tB \cap \mathbb{Z}^d|} \sum_{x \in tB \cap \mathbb{Z}^d} e^{-2\pi i x \cdot \xi} = \frac{1}{|tB \cap \mathbb{Z}^d|} \sum_{(\tau \circ x) \in tB \cap \mathbb{Z}^d} e^{-2\pi i (\tau \circ x) \cdot \xi}$$

$$= \frac{1}{|tB \cap \mathbb{Z}^d|} \sum_{x \in tB \cap \mathbb{Z}^d} e^{-2\pi i (\tau \circ x) \cdot \xi}$$

$$= \frac{1}{|tB \cap \mathbb{Z}^d|} \sum_{x \in tB \cap \mathbb{Z}^d} \frac{1}{|\mathrm{Sym}(d)|} \sum_{\tau \in \mathrm{Sym}(d)} e^{-2\pi i (\tau \circ x) \cdot \xi}$$

This in a way matters reduce to estimating

$$\boxed{\frac{1}{|\mathrm{Sym}(d)|} \sum_{\tau \in \mathrm{Sym}(d)} e^{-2\pi i\, (\tau \circ x)\cdot \xi}}$$

for $x \in tB \cap \mathbb{Z}^d$ and $\xi \in \mathbb{T}^d$

IDEA 2: For $x \in tB \cap \mathbb{Z}^d$ let $I_x = \{ i \in \{1, \dots, d\} : |x_i| = 1 \}$. Then

$$\frac{|\{ tB \cap \mathbb{Z}^d : |I_x| < \frac{d}{10} \}|}{|tB \cap \mathbb{Z}^d|} \leq C\, 2^{-t^2} = C\, 2^{-k^2 d} \leq C\, 2^{-k^2 |\xi|}$$

Thus in a way we may only focus on $x \in tB \cap \mathbb{Z}^d$ such that $|I_x| \geq \frac{d}{10}$. Oversimplifying say $|I_x| = d$. We need to control

$$\frac{1}{|\mathrm{Sym}(d)|} \sum_{\tau \in \mathrm{Sym}(d)} e^{-2\pi i (\tau \circ x) \cdot \xi} \quad , \quad \xi \in \mathbb{T}^d$$

but this time for $\boxed{x \in \{-1,1\}^d \Longleftrightarrow \tau \circ x \in \{-1,1\}^d}$

IDEA 3: For $S \subseteq \{1, \dots, d\}$ denote

$$a_S(\xi) = \prod_{j \in S} \sin^2 \pi \xi_j \prod_{i \in \{1,\dots,d\} \setminus S} \cos^2 \pi \xi_i \,, \quad \xi \in \mathbb{T}^d$$

and, for $n \leq d$,

$$K_n^{(d)}(S) = \frac{1}{\binom{d}{n}} \sum_{j=0}^{n} (-1)^j \binom{|S|}{j} \binom{d-|S|}{n-j} .$$

$K_n^{(d)}$ is called Krawtchouk polynomial

It can be justified that for $x \in \{-1,1\}^d$

$$\boxed{\frac{1}{|\mathrm{Sym}(d)|} \sum_{\tau \in \mathrm{Sym}(d)} e^{-2\pi i (\tau \circ x)\cdot \xi} = \sum_{S \subseteq \{1,\dots,d\}} a_S(\xi) K_n^{(d)}(S)},$$

where $n = t^2 \in \mathbb{N}$.

Appropriate (dimension-free) estimates for $K_n^{(d)}(S)$

were proved [HARROW, KOLLA, SCHULMAN, 12]. This allows for a completion of

the estimate for $m_t^\beta(\xi)$.

Thank you for your attention!