METRIC COMPACTNESS CRITERIA INVOLVING SEQUENCES OF MAPPINGS AND A PROOF OF THE ASCOLI–ARZELÀ THEOREM WITH THE USE OF BERNSTEIN POLYNOMIALS

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Abstract. We establish inter alia a compactness criterion in metric spaces involving a sequence of completely continuous mappings, which is continuously convergent, in the sense of H. Hahn (see, e.g., [6, p. 197]), to the identity mapping. For Banach spaces, the linear version of that result coincides with the compactness theorem due to S. Mazur, which was first mentioned, without a proof, in Banach's French monograph [1, p. 237]. We also present probably a new proof of the Ascoli–Arzelà theorem, in which we use the above compactness criterion applied to the sequence of Bernstein operators. Let us note that in classical proofs of the Ascoli–Arzelà theorem either a finite ε-net for a suitable family of functions is constructed (see, e.g., [3] or [8, p. 394]), or a diagonalization argument is used as done, e.g., in [5, p. 154]. There are also other lesser-known approaches: the proof given by Ullrich [9] is based on the Tychonoff compactness theorem; Nagy [7] presented a functional analytic proof with the help of the Banach–Alaoglu theorem; Beer [2] derived the result from the Zarankiewicz compactness theorem for sequences of closed sets in a separable metric space; Hanche-Olsen and Holden [4] proved the theorem via a clever simple lemma on metric compactness, which, however, is completely different from our criterion. At last, recently, Wójtowicz [10] provided yet another proof that uses the Stone–Čech compactification technique.

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