On Δ-spaces X and their characterization in terms of spaces $C_p(X)$

ABSTRACT

Reed (see [6], [4]) studied those uncountable subsets D (under the name Δ-sets) of the reals $\mathbb{R}$ (with the natural topology) having the following property:

For any decreasing sequence $(H_n)_n$ *of subsets of* D *with* $\bigcap_n H_n = \emptyset$ *there is a sequence* $(V_n)_n$ *of* G_δ*-subsets of* D *such that* $H_n \subset V_n$, $n \in \mathbb{N}$, *and* $\bigcap_n V_n = \emptyset$. Research about Δ-spaces is strictly connected with a study of $\mathbb{Q}$-sets, one of the most mysterious objects in $\mathbb{R}$. In [2] the concept of a Δ-set has been extended to arbitrary topological spaces: A topological space X is called a Δ-*space* if for every decreasing sequence $(D_n)_n$ of subsets of X with $\bigcap_n D_n = \emptyset$, there is a decreasing sequence $(V_n)_n$ of open subsets of X, $D_n \subset V_n$ for every $n \in N$ and $\bigcap_n V_n = \emptyset$. In [2] we proved that X is a Δ-space if and only the dual of $C_p(X)$ endowed with the topology of the uniform convergence on $C_p(X)$-bounded sets carries the finest locally convex topology. This analytic approach provided several new results about Δ-sets and Δ-spaces [2], [3] [5]. Some alternative characterization was also presented in [1]. Applications for Banach spaces $C(K)$ and spaces $C_p(K)$ are provided.

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[3] J. Kąkol, A. Leiderman, *Basic properties of X for which spaces* $C_p(X)$ *are distinguished*, Proc. Amer. Math. Soc. 8 (2021), 257–280.

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