On geometric complexity of Julia sets - III

Będlewo, 26.09-1.10.2021

Abstracts of talks

Mañé-Sad-Sullivan and meromorphic dynamics
Anna Miriam Benini

This is joint work with M. Astorg and N. Fagella. We present a version of Mañé-Sad-Sullivan Theorem for natural families of meromorphic maps with finitely many singular values. One of the main differences with respect to the rational case (Lyubich, Mañé-Sad-Sullivan) and to the entire transcendental case (Eremenko, Lyubich) is that some points in periodic cycles can disappear at infinity, creating so-called virtual cycles.

The Hausdorff dimension of Julia sets of meromorphic functions in the Speiser class
Walter Bergweiler

We show that for each d in $(0, 2]$ there exists a meromorphic function f such that the inverse function of f has three singularities and the Julia set of f has Hausdorff dimension d. This is joint work with Weiwei Cui.

Higher bifurcations for polynomial skew products
Fabrizio Bianchi

Given a holomorphic family of rational maps on the Riemann Sphere, one can decompose the parameter space into a stability locus and a bifurcation locus. The latter corresponds to maps whose global dynamics are very sensitive to a perturbation of the parameter and is characterized as the support of the so-called bifurcation current.

The changes in the global dynamics are dictated by changes in the dynamics of the critical set. When several critical points are present, it makes sense to define a stratification of the bifurcation locus, depending on how many critical points bifurcate independently. The good framework to do this is by means of the self-intersections of the bifurcation current, and one can prove that the resulting stratification is strict.

We consider in this talk dynamical systems on $\mathbb{C}^2$ of the form $f(z, w) = (p(z), q(z, w))$, for suitable polynomials p, q. The stability/bifurcation dichotomy in higher dimensions was developed in previous joint work with Berteloot and Dupont. We prove here that, in contrast with the one-dimensional case, all the self-intersections of the bifurcation current have the same support.

A key point in the proof is the construction of a sufficiently large hyperbolic set. This is achieved by exploiting a recent result by Przytycki-Zdunik on non completely disconnected Julia sets and by extending techniques coming from the thermodynamic formalism of rational maps to this skew-product setting.

Joint work with Matthieu Astorg, Orleans.

Dimensions of transcendental Julia sets
Christopher Bishop

This is a survey of some known results about Hausdorff and packing dimension of Julia sets of transcendental entire functions. After reviewing the basic definitions and some analogous results for polynomials, I will discuss Baker's theorem that the Hausdorff dimension is always at least 1, and describe examples showing that all values in the interval [1,2] can be attained. Whereas in polynomial dynamics it is hard to construct Julia sets with dimension 2 or positive area, in transcendental dynamics the difficult problem is to build "small" examples, e.g., dimension close to 1 or having finite (spherical) length. I will end by stating some open problems.

On the geometry of simply connected wandering domains
Luka Boc Thaler

In this talk we will construct an entire function $f : \mathbb{C} \to \mathbb{C}$ for which the unit disk $\mathbb{D}$ is a wandering domain. The construction relies on the approximation techniques and can be generalized so that the unit disk $\mathbb{D}$ may be replaced by any bounded connected regular open set U, whose closure has a connected complement. In particular this implies that every simply connected Jordan domain is an wandering domain of some entire function. These wandering domains can either be escaping or oscillating [2]. Similar approach was used earlier in [1] to show that a unit ball $\mathbb{B} \subset \mathbb{C}^m$ is a wandering domain of some automorphism of $\mathbb{C}^m$.

References

[1] L. Boc Thaler, *Automorphisms of $\mathbb{C}^m$ with bounded wandering domains*, Annali di Matematica Pura ed Applicata (2021), doi: 10.1007/s10231-020-01057-3

[2] L. Boc Thaler, *On the geometry of simply connected wandering domains*, Bulletin of the London Mathematical Society, (2021), https://doi.org/10.1112/blms.12518

Eremenkos Conjecture, Devaneys Hairs, and the Growth of Counterexamples
Andrew Brown

Fatou noticed in 1926 that certain transcendental entire functions have Julia sets in which there are curves of points that escape to infinity under iteration and he wondered whether this might hold for a more general class of functions. In 1989, Eremenko carried out an investigation of the escaping set of a transcendental entire function f, $I(f) = \{z \in \mathbb{C}\colon |f^n(z)| \to \infty\}$ and produced a conjecture with a weak and a strong form. The strong form asks if every point in the escaping set of an arbitrary transcendental entire function can be joined to infinity by a curve in the escaping set.

This was answered in the negative by the 2011 paper of Rottenfusser, Rckert, Rempe, and Schleicher (RRRS) by constructing a tract that produces a function that cannot contain such a curve. In the same paper, it was also shown that if the function was of finite order, that is, $\log\log|f(z)| = \mathcal{O}(\log|z|)$ as $|z| \to \infty$, then every point in the escaping set can indeed be connected to infinity by a curve in the escaping set.

The counterexample f used in the RRRS paper has growth such that $\log\log|f(z)| = \mathcal{O}\left((\log|z|)^K\right)$ where $K > 12$ is an arbitrary constant. The question is, can this exponent, K, be decreased and can explicit calculations and counterexamples be performed and constructed that improve on this?

Analytic symmetries of parabolic and elliptic elements
Davoud Cheraghi

In this talk we discuss the analytic centralisers of parabolic and elliptic analytic maps. We explain the analytic centraliser for a number of classes of maps, in particular, we show that for a dense set of irrational numbers α, the analytic centraliser of the map $e^{2\pi i\alpha}z + z^2$ near 0 is trivial. We also present the first examples of analytic circle diffeomorphisms, with irrational rotation numbers, which have trivial centralisers.

Bounded type Siegel disks of finite type maps with few singular values
Arnaud Chéritat

Let U be an open subset of the Riemann sphere $\widehat{\mathbb{C}}$. We give sufficient conditions for which a finite type map $f : U \to \widehat{\mathbb{C}}$ with at most three singular values has a Siegel disk compactly contained in U and whose boundary is a quasicircle containing a unique critical point. The main tool is quasiconformal surgery à la Douady-Ghys-Herman-Świątek. We also give sufficient conditions for which, instead, Δ has not compact closure in U. The main tool is the Schwarzian derivative and area inequalities à la Graczyk-Świątek.

Hausdorff dimension of escaping sets for Speiser functions with few singular values
Weiwei Cui

A meromorphic function $f \in \mathbb{C} \to \hat{\mathbb{C}}$ is called a Speiser function, if it has finitely many singular values. In the talk we will talk about constructing Speiser functions with only few singular values. The purpose is to show that to attain each possible value of Hausdorff dimension of escaping sets for Speiser functions, it is enough to consider functions with only four singular values. This is a joint work with Magnus Aspenberg.

How to use box mappings as “black boxes”
Kostiantyn Drach

The concept of a complex box mapping (aka puzzle mapping) is a generalization of the classical notion of polynomial-like map to the case when one allows for countably many components in the domain and finitely many components in the range of the mapping. In one-dimensional dynamics, box mappings appear naturally as first return maps to certain nice sets intersecting the critical set of the map. In this talk, we will discuss various features of general box mappings, as well as so-called dynamically natural box mappings, which will include local connectivity of their Julia sets, ergodicity, etc. We will then show how these results can be used almost as “black boxes” to conclude similar properties in those families of rational maps where non-trivial box mappings can be extracted. Among the key examples for us will be complex polynomials of arbitrary degrees and their Newton maps. (The talk is based on joint work with Trevor Clark, Oleg Kozlovski, Dierk Schleicher and Sebastian van Strien.)

Expanding and relatively expanding Thurston maps
Dzmitry Dudko

One of the prominent features of post-critically finite rational maps is that they expand the hyperbolic metric of the complement of the postcritical set. More generally, a non-invertible Thurston map f is isotopic to an expanding map if and only if f admits no Levy cycle (unless an “exceptional” torus endomorphism doubly covers f). Even if there is a Levy cycle, f may still have an expanding “cactoid” quotient. We will discuss this "relative expansion" property and its relation to the mating and Thurston decidability problems. Based on joint work with Laurent Bartholdi and Kevin Pilgrim.

Boundary dynamics of wandering domains: sufficient conditions for uniform behaviour 2
Vasiliki Evdoridou

This is the third of a series of talks on this topic. In this talk we focus on the boundary behaviour of contracting wandering domains. For such a wandering domain U of a transcendental entire function f, we show that the Denjoy-Wolff set of $(f^n|_U)$ has either full or zero harmonic measure. In fact, we prove a more general result concerning compositions of holomorphic maps, which is inspired, and somehow extends, a result by Pommerenke on compositions of inner functions. We also give an example which shows that the contracting condition is necessary in the general case. This is joint work with A.M. Benini, N. Fagella, P. Rippon and G. Stallard.

Virtual centers in the parameter space of meromorphic maps
Núria Fagella

(Joint work with Anna Miriam Benini and Matthieu Astorg.) We present new results about the bifurcation loci of natural families of meromorphic transcendental maps. We describe some special types of parameter values and their role in the bifurcation locus. In particular we prove (in great generality) that the set of parameters for which an asymptotic value is a prepole of order n, coincides with the set of parameters for which an attracting periodic of period $n+1$ disappears to infinity (virtual centers).

Projecting Newton maps of Entire functions via the Exponential map
Robert Florido Llinàs

Lifting maps by the exponential function is a fruitful procedure to construct examples of escaping wandering domains. This method relies on a theorem by Bergweiler relating the Julia sets of an entire map and its projection, which was extended by Zheng in 2005 to meromorphic functions outside a small set. We investigate the Fatou components of a particular class of transcendental Newton maps and the corresponding projections, and we seek conditions for the existence of possible wandering domains. Work in progress.

Dynamics of the secant map
Antonio Garijo

We investigate the root finding algorithm given by the secant method applied to a real polynomial p as a discrete dynamical system defined on the real plane. In particular, given a simple root of p we show the existence of a four cycle related to the immediate basin of attraction. This is a joint work with Ernest Fontich, Laura Gardini and Xavier Jarque.

Shapes of trees
Oleg Ivrii

A finite tree in the plane is called a true tree if every side of every edge has the same harmonic measure as seen from infinity. It is well known that any finite tree has a conformally balanced shape, unique up to scale. In this talk, we study shapes of infinite trees, focusing on the case of an infinite trivalent tree. To conformally balance the infinite trivalent tree, we truncate it at level n, form the true tree $\mathcal{T}_n$ and take $n \to \infty$. We show that the Hausdorff limit of the $\mathcal{T}_n$ contains the boundary of the developed deltoid, the domain obtained by repeatedly reflecting the deltoid in its sides. We also give a sequence of trees which produces the Cauliflower, the Julia set of $z^2 + 1/4$. (This is joint work with P. Lin, S. Rohde and E. Sygal.)

On the basins of attraction of a one-dimensional family of root finding algorithms: From Newton to Traub

Xavier Jarque

In this paper we study the dynamics of damped Traub's methods T_δ when applied to polynomials. The family of damped Traub's methods consists of root finding algorithms which contain both Newton's ($\delta = 0$) and Traub's method ($\delta = 1$). Our goal is to obtain several topological properties of the basins of attraction of the roots of a polynomial p under T_1, which are used to determine a (universal) set of initial conditions for which convergence to all roots of p can be guaranteed. We also numerically explore the global properties of the dynamical plane for T_δ to better understand the connection between Newton's method and Traub's method.

Dynamics on the boundary of Fatou components

Anna Jové Campabadal

In this talk, we compile the known results about the dynamics on the boundary of invariant simply connected Fatou components, as well as the questions which are still open concerning the topic. We focus on ergodicity and recurrence. One of the main tools to deal with this kind of questions is to study the boundary behaviour of the associate inner functions. Therefore, first ergodicity and recurrence are studied for inner functions. Second, these results are applied to study the dynamics on the boundary of invariant simply connected Fatou components.

Moreover, we study the concrete example $f(z) = z + e^{-z}$, which presents infinitely many invariant doubly-parabolic Baker domains U_k. Making use of the associate inner function, which can be computed explicitly, we give a complete characterization of the periodic points in ∂U_k and prove the existence of uncountably many curves of non-accessible escaping points.

Fatou-Shishikura inequality for transcendental entire functions in the Speiser class

Masashi Kisaka

We discuss the following realizability problem: For given numbers which satisfy the Fatou-Shishikura inequality, is there a transcendental entire function in the Speiser class S with the given numbers of Fatou components?

On hyperbolic sets of polynomials, II

Genadi Levin

Let f be an infinitely-renormalizable quadratic polynomial and J_∞ the intersection of orbits of 'small' Julia sets of simple renormalizations of f. In [LP] we show that the restriction map $f : J_\infty \to J_\infty$ has no hyperbolic sets. I plan to talk about a more general question: can the Lyapunov exponent of an invariant measure of the map $f : J_\infty \to J_\infty$ be positive? Joint work (in progress) with Feliks Przytycki.

[LP] G. Levin, F. Przytycki, *On hyperbolic sets of polynomials.* arXiv:2107.11962

Quasisymmetric Uniformization, quasi-visual approximations, and Thurston maps
Daniel Meyer

Quasisymmetric maps are generalizations of conformal maps and may be viewed as a global versions of quasiconformal maps. Originally, they were introduced in the context of geometric function theory, but appear now in geometric group theory and analysis on metric spaces among others. The quasisymmetric uniformization problem asks when a given metric space is quasisymmetric to some model space. Of particular importance is the case where the model space is the 2-sphere. The reason is that Cannon's conjecture, and Thurston's characterization of rational maps, may be expressed that certain metric spaces are quasisymmetrically equivalent to the standard 2-sphere. A quasi-visual approximation is a sequence of discrete approximations of a given metric space. There is a necessary and sufficient condition for quasisymmetric equivalence in terms of quasi-visual approximations. This has applications to the quasisymmetric uniformization of trees as well as to Thurston maps. This is joint work with Mario Bonk and Mikhail Hlushchanka.

Orbits and bungee sets
Dan Nicks

In the study of discrete dynamical systems, we typically start with a function from a space into itself, and ask questions about the properties of sequences of iterates of the function. In the first part of this talk we reverse the direction of this study by starting with a sequence of points and studying the functions (if any) for which this sequence is an orbit under iteration. This gives rise to questions of existence and of uniqueness. The answers depend on the class of functions considered: holomorphic functions, quasiregular functions, continuous functions, etc.

In the second part of the talk, we consider the *bungee set* of a function f; that is, the set of points x for which the orbit $(f^n(x))_{n\geq 0}$ has both bounded and unbounded subsequences. For a quasiregular map f we make some connections between the bungee set and the Julia set of f.

Joint work with David Sixsmith.

Newton-like behaviour in the Cebyshev-Halley family of degree n polynomials
Dan Paraschiv

Previously, Campos, Canela, and Vindel have studied the family of maps obtained by applying the Cebyshev-Halley family of numerical methods to degree n unicritical polynomials. More precisely, they have studied the possible connectivities of Fatou components in a general setting and the dynamical/ parameter plane for $n = 2$. We prove that for $n > 2$, there exist parameters such that there exists a connected component of the Julia set which is a quasiconformal copy of the Julia set of a Newton's map for degree n unicritical polynomials. Work in progress.

The important set for non-autonomous exponential map
Łukasz Pawelec

In studying the behaviour of exponential map it is often useful to consider the set of points whose trajectory do not leave the strip $\mathbb{R} \times (-\pi, +\pi)$. The same holds for non-autonomous maps $\lambda_n e^z$. We will visit some properties of the set, such as its Hausdorff dimension and its topology. We will sketch the proof that the dimension is equal to one under some assumptions on lambda.

Zeros of the Independence polynomial for graphs of large degrees
Han Peters

I will report on joint work with Pjotr Buys and Ferenc Bencs, both from the University of Amsterdam (UvA) . The goal of our project is to accurately describe the maximal zero-free component of the independence polynomial for graphs of bounded degree, for large degree bounds. In previous work with David de Boer, Lorenzo Guerini and Guus Regts (all UvA) we demonstrated that this component coincides with the normality region of a discrete semi-group generated by finitely many rational maps. By passing through the infinite degree limit this normality region translates to the boundedness region for a continuous semi-group generated by infinitely many exponential maps. The key observation is that in the interior of the boundedness component, dominating singular orbits are strictly invariant under all the exponential generators, and hence provide bounded invariant sets for the rational semi-groups for sufficiently large degrees.

We prove that away from the real axis, the exponential boundedness component avoids a neighborhood of the limit cardioid, answering a recent question posed by Andreas Galanis (Oxford). We also describe the boundary of the exponential boundedness component near each of the real boundary points. Finally, we show that properties of the singular orbit can be used for rigorous computer computations of the exponential boundedness component.

Boundary dynamics of wandering domains: sufficient conditions for uniform behaviour 1
Phil Rippon

This is the second of a series of talks on this topic. In this talk we prove a theorem which shows that if the orbit under a holomorphic map f of an interior point in a wandering domain of f converges to the boundary of the corresponding wandering domains sufficiently quickly, depending on the geometry of the domains, then almost all points on the boundary have the same convergence property; that is, the so-called Denjoy-Wolff set of $(f^n|_U)$ has full harmonic measure. In fact, we prove such a result in a much more general setting. This generalises one part of a dichotomy due to Aaronson and Doering & Mañé, concerning the boundary dynamics of inner functions, and we give examples relating to the other part of the dichotomy.

Uniformity in internal dynamics of wandering domains
Gustavo Rodrigues Ferreira

In 2019, Benini et al. showed that, in a simply connected wandering domain, all pairs of orbits behave the same way relative to the hyperbolic metric, thus giving us our first insight into the general internal dynamics of such domains. After the more recent observation that the same is not true for multiply connected wandering domains, we ask ourselves: how inhomogeneous can multiply connected wandering domains get? We give an answer to this question, in that we show that whatever happens inside an open subset of the domain generalises (in some sense) to the whole wandering domain. After that (time allowing), we will show an application of this result towards the construction of new examples.

Postsingularly finite entire functions: combinatorics, complexity, Thurston theory
Dierk Schleicher

We describe the dynamics of post-singularly finite transcendental entire functions, reporting on work by and with David Pfrang, Roman Chernov, Malte Hassler, and Sergey Shemyakov. We show that every such function has a Homotopy Hubbard Tree that allows to distinguish different maps within any given parameter space from each other. Via these Hubbard trees, one can define “core entropy” as a measure of complexity of the dynamics. Unlike topological entropy, we show that core entropy is always finite. For certain families of maps such as exponential maps, the entropy is always uniformly bounded (for exponential maps, by $\log 2$). For other families, such as the cosine family, there is no uniform bound throughout the family. The main focus of this talk is on extending Thurston theory from rational maps to a certain family of transcendental entire functions. Together with the existence of Hubbard trees, this provides a possibility to actually classify certain explicit families of entire functions in terms of their Hubbard tree. Some parts of this presentation has the character of an overview on recent work, and others describe work in progress.

TBA
Mitsuhiro Shishikura

Boundary dynamics of wandering domains: overview
Gwyneth Stallard

Although the dynamical behaviour of periodic Fatou components is well understood, far less is known about the behaviour of wandering domains. Recently, we showed that the internal dynamics of wandering domains can be classified into nine possible types. Now we give several results concerning the relationship between the behaviour of interior points and points on the boundary. We state our results in the much more general setting of sequences of holomorphic maps between simply connected domains, generalising classical results about iterates of self-maps of the unit disc. Motivated by the Denjoy-Wolff theorem, we introduce the notion of the Denjoy-Wolff set; those points on the boundary whose images have the same limiting behaviour as the images of all interior points. We state several results about the possible size of this set. The two subsequent talks will give details of some of the proofs and construct examples to show the rich variety of possible behaviours that can occur in our setting.

This is joint work with Anna Miriam Benini, Vasiliki Evdoridou, Nuria Fagella and Phil Rippon.

Explosion points of Zorich maps
Athanasios Tsantaris

In the theory of one dimensional holomorphic dynamics, one of the most well studied families of maps is the exponential family $E_\lambda(z) := \lambda e^z$, $\lambda \in \mathbb{C} \setminus \{0\}$. Zorich maps are the quasiregular higher dimensional analogues of the exponential map on the plane. For the exponential family $E_\lambda(z) := \lambda e^z$, $\lambda > 0$ it is generally well known that for $0 < \lambda \geq 1/e$ the Julia set of E_λ is a collection of disjoint curves. Mayer has shown that the set of endpoints of those curves together with the point at infinity form a connected set but the endpoints themselves are totally separated. In this talk we will discuss how we can generalize this result to the higher dimensional setting of Zorich maps.

Wandering Lakes of Wada
James Waterman

Whether Lakes of Wada continua can arise in complex dynamics is a long standing open problem, an analogue of which, Fatou first posed in 1920 concerning Fatou components of rational functions. We discuss constructing a transcendental entire function for which infinitely many Fatou components share the same boundary, answering this question. Our theorem also provides the first example of an entire function having a simply connected Fatou component whose closure has a disconnected complement, answering a recent question of Boc Thaler. Using the same techniques, we discuss giving new counterexamples to a conjecture of Eremenko concerning curves in the escaping set of a transcendental entire function. This is joint work with David Martí-Pete and Lasse Rempe.

Harmonic measures, Julia sets, and computability
Michael Yampolsky

The interplay between computability theory and complex analysis leads to interesting results in both disciplines. I will discuss some examples relating to computable properties of harmonic measures.

Integral means spectrum
Michel Zinsmeister

In this talk I will
1) survey some known facts and conjectures about integral means spectrum with a focus on their connection with holomorphic dynamics (thru the notion of pressure),
2) discuss some cases where this spectrum can be explicitely computed. (Joint work with B.Duplantier, Han Yong, Nguyen Thi Phung Chi)