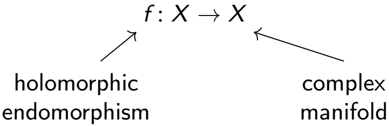
Higher bifurcations in families of polynomial skew products

Fabrizio Bianchi

CNRS and Laboratoire Paul Painlevé, Lille

October 1, 2021

Holomorphic dynamics



$$f \colon X \to X$$

holomorphic
endomorphism

complex
manifold

General questions: describe orbits $z, f(z), f \circ f(z), \ldots$

Deeper questions: how do orbits for a *generic* polynomial look like? How large is the set of maps with a given property P?

- e.g., $P =$ hyperbolicity (Fatou conjecture)

Dichotomies

1: Fatou - Julia dichotomy (in the dynamical space)

F = the dynamics stays the same after a small perturbation
(the iterates $\{f^n\}$ form a normal family on F)
J = drastic change of the dynamics after a small perturbation

<u>of the point</u>

... and Julia = closure of the repelling periodic points
= support of the unique measure of maximal entropy

2: Stability - Bifurcation dichotomy (in the parameter space)

Stab = the dynamics stays the same after a small perturbation
Bif = drastic change of the dynamics after a small perturbation

<u>of the parameter</u>

... and Bif is the closure of ... ? is the support of ... ?

Lyubich, Mané-Sad-Sullivan ('80s)

There is a natural dichotomy Stability - Bifurcation

More precisely, the following assertions are equivalent:

1. the periodic expanding points vary holomorphically
2. the Julia sets vary holomorphically
3. the Julia sets vary continuously
4. ...

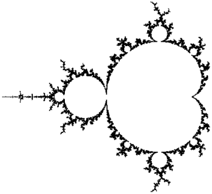
Lyubich, Mané-Sad-Sullivan ('80s)

There is a natural dichotomy Stability - Bifurcation

More precisely, the following assertions are equivalent:

1. the periodic expanding points vary holomorphically
2. the Julia sets vary holomorphically
3. the Julia sets vary continuously
4. ...

Example: $z^2 + \lambda$



- Stab is dense
- Shishikura, McMullen: the Hausdorff dimension of Bif is maximal

Bifurcations and critical dynamics (cf. Levin, McMullen...)

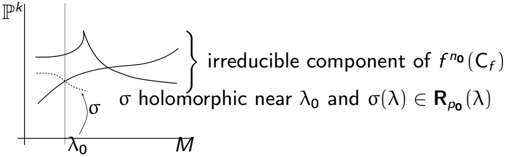
The global stability is dictated by the behaviour of the critical orbits

Activity

A critical point $c(\lambda)$ is *active* at λ_0 is the sequence $\lambda \mapsto f_\lambda^n(c(\lambda))$ is not normal near λ_0. The critical point is *passive* otherwise.

Misiurewicz parameter

A parameter λ_0 is *Misiurewicz* if some critical orbit intersects the motion of some repelling point non persistently at λ_0.



irreducible component of $f^{n_0}(C_f)$

σ holomorphic near λ_0 and $\sigma(\lambda) \in \mathbf{R}_{p_0}(\lambda)$

Misiurewicz parameters
are contained and dense
in the bifurcation locus

Higher bifurcations

$$(f_\lambda)_{\lambda\in\Lambda}, \dim \Lambda \geqslant 2$$

(e.g., the family of all rational maps of degree d)

Question

Can we identify (and characterize) the locus where several critical points bifurcate *independently* at the same moment? Does this give a stratification of the bifurcation locus?

Non-examples

- $c_1 \mapsto \ldots \mapsto c_2 \mapsto \ldots$

Bifurcations - potential

The Lyapunov function:

$$L(\lambda) = \int_{\mathbb{P}^1(\mathbb{C})} \log \|f_\lambda'\| d\mu_\lambda$$

- measure of the *expansiveness* of f_λ on $J(f_\lambda)$:

$$\boxed{(f_\lambda^n)' \sim e^{nL} \text{ for } \mu_\lambda - a.e.x \in J(f_\lambda)}$$

Bifurcations - potential

The Lyapunov function:

$$L(\lambda) = \int_{\mathbb{P}^1(\mathbb{C})} \log \|f_\lambda'\| d\mu_\lambda$$

- measure of the *expansiveness* of f_λ on $J(f_\lambda)$:

$$\boxed{(f_\lambda^n)' \sim e^{nL} \text{ for } \mu_\lambda - a.e. x \in J(f_\lambda)}$$

- continuous subharmonic function of $\lambda \in \Lambda$ ($dd^c L \geqslant 0$)

Przytycki, Sibony (polynomials), De Marco (rational maps)

The bifurcation set is equal to the support of $dd^c L =: T_{bif}$

- quantitative way to study bifurcations
- $dd^c L$ is an analogous of the measure of maximal entropy μ

Explanation of previous slide for polynomials - Przytycki's formula

Ingredients:

1. $\delta_a = dd^c \log|z-a|$

$$f' = d \cdot \prod_{i=1}^{d-1}(z-c_i) \Rightarrow \quad \log|f'| = \log d + \sum_{i=1}^{d-1} \log|z-c_i|$$

$$\Rightarrow dd^c \log|f'| = [C] = \delta_{c_1} + \cdots + \delta_{c_{d-1}}$$

2. $\mu = dd^c G$

Explanation of previous slide for polynomials - Przytycki's formula

Ingredients:

1. $\delta_a = dd^c \log|z-a|$

$$f' = d \cdot \prod_{i=1}^{d-1}(z-c_i) \Rightarrow \quad \log|f'| = \log d + \sum_{i=1}^{d-1} \log|z-c_i|$$

$$\Rightarrow dd^c \log|f'| = [C] = \delta_{c_1} + \cdots + \delta_{c_{d-1}}$$

2. $\mu = dd^c G$

$$L(f) = \int \log|f'|\mu = \int \log|f'| dd^c G = \log d + \int G\, dd^c \log|f'| = \log d + \sum_{i=1}^{d-1} G(c_i)$$

Explanation of previous slide for polynomials - Przytycki's formula

Ingredients:

1. $\delta_a = dd^c \log|z-a|$

$$f' = d \cdot \prod_{i=1}^{d-1}(z-c_i) \Rightarrow \quad \log|f'| = \log d + \sum_{i=1}^{d-1} \log|z-c_i|$$

$$\Rightarrow dd^c \log|f'| = [C] = \delta_{c_1} + \cdots + \delta_{c_{d-1}}$$

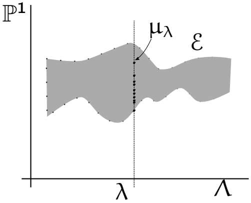
2. $\mu = dd^c G$

$$L(f) = \int \log|f'|\mu = \int \log|f'| dd^c G = \log d + \int G\, dd^c \log|f'| = \log d + \sum_{i=1}^{d-1} G(c_i)$$

$$dd^c L(\lambda) = \text{"}\, dd^c_\lambda \int \log|f'_\lambda|\mu_\lambda\text{"} = \sum_{i=1}^{d-1} dd^c G(c_i(\lambda)) \geqslant 0$$

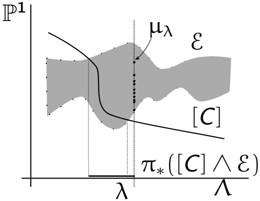
Why Misiurewicz parameters are in the support of $dd^c L$

- $\exists$ a current $\mathcal{E}$ on $\Lambda \times \mathbb{P}^1(\mathbb{C})$ such that $\mathcal{E}_\lambda = \mu_\lambda$



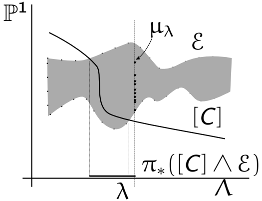
Why Misiurewicz parameters are in the support of $dd^c L$

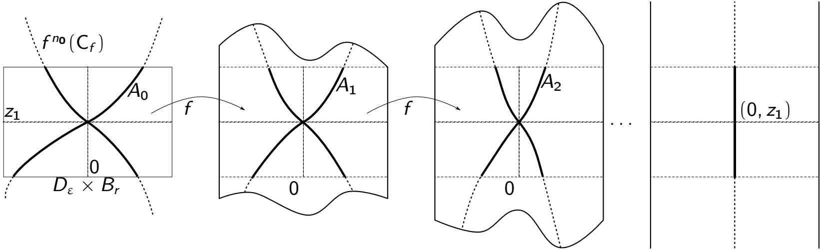
- $\exists$ a current $\mathcal{E}$ on $\Lambda \times \mathbb{P}^1(\mathbb{C})$ such that $\mathcal{E}_\lambda = \mu_\lambda$
- $dd^c L = \pi_*([C] \wedge \mathcal{E}) \quad ([C] = dd^c \log |f'|)$



Why Misiurewicz parameters are in the support of $dd^c L$

- $\exists$ a current $\mathcal{E}$ on $\Lambda \times \mathbb{P}^1(\mathbb{C})$ such that $\mathcal{E}_\lambda = \mu_\lambda$
- $dd^c L = \pi_*([C] \wedge \mathcal{E}) \qquad ([C] = dd^c \log |f'|)$
- $f_\lambda^*(\mu_\lambda) = d\mu_\lambda \Rightarrow dd^c L = \pi_*([f^n C] \wedge \mathcal{E})$





Higher bifurcations

$\dim \Lambda \geqslant 2; \quad L \text{ psh} \Rightarrow T_{bif} := dd^c L$ (1, 1 current)

$$T_{bif}^m := T_{bif} \wedge \ldots (m) \cdots \wedge T_{bif}$$
$$Bif_m := \operatorname{supp} T_{bif}^m$$

(Bassanelli-Berteloot)

Higher bifurcations

$\dim \Lambda \geqslant 2; \quad L$ psh $\Rightarrow T_{bif} := dd^c L$ (1, 1 current)

$$T^m_{bif} := T_{bif} \wedge \ldots (m) \cdots \wedge T_{bif}$$
$$Bif_m := \mathrm{supp}\, T^m_{bif}$$

(Bassanelli-Berteloot)

Bif_2 is *not* the locus where 2 critical points are active

Example

Cubic polynomials $z^3 + az^2 + b$. Then $(1/2, 1) \in \mathrm{Bif} \setminus \mathrm{Bif}_2$

In this example, the critical points are conjugated, and cannot bifurcate independently

Higher bifurcations

$\dim \Lambda \geqslant 2$; L psh $\Rightarrow T_{bif} := dd^c L$ (1, 1 current)

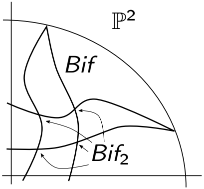
$$T_{bif}^m := T_{bif} \wedge \dots (m) \dots \wedge T_{bif}$$
$$Bif_m := \operatorname{supp} T_{bif}^m$$

(Bassanelli-Berteloot)

Bassanelli-Berteloot, Buff-Epstein

Let M_d be the family of all rational maps (resp. polynomials) of degree d, modulo holomorphic conjugation and take $m \leqslant \dim M_d$. Then Bif_m is the closure of the set of parameters with m Misiurewicz relations (critical points preperiodic to repelling points).

$$\mathrm{Bif}_{\dim M_d} \subset \dots \subset \mathrm{Bif}_2 \subset \mathrm{Bif}_1$$



General question/goal

Can all this picture (bifurcation, dictionary, density of stability, higher bifurcations, stratification...) be generalized to higher dimensions?

Higher dimensions, families of endomorphisms of $\mathbb{P}^k(\mathbb{C})$

$$f_\lambda \colon \mathbb{P}^k(\mathbb{C}) \to \mathbb{P}^k(\mathbb{C}), \lambda \in \Lambda$$

where each f_λ is of the form

$$[z_0, \ldots, z_k] \mapsto [F_0, \ldots, F_k]$$

with the F_i's homogeneous polynomials in $z_0, \ldots z_k$ (with 0 as only common zero)

Fornaess-Sibony, Briend-Duval, Dinh-Sibony '90-'00

$$J := \mathrm{Supp}(\mu) \subsetneq \overline{\{\text{ periodic repelling points }\}}$$

Remark: the critical set is a complex hypersurface $C = \{\mathrm{Jac}\ Df = 0\}$

Lyapunov exponents

$$L(\lambda) = \int \log |\text{Jac } Df_\lambda| \mu_\lambda(f)$$

$$L = \sum_{i=1}^k L_i, \text{ where } Df_x^n \sim \begin{pmatrix} e^{nL_1} & & \\ & \ddots & \\ & & e^{nL_k} \end{pmatrix} \text{ for } \mu - a.e.\, x \in J$$

Bassanelli-Berteloot, Dinh-Sibony, Pham

The function $L(\lambda)$ is continuous and (pluri)subharmonic.

Families of endomorphisms of $\mathbb{P}^k(\mathbb{C})$

Berteloot-B.-Dupont, B.

There is a natural dichotomy Stability - Bifurcation valid in any dimension.

More precisely, the following notions are equivalent:

1. *asymptotically all* the repelling periodic points in J vary holomorphically
2. μ-almost all points in J vary holomorphically
3. $dd^c L \equiv 0$
4. there are no Misiurewicz parameters [recall: the critical set is a hypersurface!]
5. ...

Families of endomorphisms of $\mathbb{P}^k(\mathbb{C})$

Berteloot-B.-Dupont, B.

There is a natural dichotomy Stability - Bifurcation valid in any dimension.

More precisely, the following notions are equivalent:

1. *asymptotically all* the repelling periodic points in J vary holomorphically
2. μ-almost all points in J vary holomorphically
3. $\boxed{dd^c L \equiv 0}$
4. there are no Misiurewicz parameters [recall: the critical set is a hypersurface!]
5. ...

- Central role of the condition $dd^c L = 0$

Tools and techniques

Dimension 1

- Riemann uniformization
- distortions and Koebe
- Montel theorems

Several variables

- pluripotential theory
- complex geometry
- ergodic theory

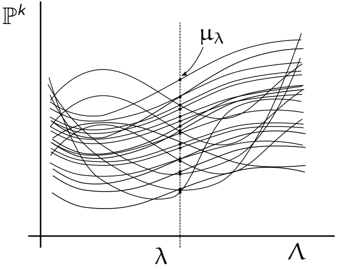
Tools and techniques

Dimension 1

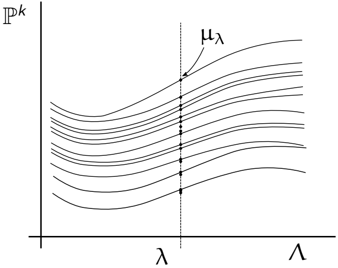
- Riemann uniformization
- distortions and Koebe
- Montel theorems

Several variables

- pluripotential theory
- complex geometry
- ergodic theory



$\Rightarrow$



Bifurcations: new phenomena

Dimension 1

Stab is dense

Bifurcations: new phenomena

Dimension 1

Stab is dense

B.-Taflin

Explicit example of family of endomorphisms of $\mathbb{P}^2$ with $\mathring{\mathrm{Bif}} \neq \emptyset$

Dujardin, Taflin, Biebler

$\mathring{\mathrm{Bif}} \neq \emptyset$ in the family $\mathcal{H}_d(\mathbb{P}^k)$

Bifurcations: new phenomena

Dimension 1

Stab is dense

B.-Taflin

Explicit example of family of endomorphisms of $\mathbb{P}^2$ with $\mathring{\mathrm{Bif}} \neq \emptyset$

Dujardin, Taflin, Biebler

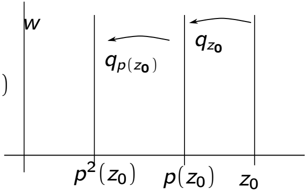
$\mathring{\mathrm{Bif}} \neq \emptyset$ in the family $\mathcal{H}_d(\mathbb{P}^k)$

Question

What does $(dd^c L)^m$ detect this time? Do we still have a stratification of Bif?

Polynomial skew products

$$f(z,w) = (p(z), q(z,w)) = (p(z), q_z(w))$$



- "1,5" variables: partial use of 1-dimensional techniques
- many new phenomena already, with respect to the one-dimensional setting (example: wandering domains, Astorg-Buff-Dujardin-Peters-Raissy)
- naturally defined subfamily $SK_d(p)$ of all the endomorphisms of a given degree

$$f_\lambda(z,w) = (p(z), q_\lambda(z,w))$$

Results

$$f_\lambda(z,w) = (p(z), q_\lambda(z,w))$$

Theorem (Astorg-B.)

$p \notin \{z^d$, Chebyshev, Julia totally disconnected $\}$

$$\text{Supp } T_{bif} \equiv \text{Supp } T_{bif}^2 \equiv \cdots \equiv \text{Supp } T_{bif}^{\dim \Lambda}$$

Corollary (using Dujardin-Taflin)

$\text{Supp} \mathring{T}_{bif}^{\dim \Lambda} \neq \emptyset$

Corollary (of the proof)

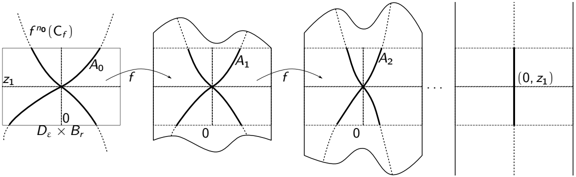
There exist (large) families (hypersurfaces) such that $\mathring{\text{Supp}}\, T_{bif} \neq \emptyset$

Ingredients of the proof

Let $\lambda_0 \in SK_d(p)$ be such that f_{λ_0} has a Misiurewicz relation $f_{\lambda_0}^{n_1}(z_1, c_1(\lambda_0))$ is repelling and l_1-periodic. We denote by $M_1 \subset SK_d(p)$ the analytic subset where $f_\lambda^{n_1}(z_1, c_1(\lambda))$ is repelling and l_1-periodic.

Proposition 1 - analytic criterion

Let $\lambda \in SK_d(p)$ be such that f_λ has m Misiurewicz relations, and $M_1, \ldots, M_m$ satisfy $\mathrm{codim}(M_1 \cap \cdots \cap M_m) = m$. Then $\lambda \in \mathrm{Bif}_m$.



Buff-Epstein, Gauthier, Berteloot-B.-Dupont, Astorg-Gauthier-Mihalache-Vigny

Ingredients of the proof

Proposition 1 - analytic criterion

Let $\lambda \in SK_d(p)$ be such that f_λ has m Misiurewicz relations, and $M_1, \ldots, M_m$ satisfy $\mathrm{codim}(M_1 \cap \cdots \cap M_m) = m$. Then $\lambda \in \mathrm{Bif}_m$.

Key proposition 2 - Geometric construction

Let M be a family of polynomial skew products with a *good* Misiurewicz relation. Then near any $\lambda \in M$ there exists a new (non-persistent) *good* Misiurewicz relation.

Good Misiurewicz relations

Notation: $f^n(z,w) = (p^n(z), Q_z^n(w)) = (p^n(z), q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z(w))$

Definition

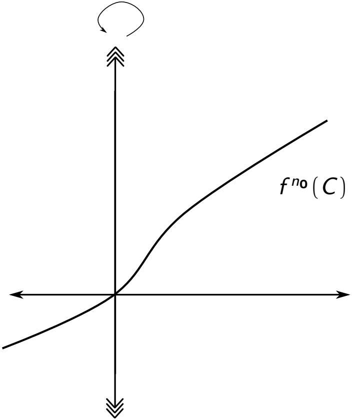
A repelling $\ell-$periodic point (z_0, w_0) is *vertical-like* if its vertical multiplier $\frac{\partial}{\partial w} Q_z^\ell(w_0)$ is larger than $(p^\ell)'(z_0)$.

Definition

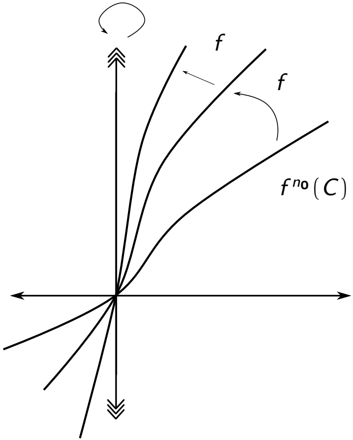
Let $M \subset SK_d(p)$ be a family of polynomial skew products. A Misiurewicz relation $f^{n_0}(z, c(\lambda)) = (z_1, w_1(\lambda))$ is *good* if for all $\lambda \in M$

1. (z_1, c_1) is a vertical-like repelling periodic point;
2. its vertical eigenvalue $B(\lambda)$ is non-constant in M;
3. $f^{n_0}(C(f_\lambda))$ is not tangent to any eigenspace of f_λ^ℓ at $(z_1, c_1(\lambda))$.

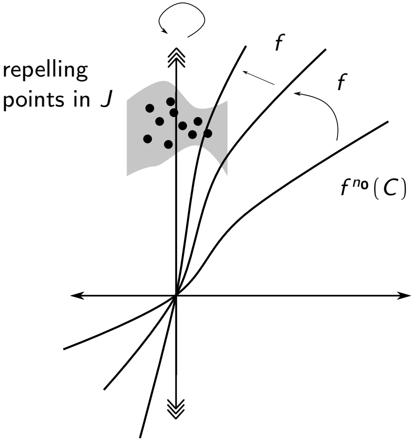
Creation of new Misiurewicz parameters



Creation of new Misiurewicz parameters



Creation of new Misiurewicz parameters

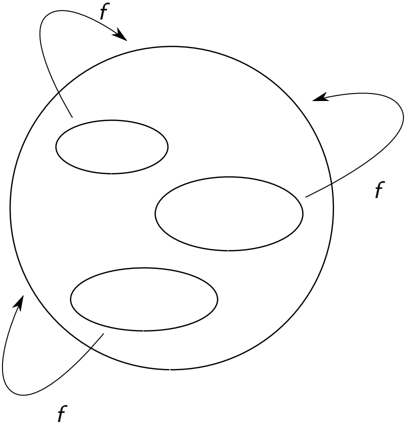


Vertical-like hyperbolic sets

Idea: construct a measure with large ($> \log d$) entropy and whose Lyapunov exponents satisfy $L_{ver} > L_{hor}$ (+ mixing + "controlled" Jacobian...)

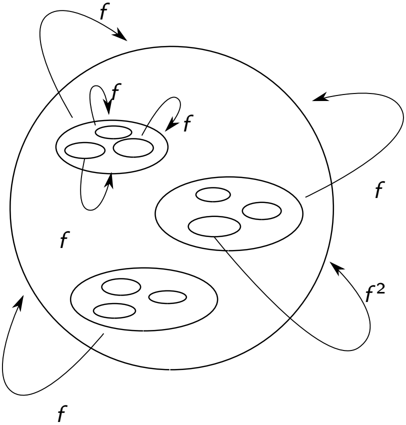
Vertical-like hyperbolic sets

Idea: construct a measure with large ($> \log d$) entropy and whose Lyapunov exponents satisfy $L_{ver} > L_{hor}$ (+ mixing + "controlled" Jacobian...)



Vertical-like hyperbolic sets

Idea: construct a measure with large ($> \log d$) entropy and whose Lyapunov exponents satisfy $L_{ver} > L_{hor}$ (+ mixing + "controlled" Jacobian...)



Vertical-like hyperbolic sets

Idea: construct a measure with large ($> \log d$) entropy and whose Lyapunov exponents satisfy $L_{ver} > L_{hor}$ (+ mixing + "controlled" Jacobian...)

Theorem (Przytycki-Zdunik)

If $p \notin \{z^d$, Chebyshev, Julia totally disconnected $\}$, there exists a (compact) hyperbolic set $H \subset J_p$ with $\dim_H H > 1$

- 1D thermodynamical formalism implies the existence of a measure $\tilde{\nu}$ on J_p with $L(\tilde{\nu}) = \frac{\text{entropy}(\tilde{\nu})}{\dim_H H} < \log d$.

$$\nu := \int_{z \in H} \mu_z \tilde{\nu}(z) \qquad\qquad \left(\mu = \int_{z \in J_p} \mu_z \mu(z)\right)$$

- the vertical exponent is $\geqslant \log d$; the horizontal exponent is $= L(\tilde{\nu}) < \log d$
- ν is mixing, and with controlled Jacobian

Thermodynamical formalism in higher dimension: Urbanski-Zdunik, B.-Dinh

Thank you for your attention!