Eremenko's Conjecture, Devaney's Hairs, and the Growth of Counterexamples

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Some basics

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a transcendental entire function (i.e. not a polynomial).

- $F(f)$, the Fatou set of f, is the subset of $\mathbb{C}$ where f exhibits stable behaviour.
- $J(f) = \mathbb{C} \setminus F(f)$, the Julia set of f, is the subset of $\mathbb{C}$ where f exhibits chaotic behaviour.
- $I(f) = \{z\colon |f^n(z)| \to \infty\}$, the escaping set of f

Singular values

Where do we have issues inverting the transcendental entire function f?

Critical values, $CV(f)$

Asymptotic values. α is an asymptotic value of f if there exists a curve $\gamma\colon\ [0,\infty] \to \mathbb{C}$ such that $\lim_{t\to\infty} |\gamma(t)| = \infty$ and $\alpha = \lim_{t\to\infty} f(\gamma(t))$. The set of these values is denoted by $AV(f)$.

Example: Let $f(z) = exp(z)$ and $\gamma(t) = -t$, corresponding to $\alpha = 0$.

We then define the set of singular values to be

$$S(f) = \overline{CV(f) \cup AV(f)}.$$

For any neighbourhood U of $\alpha \in S(f)$, there exists a component V of $f^{-1}(U)$ such that $f\colon V \to U$ is not bijective.

Classes

We can classify transcendental entire functions according to the nature of $S(f)$.
Two in particular include:

The Speiser class $\mathcal{S}$, and

The Eremenko–Lyubich class $\mathcal{B}$.

Class $\mathcal{S}$

A transcendental entire function f is in Class $\mathcal{S}$ if $S(f)$ is finite.
Example: $S(\cos) = \{-1, 1\}$

Class $\mathcal{B}$

A transcendental entire function f is in Class $\mathcal{B}$ if $S(f)$ is bounded.
Example: $\frac{\sin z}{z}$ has infinitely many singular values yet they are all bounded by 1. This example shows that $\mathcal{S} \neq \mathcal{B}$.

So what?

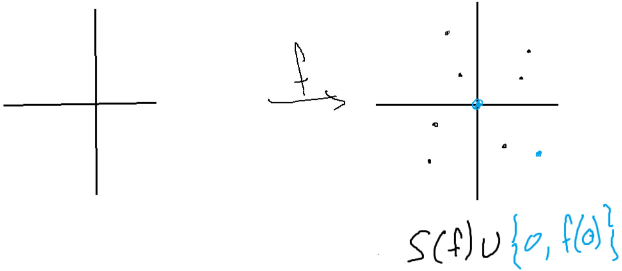
We now have, with a function in class $\mathcal{B}$, the ability to work outside of our region of difficulty and can consider pre-images.

Pre-images and Tracts

Finally, pictures.

Pre-images and Tracts

If we let D be a ball that contains $= S(f) \cup \{0, f(0)\}$.



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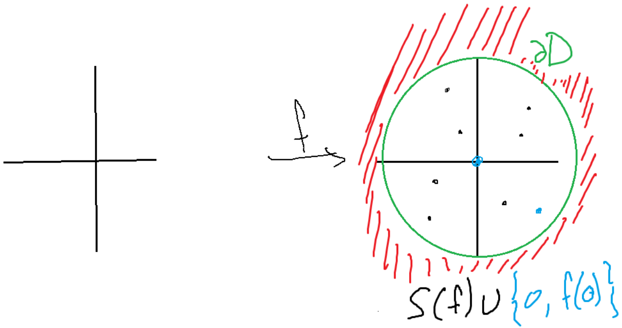
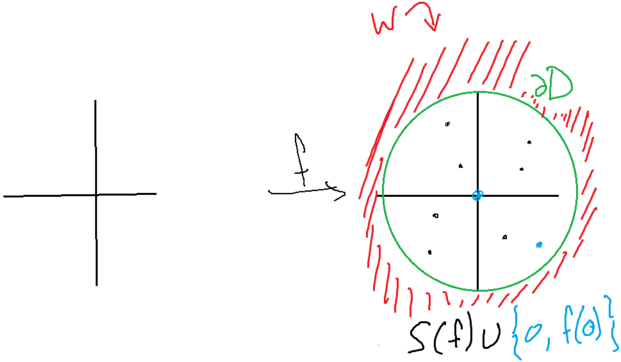
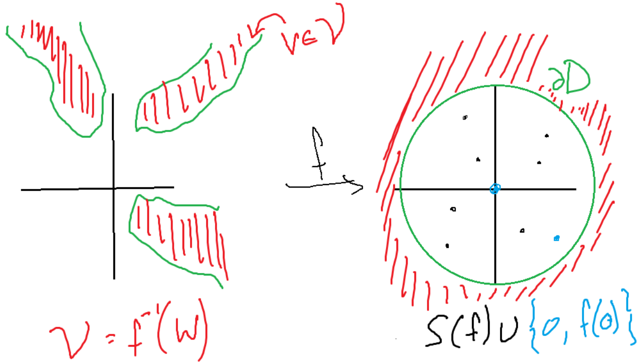


If we let D be a ball that contains $= S(f) \cup \{0, f(0)\}$. Consider $W = \mathbb{C} \setminus \overline{D}$.



If we let D be a ball that contains $= S(f) \cup \{0, f(0)\}$. Consider $W = \mathbb{C} \setminus \overline{D}$. Now also consider $f^{-1}(W) = \mathcal{V}$ and let V be a component of $\mathcal{V}$



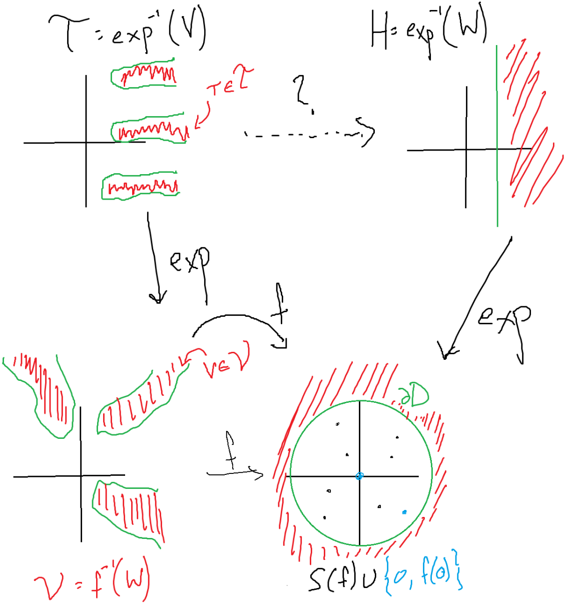
Pre-images and Tracts

Why include $\{0, f(0)\}$?
So we can take logarithms.

$exp^{-1}(W) = H$

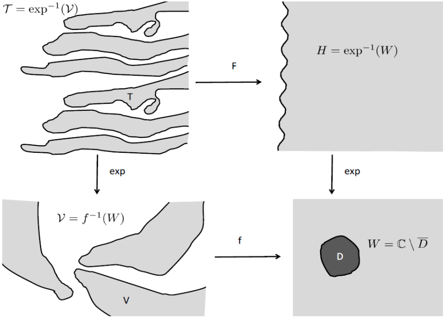
Choose a component V from $\mathcal{V}$ and then we get $\mathcal{T} = exp^{-1}(V)$. From this, choose a component $T \in \mathcal{T}$. These components, T, are called *tracts* and these are simply connected Jordan domains.

Pre-images and Tracts



Can we make this diagram commute with a function from T to H?
Yes.

Pre-images and Tracts



From “Dynamics in the Eremenko–Lyubich Class”, Sixsmith

Pre-images and Tracts

In the previous, $F\colon T \to H$ is called the *logarithmic transform* of f and is analytic.
We usually work with tracts $\mathcal{T}$ and (right) half-planes H such that $\overline{\mathcal{T}} \subset H$ and we say that the corresponding F is of *disjoint type*.

Functions from tracts

In §7 of the paper by Rottenfusser, Rückert, Rempe, and Schleicher (2011 - RRRS) it is shown, via approximation theory and Cauchy integrals, that in constructing the tracts in this paper, there is indeed a corresponding entire Class $\mathcal{B}$ function.
In constructing a tract T, we can achieve a Riemann map F from T to a right half-plane H which will have a continuous extension to the boundary, by Carathéodory's Theorem.

In 1926, Fatou noticed that the Julia set of $r\sin(z)$, for $r \in \mathbb{R}$ contains curves of points that escape to infinity under iteration.
In 1989, Eremenko, by directly studying $I(f)$ and showing that every component of $\overline{I(f)}$ is unbounded for an arbitrary transcendental entire function, conjectured that $I(f)$ is comprised of unbounded components.
This is referred to as the weak form of Eremenko's conjecture.

There is also a strong form of Eremenko's conjecture which states: "It is plausible that the set $I(f)$ always has the following property: every point $z \in I(f)$ can be joined with ∞ by a curve in $I(f)$."
These curves are often called 'Devaney Hairs'.

Eremenko's Conjecture

The strong form of Eremenko's conjecture was shown to not hold in general, even for some functions within Class $\mathcal{B}$ in the RRRS paper. This was shown by a counterexample.

Finite Order

It's not all bad news.
In the same paper, it was shown that Eremenko's conjecture actually does hold for Class $\mathcal{B}$ functions f of finite order, that is,

$$\log\log|f(z)| = \mathcal{O}(\log|z|) \text{ as } |z| \to \infty.$$

($f = \mathcal{O}(g)$ if there is an $M > 0$ and $R > 0$ such that $|f(z)| \leq M|g(z)|$ whenever $|z| > R$)

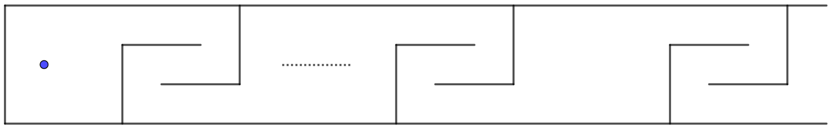
RRRS Counterexample

The counterexample was found by constructing a tract with 'wiggles' that, along with considering the semi-circular geodesics of the right-half plane (with the hyperbolic metric) and their pre-images within the tract. How do we get our contradicicition?

RRRS Counterexample

There is a sequence of geodesics constructed in such a way that there cannot be an escaping ray (details in §6 of RRRS).

Sketch of RRRS Counterexample



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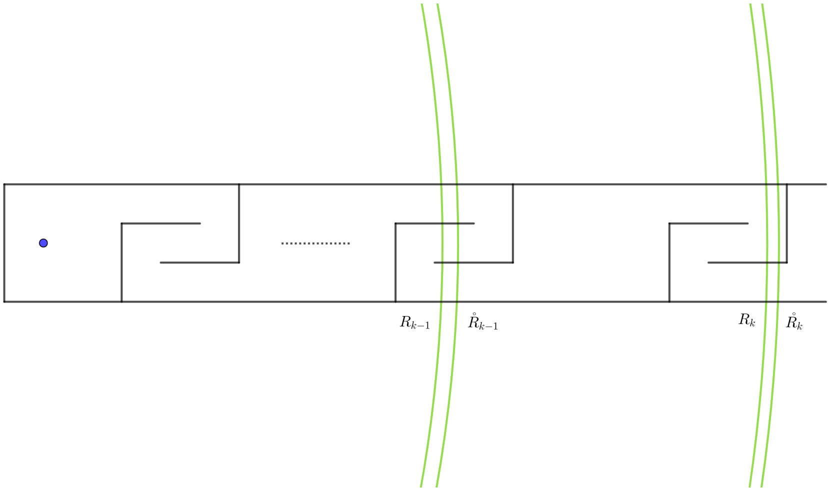

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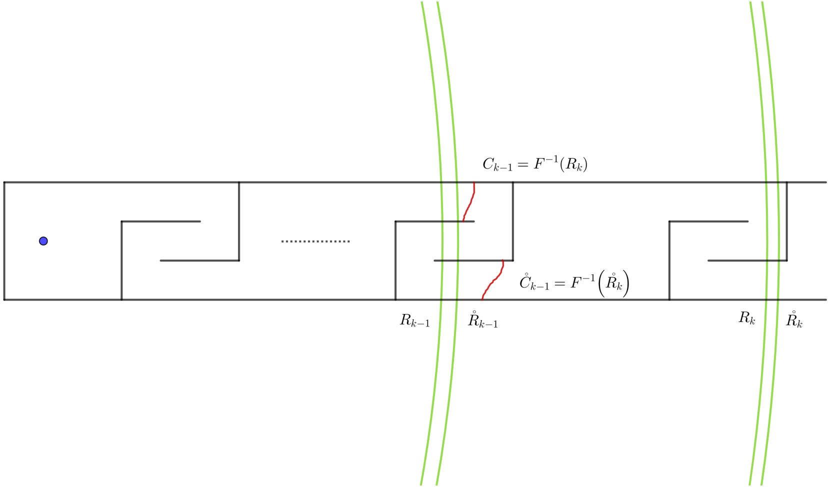

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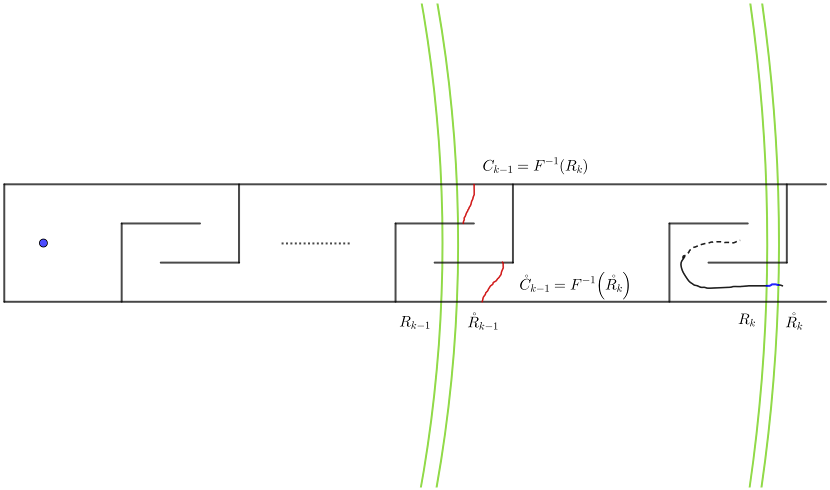

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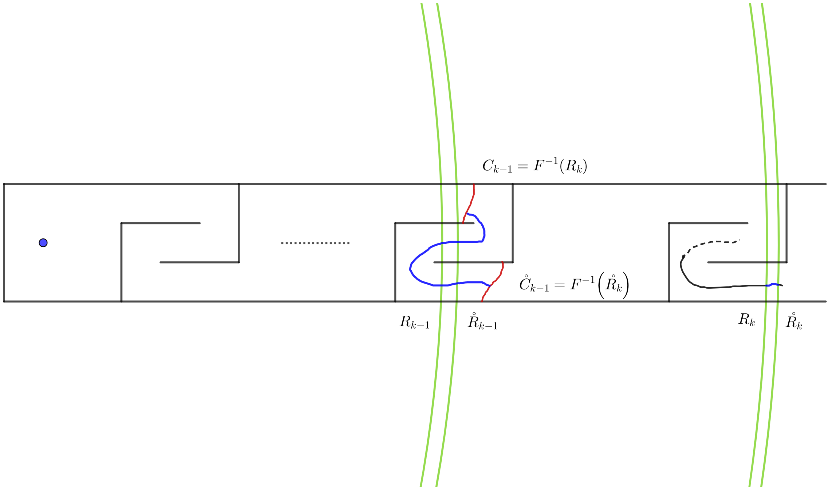


Sketch of RRRS Counterexample

This process can be iterated in such a way that the curve must contain 2^k disjoint subcurves connecting C_0 to $\mathring{C}_0$. Since this holds for a general $k > 0$, we have a contradiction.

The sketched counterexample satisfies:

$$\log Re\left(F(z)\right) = \mathcal{O}\left(\left(Re\left(z\right)\right)^{12L}\right)$$

where L is an arbitrary constant greater than 1.
How do we bring this exponent down?

We ‘meddle’ with the tract. The way that this is done in RRRS is by making the top channel of the wiggles thin. This forces any escaping curve to pass nearer and nearer to the boundary of the tract. From hyperbolic geometry, we can use the *standard estimate* on the hyperbolic metric in the tract T.

$$\frac{1}{2dist\left(z,\partial T\right)} \leq \rho_T\left(z\right) \leq \frac{2}{dist\left(z,\partial T\right)}$$

Where ρ_T is the density of the hyperbolic metric on T.

Growth

Theorem

Counterexamples of mild growth, RRRS Prop 8.2 For every $\varepsilon > 0$, there is a function $f \in \mathcal{B}$ such that $J(f)$ has no unbounded path-connected components, and such that

$$\log \log |f(z)| = \mathcal{O}\left((\log |z|)^{1+\varepsilon}\right).$$

Note: Found by working with logarithmic transform F, implying existence of f.

But wait, there's more

Couldn't we let the thin parts get thinner and thinner further and further down the line/tract? Of course we can.

Theorem

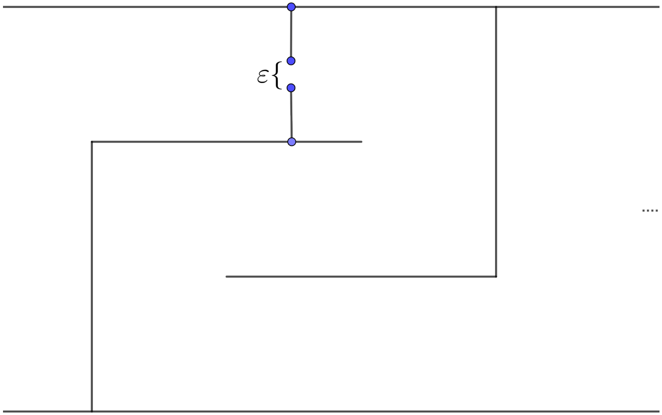
RRRS Prop 8.3 There exists a logarithmic transform function F such that $\log Re\,(F(z)) = (Re\,(z))^{1+o(1)}$ *and* $I(F)$ *contains no unbounded path-connected components.*

($f(z) = o(g(z))$ if (assuming $g \neq 0$ after a certain point) $\lim \frac{f}{g} = 0$ as $|z| \to \infty$)

Where do we go from here?

The current project aim is to get an explicit bound on the order of growth of the function and see how “close” to finite order we can get.

Further Meddling



etc.

Thank you for your time[1].

[1] No refunds.