Analytic symmetries of parabolic and elliptic elements

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On geometric complexity of Julia sets - III, Bedlewo, Sep 26 - Oct 1, 2021

joint work with Artur Avila and Alexander Eliad

Consider an orientation preserving circle diffeomorphism

$$f : \mathbb{T} \to \mathbb{T}, \qquad \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

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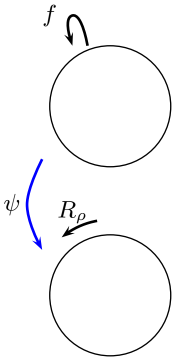
Poincare: the rotation number of f,

$$\rho(f),$$

is the average rate of rotation around $\mathbb{T}$ by an iterate of f.

Theorem (Denjoy 1932)

If $f \in \mathrm{Dif}^2_+(\mathbb{T})$ *with* $\rho = \rho(f) \in \mathbb{R} \setminus \mathbb{Q}$*,* f *is topologically conjugate to the rotation* R_ρ*.*



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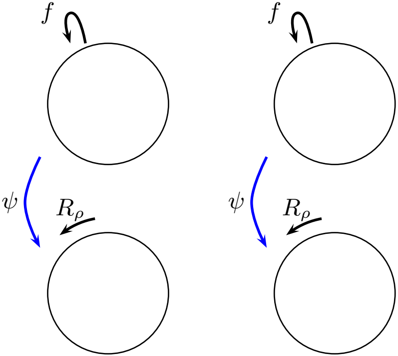

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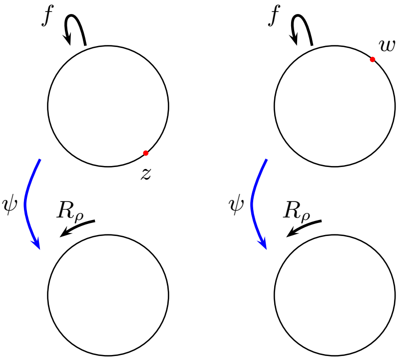

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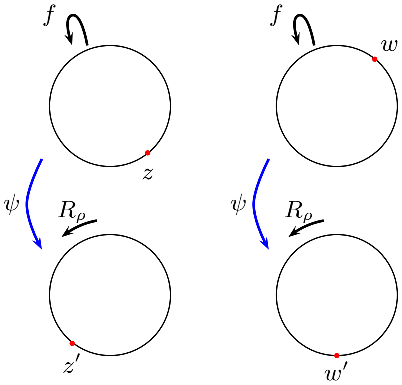

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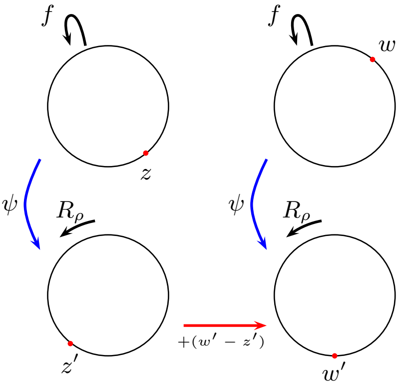

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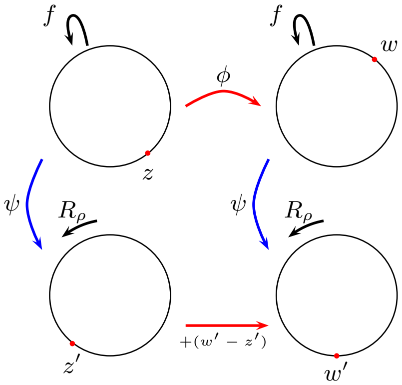

At topological level, there is total symmetry in the dynamics of f on $\mathbb{T}$.

Theorem (Herman 1979)

Every $f \in \mathrm{Dif}_{+}^{\infty}(\mathbb{T})$ *with* $\rho(f)$ *of Diophantine type is* C^{∞} *conjugate to the rigid rotation* $R_{\rho(f)}$.

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There is a set of full measure $\mathcal{H} \subset \mathbb{R} \setminus \mathbb{Q}$ *such that every* $f \in \mathrm{Dif}_+^\omega(\mathbb{T})$ *with* $\rho(f) \in \mathcal{H}$ *is* C^ω *conjugate to the rigid rotation* $R_{\rho(f)}$.

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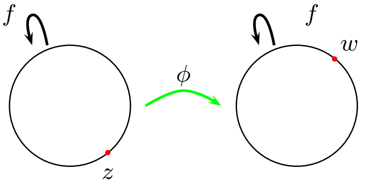

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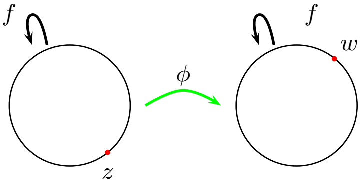

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Are there symmetries when f is not linearisable?

A measure of linearisability/nonlinearisability: the centraliser

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- Evidently, $f^{\circ n} \in \mathrm{Cent}(f)$.
- if $f \in \mathrm{Dif}^s(\mathbb{T})$ is C^s-linearisable, $\mathcal{G}^s(f) = \mathbb{R}/\mathbb{Z}$,

Lemma: if $f \circ g = g \circ f$, and g is linearisable, then f is also linearisable.

This implies that

- If $f \in C^\omega$ is not linearisable, $\mathcal{G}^\omega(f) \subset (\mathbb{R} \setminus \mathcal{H})/\mathbb{Z}$.

and

- if $f \in C^\infty$ is not linearisable, $\mathcal{G}^\infty(f) \subset (\mathbb{R} \setminus D.C.)/\mathbb{Z}$

Q: How large can $\mathcal{G}(f)$ be for a non-linearisable map f?

Q: Are there maps with trivial centralisers?

In C^∞ category:

- There exists $f \in \mathrm{Dif}^{\infty}_{+}(\mathbb{T})$ such that $\mathcal{G}^{\infty}(f)$ is uncountable (Herman 79).
- There exists $f \in \mathrm{Dif}^{\infty}_{+}(\mathbb{T})$ such that $\mathrm{Cent}^{\infty}(f)$ is trivial (Yoccoz 84).

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Theorem (Avila-C-Eliad 2020)

There exists $f \in \mathrm{Diff}^\omega_+(\mathbb{T})$ such that $\mathcal{G}^\omega(f)$ is trivial.

Our example belongs to the Arnold family of maps

$$f_{a,b} = x + a + b\sin(2\pi x) : \mathbb{R} \to \mathbb{R}.$$

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This is obtained from the following theorem, and a successive perturbation argument.

Theorem (A-C-E, 2020)

If $f_{a,b}$ has a parabolic cycle on $\mathbb{T}$ for some $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$, $Cent(f_{a,b})$ is trivial.

For

$$f(z) = e^{2\pi i\alpha} z + O(z^2),$$

$\mathrm{Cent}^\omega(f) = \{g \mid g : (\mathbb{C}, 0) \to \mathbb{C}, 0)$ is a non-constant germ with $f \circ g = g \circ f\}$.

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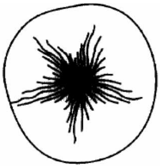

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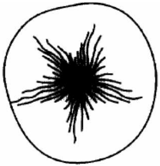

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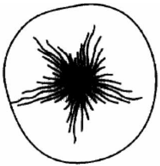

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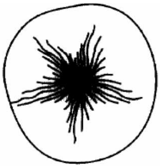

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Alexander Eliad is studying the centraliser problem for more general classes of polynomials and rational functions.

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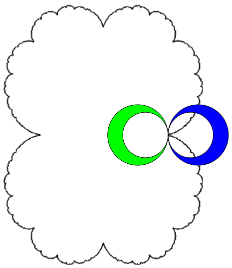

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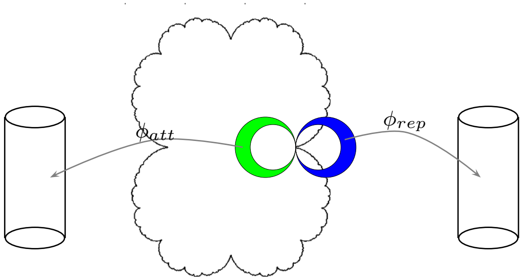

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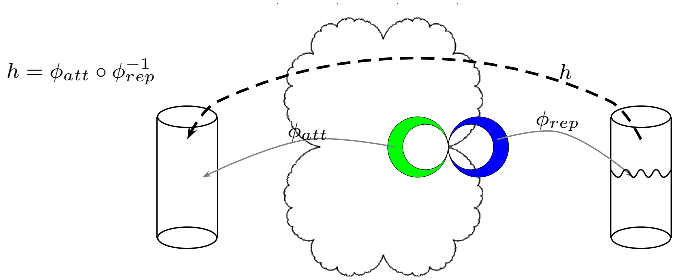

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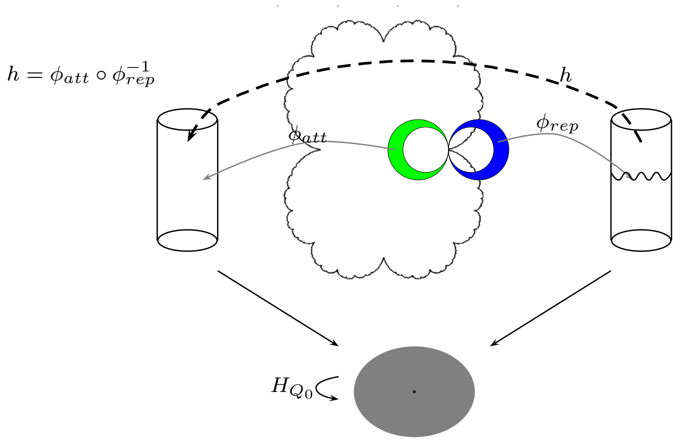

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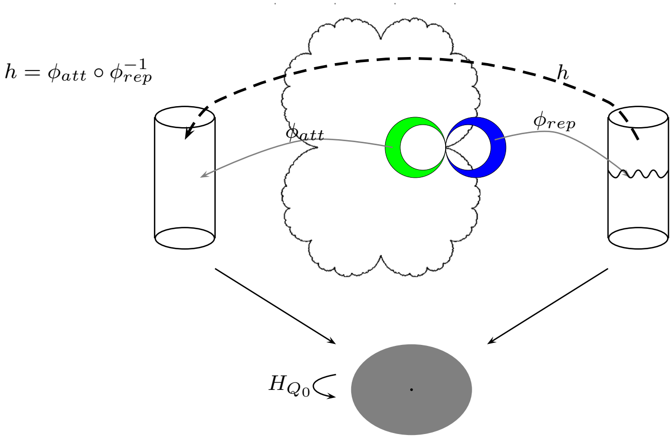

Prop. H_{Q_0} has infinitely many critical points, all mapped to the same value.

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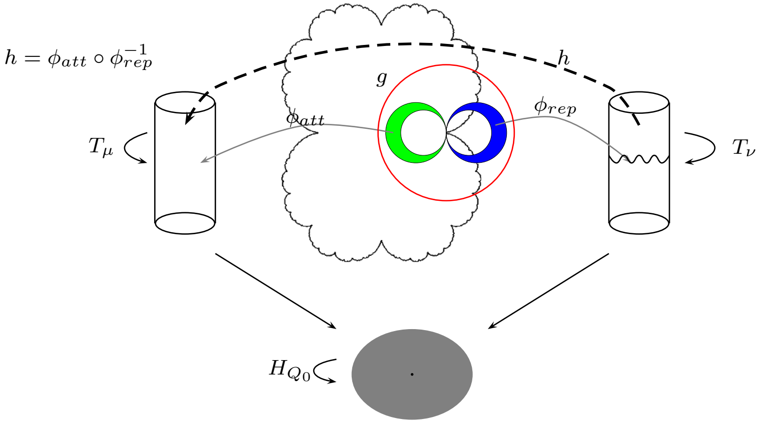

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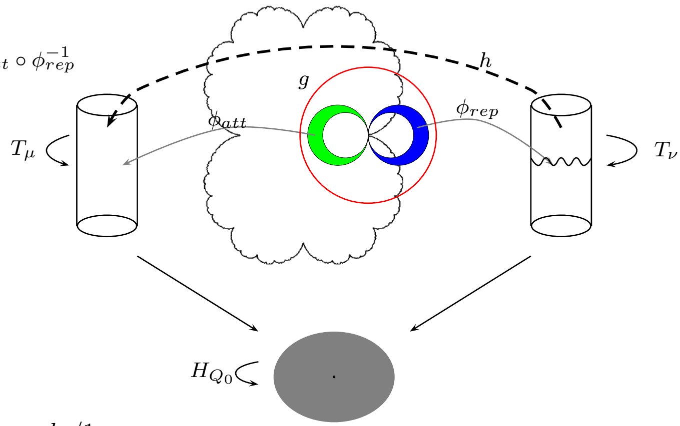

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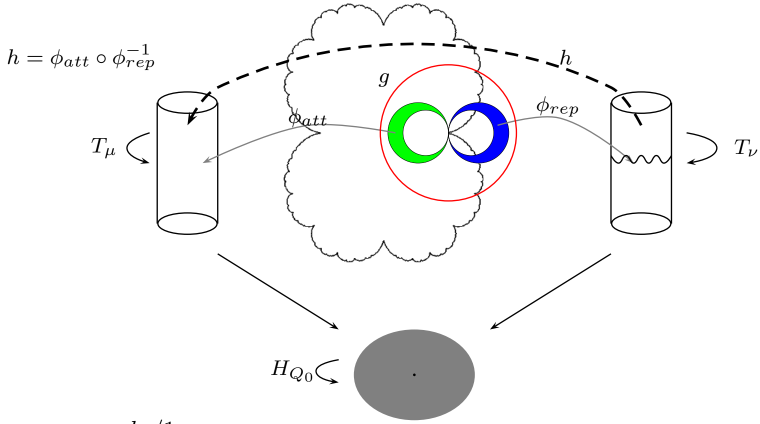

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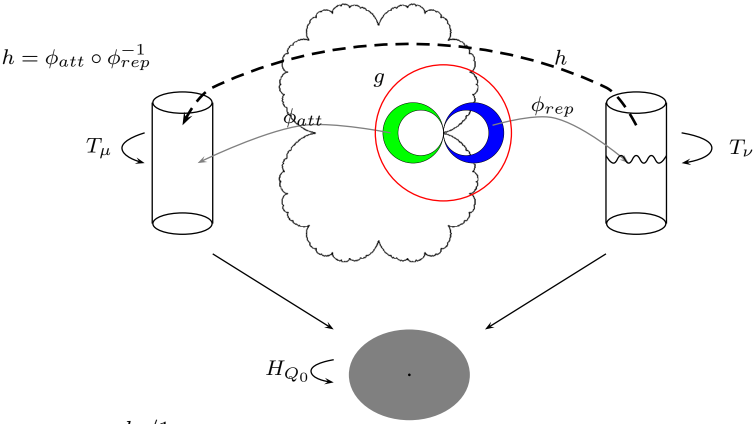

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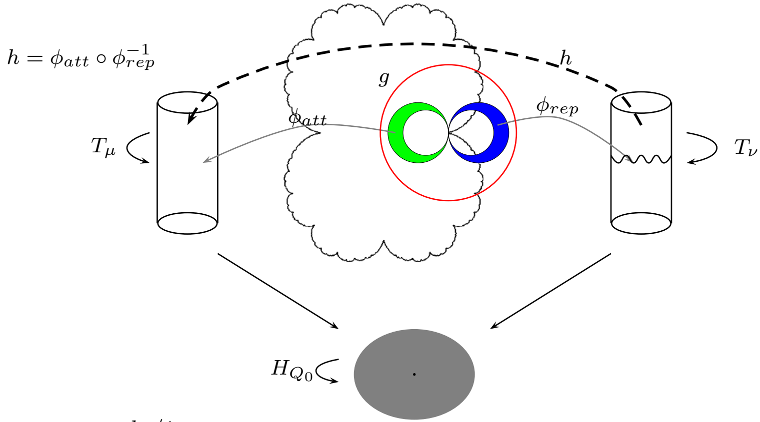

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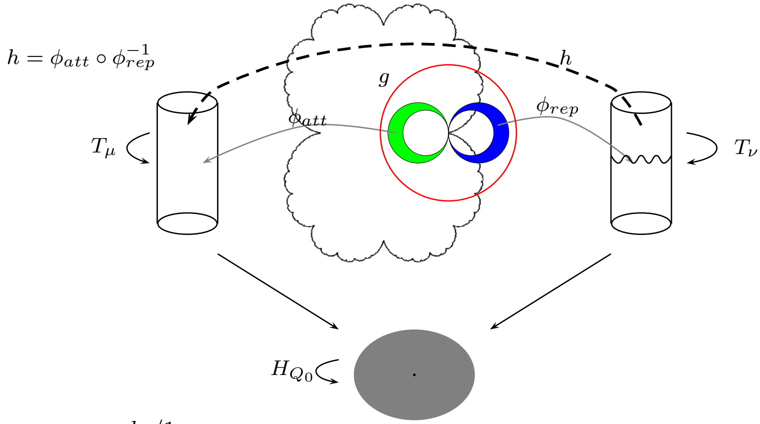

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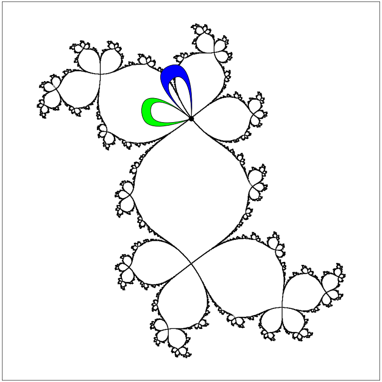

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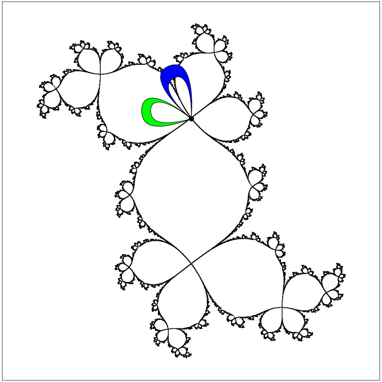

Similarly, $\mu = \nu = b_{q+1}/a_{q+1}$, ..., $H_{Q_{p/q}}$ commutes with $w \mapsto e^{2\pi i \mu} w$.

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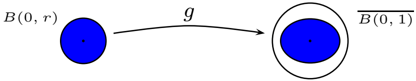

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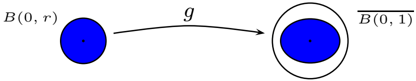

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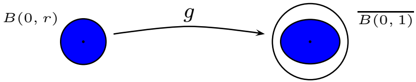

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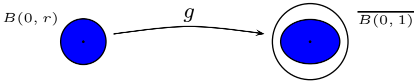

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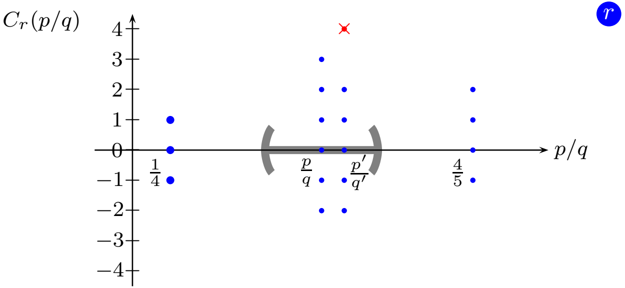
P1: for all $p/q \in \mathbb{Q}$, $C_r(p/q)$ is a finite set.

P2: for all $p/q \in \mathbb{Q}$, there is $\delta_r(p/q) > 0$ such that for all $p'/q' \in \mathbb{Q}$ satisfying $|p/q - p'/q'| \leq \delta_r(p/q)$,

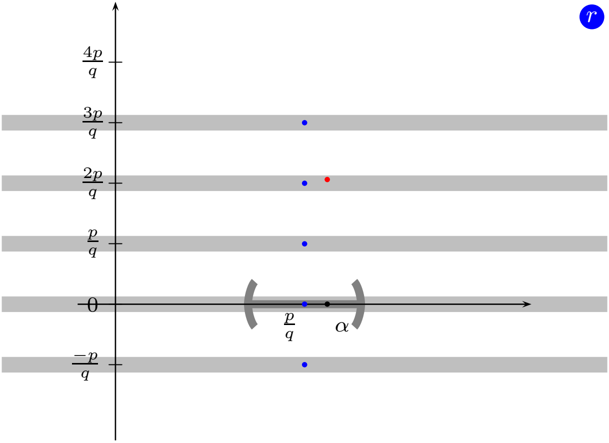
$$C_r(p'/q') \subseteq C_r(p/q).$$



P3: for every $p/q \in \mathbb{Q}$ and every $\epsilon > 0$, there is $\kappa_r(p/q, \epsilon) > 0$ satisfying the following:

For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $|\alpha - p/q| \leq \kappa_r(p/q, \epsilon)$, if $g(z) = e^{2\pi i \beta} z + O(z^2)$ commutes with Q_α and is r-good,

$$|\beta - kp/q| \leq \epsilon, \qquad \text{for some } k \in C_r(p/q)$$



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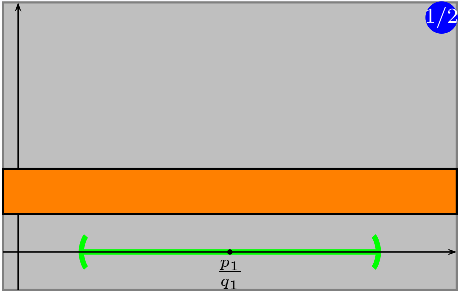

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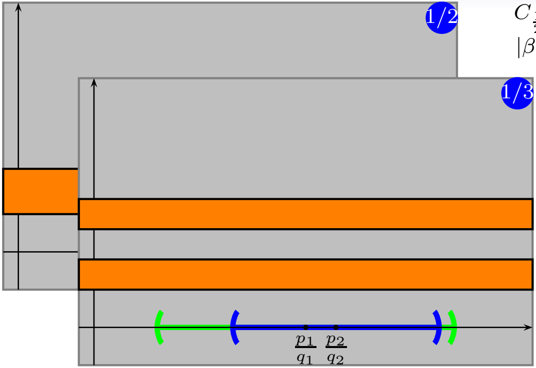

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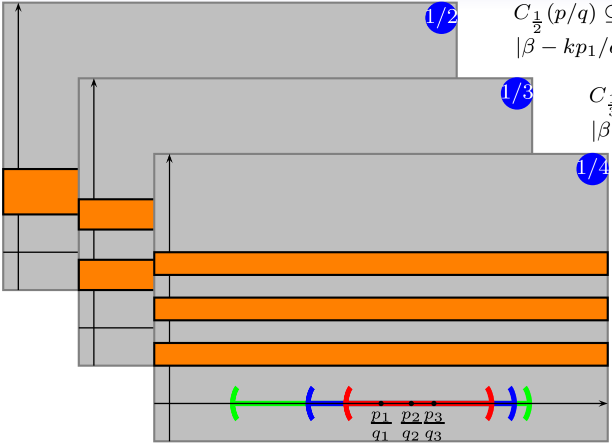

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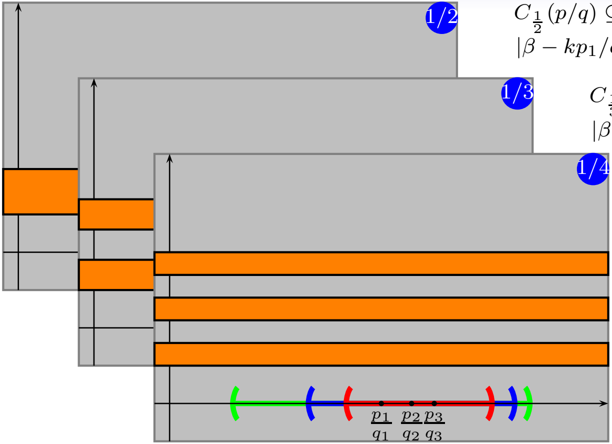

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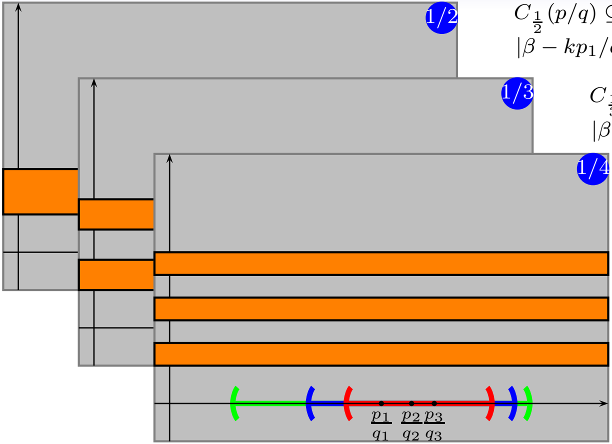

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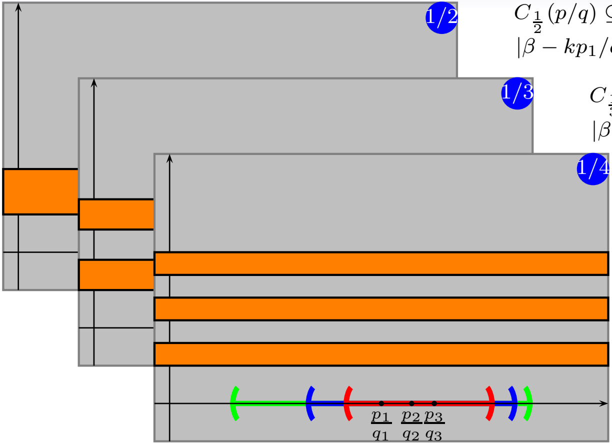

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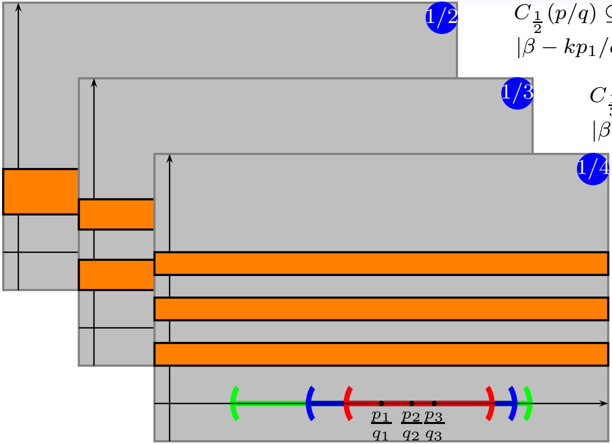

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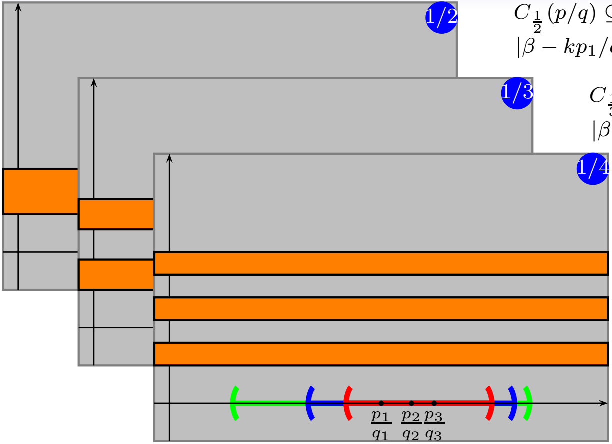

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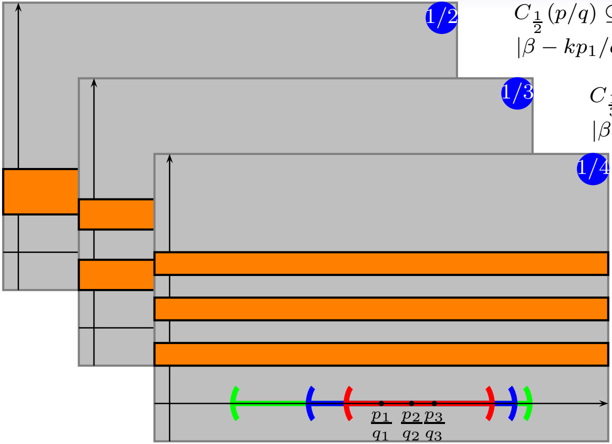

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The rest is the same, modulo, Arnold tongues.