Hausdorff dimension of escaping sets for Speiser functions with few singular values

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Outline

Singular values

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic.

$v \in \widehat{\mathbb{C}}$ is a singular value of f, if it is either critical or asymptotic:

- v is called a critical value of f, if $\exists\, z_0$ with $f(z_0) = v$ and $f^{\#}(z_0) = 0$.
- v is called an asymptotic value of f, if there exists a curve $\gamma \to \infty$ and $f(\gamma) \to v$.

Examples:

$\leadsto$ $0, \infty$ are asymptotic values of e^z.

$\leadsto$ $\pm i$ are asymptotic values of $\tan z$.

$\leadsto$ ± 1 are critical values of $\sin(z)$, while ∞ is asymptotic.

The Speiser class

Definition

$f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a Speiser function, if it has finitely many singular values.

- Examples: e^z, $\sin(z)$, $\tan(z)$, $\wp(z) \in \mathcal{S}$.
- A larger setting is the *Eremenko-Lyubich class $\mathcal{B}$*, consisting of meromorphic functions with a bounded set of finite singular values (i.e., Eremenko-Lyubich functions).

Order of growth

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic. The order of f is

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function.

Iversen: f has infinitely many poles if ∞ is not an asymptotic value.

Teichmüller: Let $f \in \mathcal{B}$. If ∞ is not an asymptotic value and the multiplicities of poles are bounded. Then

$$\rho(f) = \limsup_{r\to\infty} \frac{\log n(r, f)}{\log r},$$

where $n(r, f)$ counts the number of poles in the disk $\{|z| \leq r\}$.

Notation

$\mathcal{S}_q$ denotes Speiser functions with exactly q singular values.

E.g., $\exp \in \mathcal{S}_2$, $\sin \in \mathcal{S}_3$, $\wp \in \mathcal{S}_4$.

Definition (Equivalence)

Let $f, g \in \mathcal{S}$. We say that f is equivalent to g, if there exist two homeomorphisms $\phi, \psi : \mathbb{C} \to \mathbb{C}$ such that $\phi \circ f = g \circ \psi$.

$\mathcal{S}_2$

Fact

Let $f \in \mathcal{S}_2$. Then $f = M \circ \exp \circ A$, where M is Möbius and A is linear.

This implies:

- The order of $f \in \mathcal{S}_2$ is one.
- f is periodic.
- $f \in \mathcal{S}_2$ has two omitted values.
- There are only two equivalence classes: $[e^z]$ and $[\tan z]$.

$\mathcal{S}_q, \;\; q \geq 3$

Following $\mathcal{S}_2$, a natural question is

Question

How many equivalence classes for $\mathcal{S}_q$ if $q \geq 3$?

Uncountably many equivalence classes exist for each $q \geq 3$.

Entire case:

- Bishop (2015): quasiconformal folding.
- C. (2018): planar infinite trees with some topological condition.

Meromorphic case:

- Elfving (1934): meromorphic functions in $\mathcal{S}_3$ with rational Schwarzian derivatives.
- Speiser graphs

The escaping set

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic and transcendental.

Definition

The escaping set $\mathcal{I}(f)$ consists of $z \in \mathbb{C}$ such that

$$f^n(z) \to \infty \quad \text{as} \quad n \to \infty.$$

- Eremenko 1989, Dominguez 1998: $\mathcal{I}(f) \neq \emptyset$
- Eremenko-Lyubich 1992, Rippon-Stallard 1999: $f \in \mathcal{B} \Rightarrow \mathcal{I}(f) \subset \mathcal{J}(f)$

Escaping dimensions: the role of ∞

Theorem (Bergweiler-Karpińska-Stallard 2009)

Let $f \in \mathcal{B}$ be meromorphic and of finite order. If ∞ is an asymptotic value, then $\operatorname{HD} \mathcal{I}(f) = 2$.

Remark: If f is entire, the above result is due to Barański and Schubert.

Theorem (Bergweiler-Kotus 2012)

Let $f \in \mathcal{B}$ be meromorphic and of finite order. If ∞ is not an asymptotic value and the multiplicities of poles are bounded, then $\operatorname{HD} \mathcal{I}(f) < 2$.

Escaping dimensions: $\mathcal{B}$ vs. $\mathcal{S}$

A complete characterization of escaping dimensions in class $\mathcal{B}$ is given

Theorem (Bergweiler-Kotus 2012)

$$\{\mathrm{HD}\,\mathcal{I}(f) : f \in \mathcal{B}\} = [0, 2]$$

In class $\mathcal{S}$,

Theorem (Aspenberg-C. 2020)

$$\{\mathrm{HD}\,\mathcal{I}(f) : f \in \mathcal{S}\} = [0, 2]$$

Idea: glue two Weierstraß $\wp$-functions using quasiconformal mappings

Escaping dimensions in $\mathcal{S}_q$

The equivalence of functions in $\mathcal{S}_2$ implies

Theorem (McMullen 1987, Galazka-Kotus 2018)

$$\{\mathrm{HD}\,\mathcal{I}(f) : f \in \mathcal{S}_2\} = \{2,\ 1/2\}.$$

A closer look at previous result (Aspenberg-C. 2020) tells

Theorem

$$\{\mathrm{HD}\,\mathcal{I}(f) : f \in \mathcal{S}_7\} = [0, 2].$$

Possibly one can do this with $\mathcal{S}_q$ for some $q > 7$.

The main result

Theorem (Aspenberg-C. 2020)

$$\{\text{HD}\,\mathcal{I}(f) : f \in \mathcal{S}_4\} = [0, 2].$$

More precisely,

Theorem (Aspenberg-C. 2020)

Given $\rho \in (0, \infty)$, there exists $f \in \mathcal{S}_4$ for which ∞ is not an asymptotic value such that

$$\text{HD}\,\mathcal{I}(f) = \frac{2\rho}{1+\rho}.$$

By varying ρ, each number in $(0, 2)$ can be attained.

Idea of proof

- Glue the two ”banks” of a carefully chosen Weierstraß elliptic function using quasiconformal mappings $\rightarrow$ Quasi-meromorphic function G
- Measurable Riemann mapping theorem $\rightarrow$ meromorphic f, qc Ψ with $G = f \circ \Psi$
- Teichmüller-Wittich-Belinskii theorem $\rightarrow$ $\Psi \sim \mathrm{Id}$
- Bergweiler-Kotus $\oplus$ McMullen $\rightarrow$ dimension

Weierstraß $\wp$-functions

For a non-real τ, put

$$\Lambda := \{m + n\tau : \ m, n \in \mathbb{Z}\}.$$

The Weierstraß elliptic function with respect to Λ is defined as

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Some properties:

- $\wp$ is doubly periodic with two primitive periods 1 and τ.
- $\wp \in \mathcal{S}_4$ with four critical values at

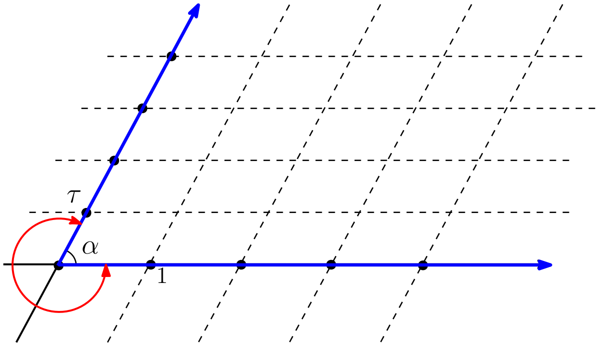
  $$\wp\left(\frac{1}{2}\right), \wp\left(\frac{1+\tau}{2}\right), \wp\left(\frac{\tau}{2}\right), \infty,$$

  and no asymptotic values.
- $\rho(\wp) = 2$.

Given $\rho \in (0, \infty)$, choose $\alpha \in (0, 2\pi]$ and $\tau \in \mathbb{C} \setminus \mathbb{R}$ such that

$$\rho = \frac{\alpha^2 + (\log|\tau|)^2}{\alpha\pi}.$$

Consider the Weierstraß $\wp$-function with two periods 1 and τ.



With

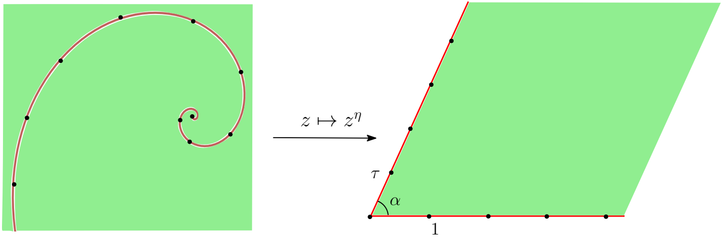
$$\eta := \frac{1}{2\pi}(\alpha - i \log |\tau|),$$

consider a spiral function

$$h : \mathbb{C} \setminus \{\text{a log. spiral } \Gamma\} \to \{z : 0 < \arg(z) < \alpha\},$$

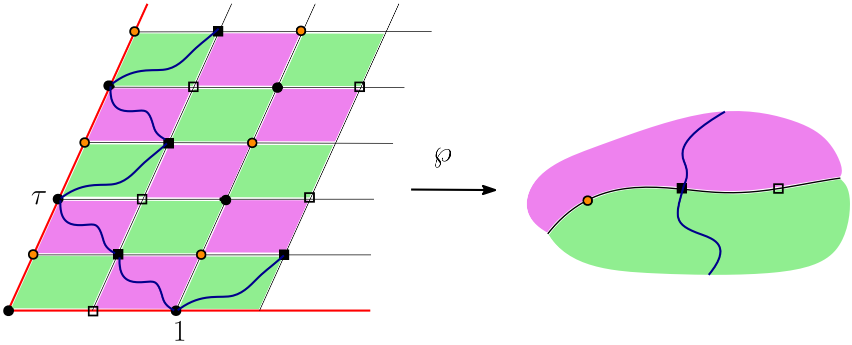
$$z \mapsto z^{\eta}$$

which can be visualized as follows.

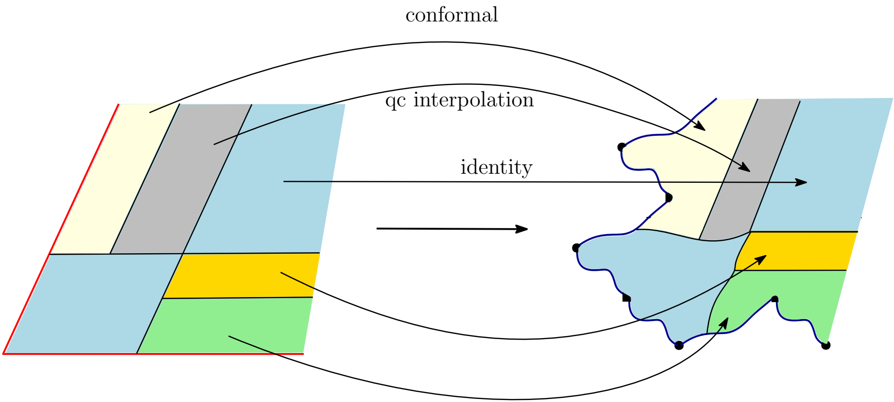


$\Rightarrow \wp(z^\eta)$ is meromorphic in $\mathbb{C}$ except for the spiral Γ

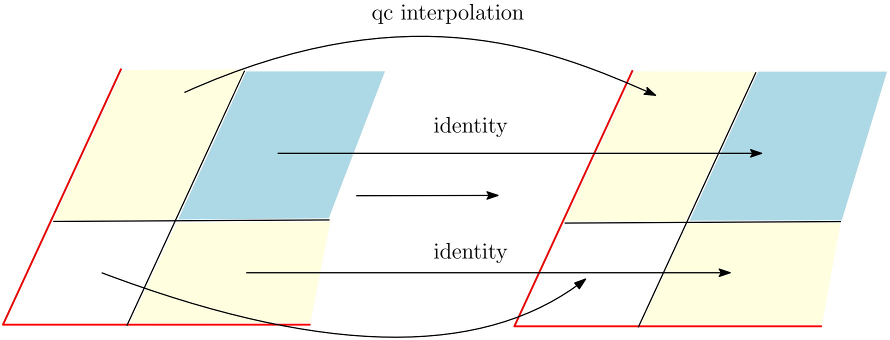
$\Rightarrow$ Discontinuity arises when extending across Γ



Straightening: ϕ_1



Correction: ϕ_2



So we have obtained

$$G := \wp \circ \phi_1 \circ \phi_2 \circ h$$

which is quasi-meromorphic in $\mathbb{C}$.

Measurable Riemann mapping theorem: $\exists$ a qc map Ψ and a meromorphic function f such that

$$G = f \circ \Psi.$$

Teichmüller-Wittich-Belinskii theorem

Theorem

Let $\Psi : \mathbb{C} \to \mathbb{C}$ *be a qc map and* K_Ψ *be its dilatation. Assume that*

$$\iint_{|z|>1} \frac{K_\Psi(z) - 1}{x^2 + y^2} dxdy < \infty.$$

Then

$$\Psi(z) \sim z \quad as\ z \to \infty.$$

$\Rightarrow$ Distribution of poles of f deduced from that of G.
$\Rightarrow$ Order of f is ρ by estimating the counting function of poles and the mentioned result of Teichmüller.

Estimate of dimension

For the obtained function f it has the following properties:

- $f \in \mathcal{S}_4$ with no asymptotic values
- multiplicities of poles $= 2$
- $f(z) \sim \left(\frac{b}{z-a}\right)^2$ as $z \to a$, where a is a pole and

$$|b| \sim |a|^{1-\rho/2}.$$

Combine results of McMullen, Bergweiler-Kotus to get

$$\dim \mathcal{I}(f) = \frac{2\rho}{1+\rho}.$$

Thank you