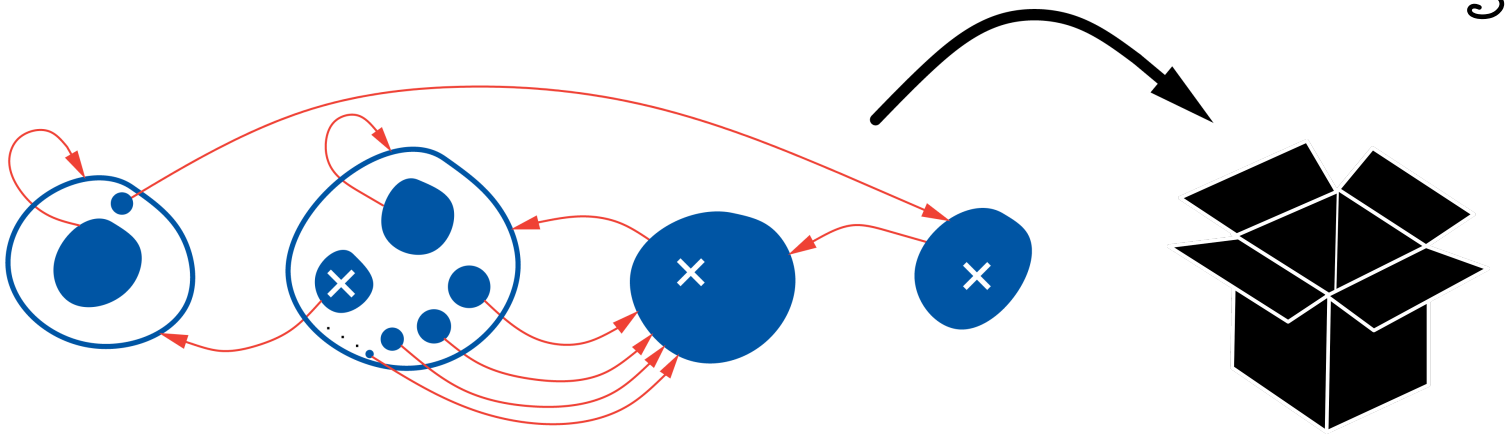
How to use box mappings as “black boxes”

joint work with :

Trevor Clark
Oleg Kozlovski
Dierk Schleicher
Sebastian van Strien

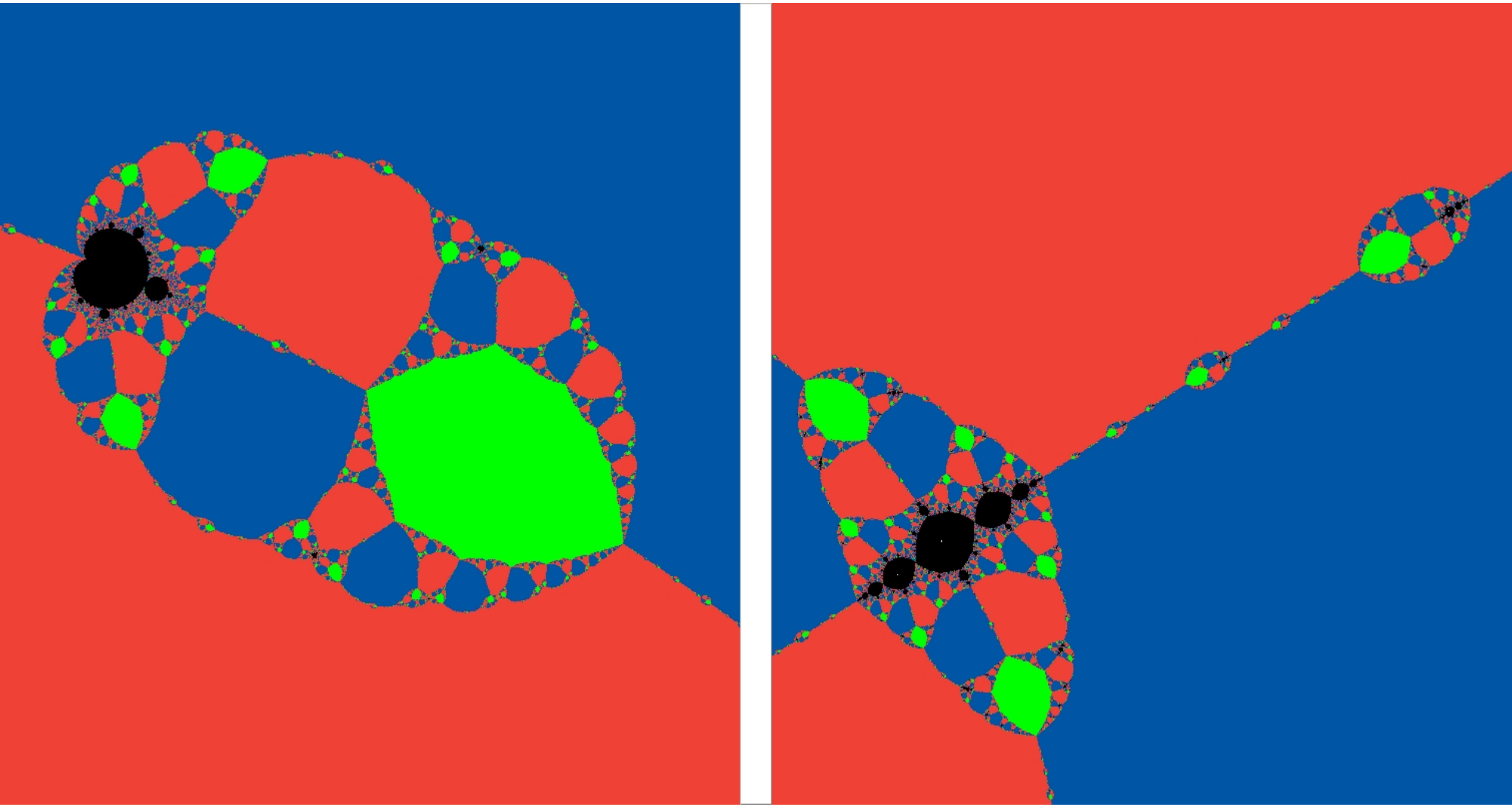
Kostya Drach
(IST Austria)



On geometric complexity of Julia sets – III

September 28, 2021

Polynomial-like maps



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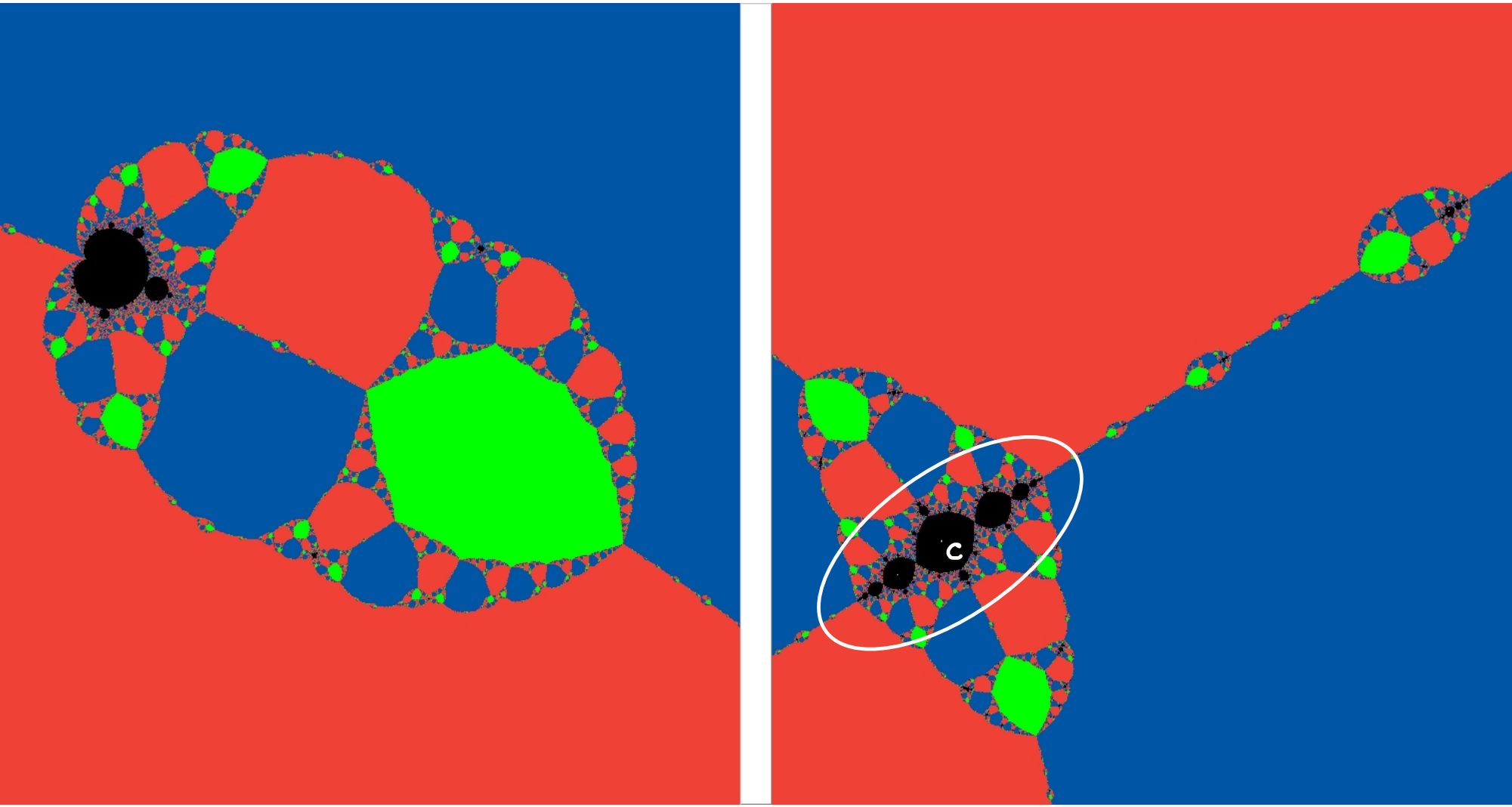

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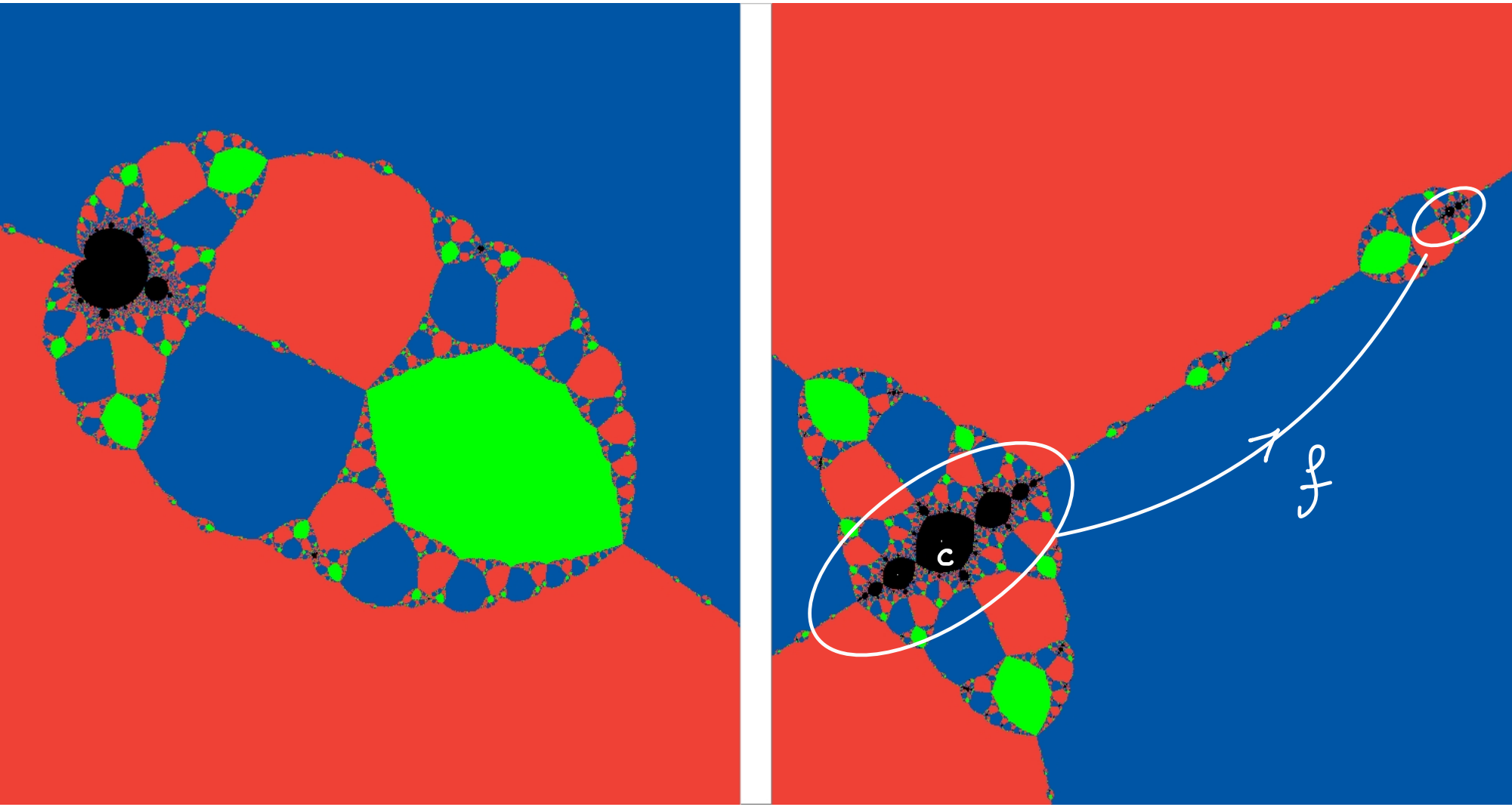

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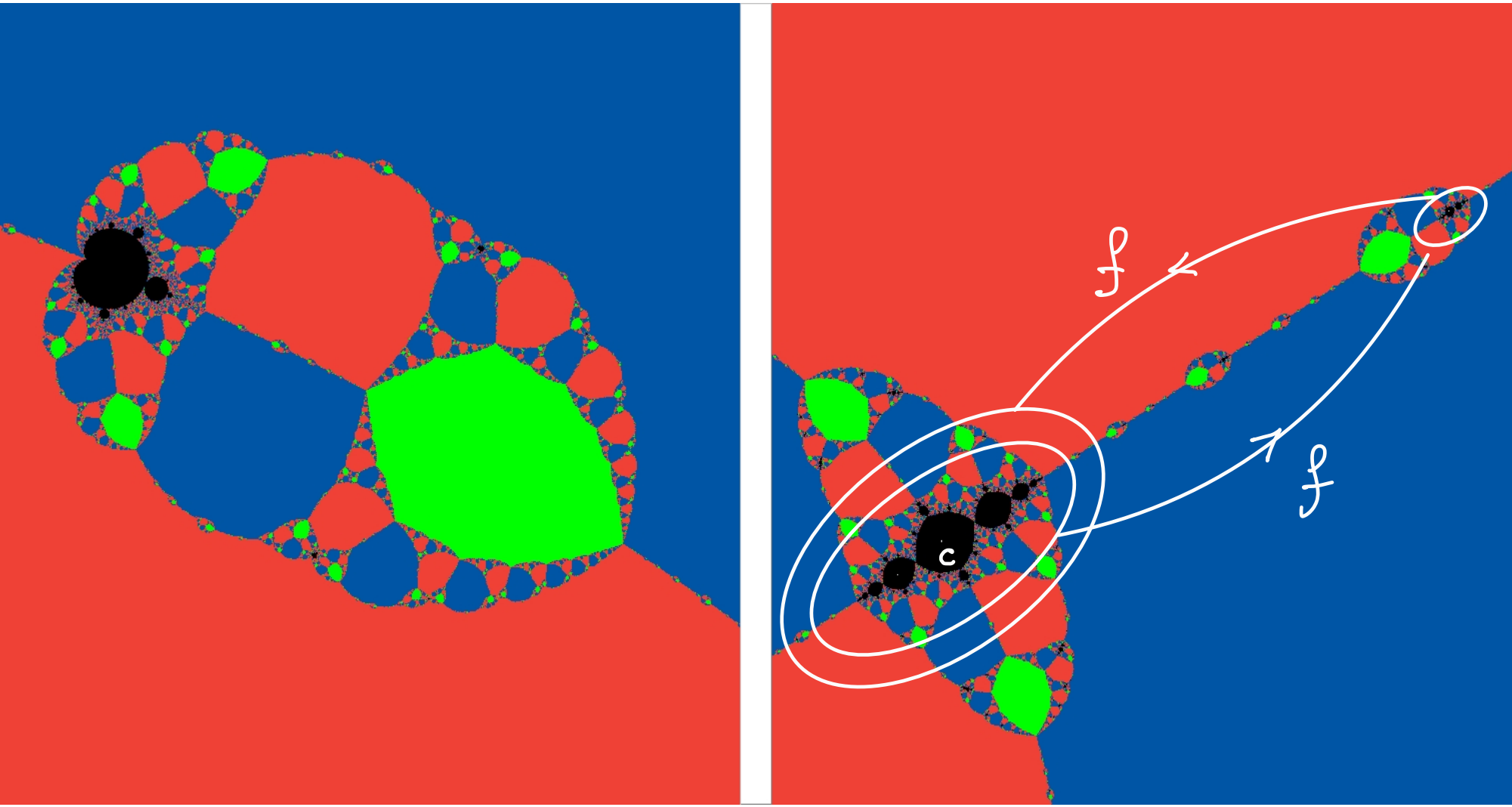

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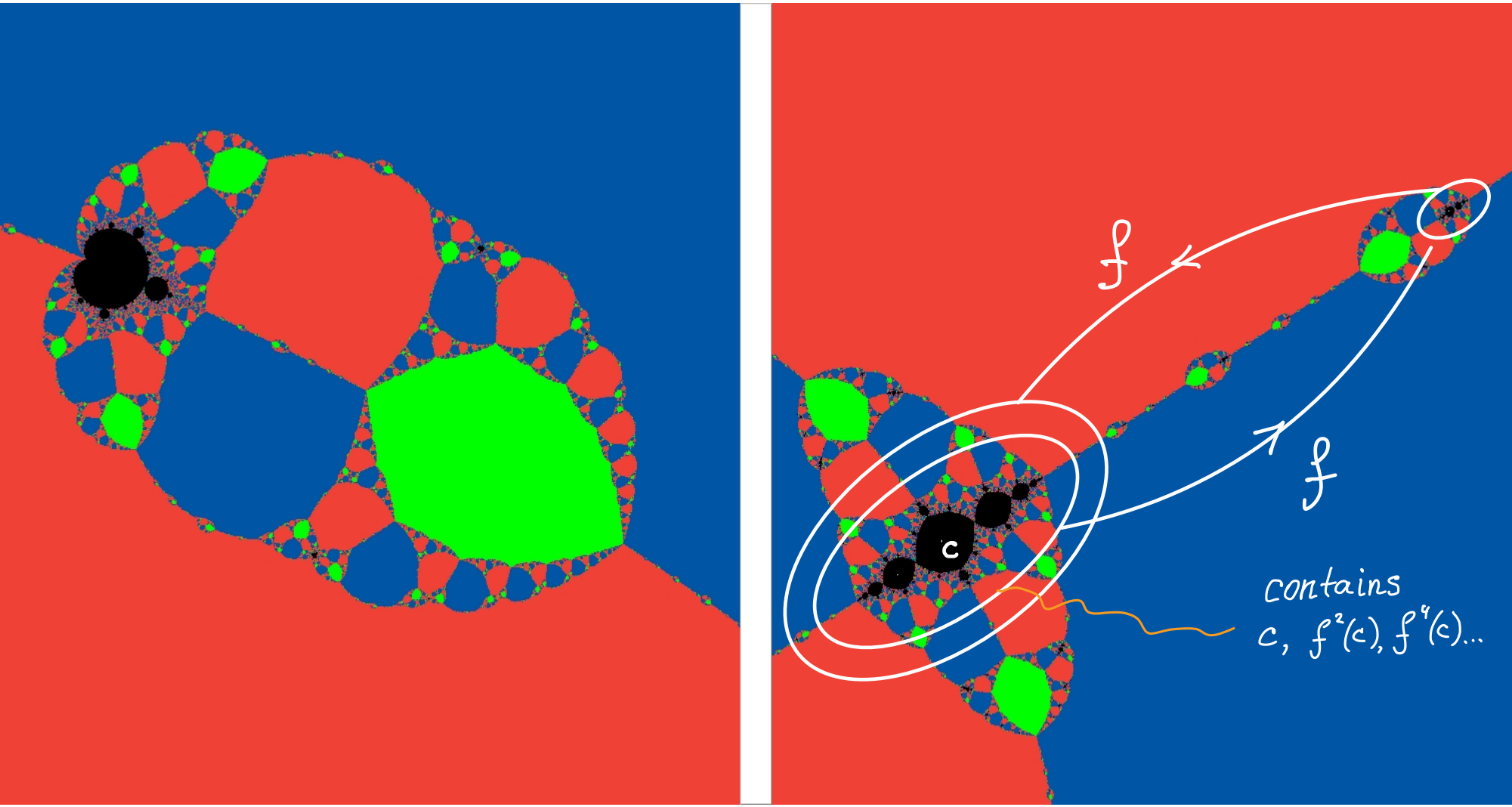

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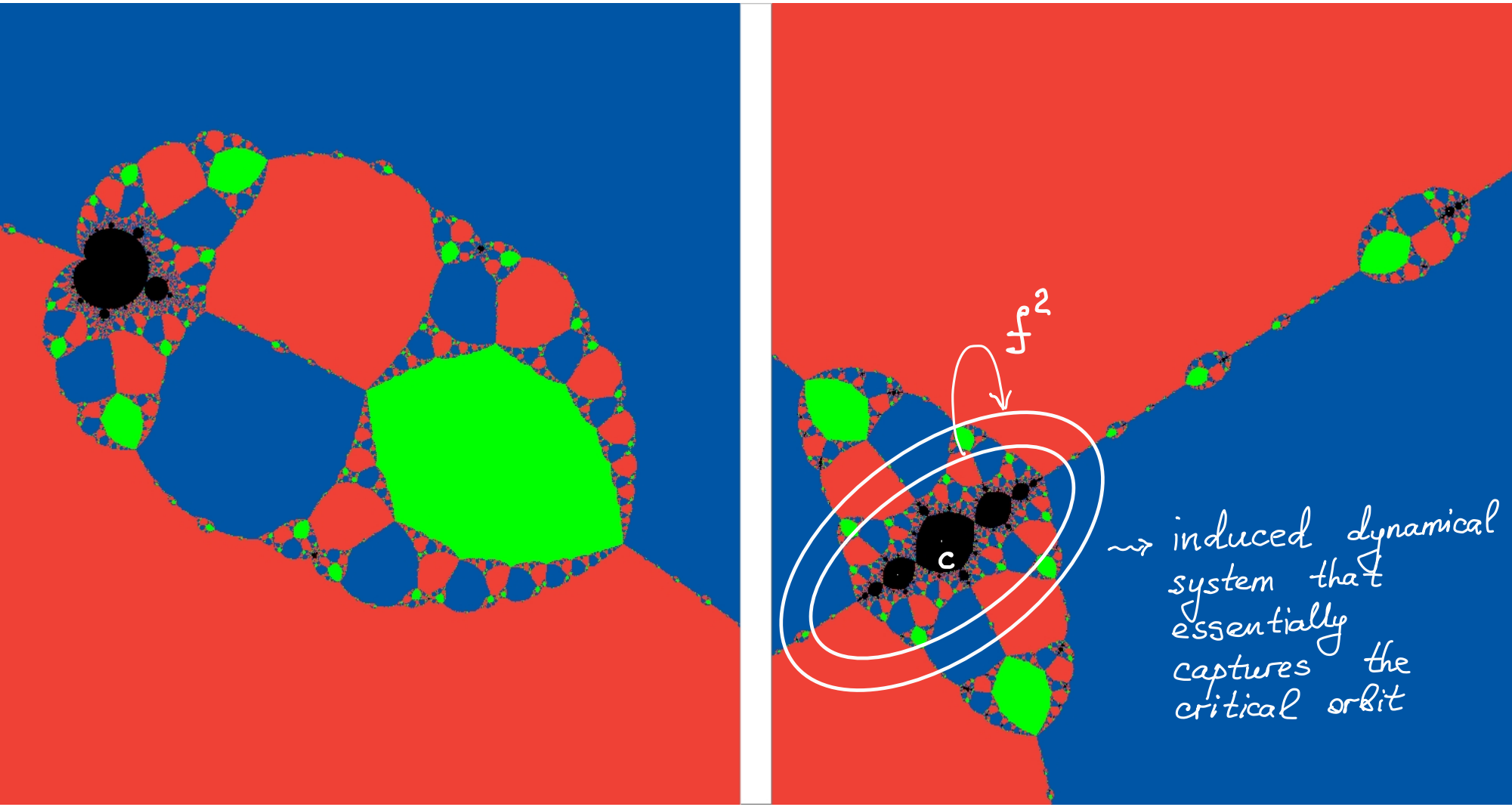

Polynomial-like maps (cont.)

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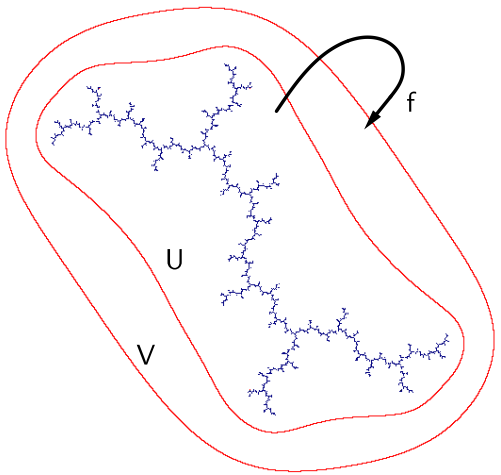
$K(f) := \bigcap_{n \geqslant 1} f^{-n}(V)$ is the **filled Julia set of** f (the non-escaping set).



The Straightening Theorem (Douady–Hubbard)

Every polynomial-like map $f : U \to V$ of degree $d \geqslant 2$ is **hybrid equivalent** *to a polynomial $p : \mathbb{C} \to \mathbb{C}$ of degree d. Moreover, if $K(f)$ is connected, then p is unique up to affine conjugation.*

(f, g are hybrid equivalent $\Leftrightarrow \exists$ quasiconformal conjugacy ψ between f and g defined on a neighborhood of their filled Julia sets with $\bar{\partial}\psi \big|_{K(f)} = 0$.)

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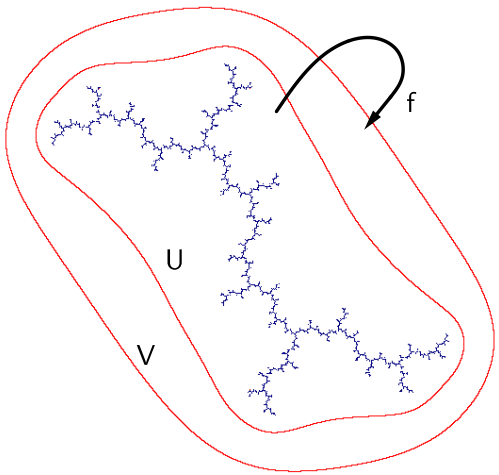

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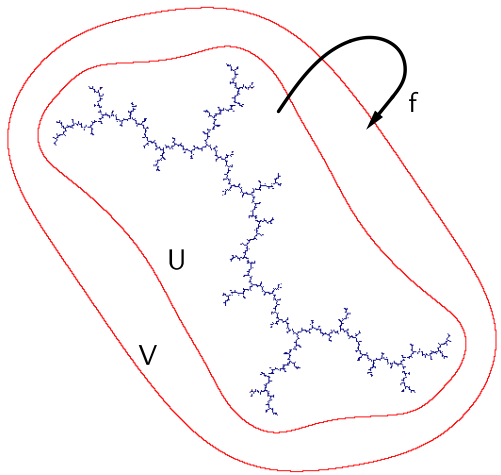

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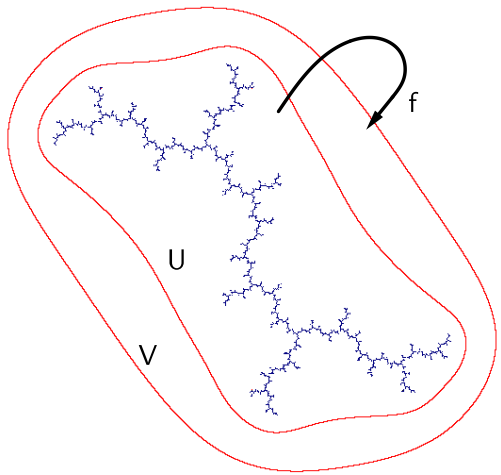

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$\rightsquigarrow$ **Q: How to construct more complicated induced dynamical systems?**

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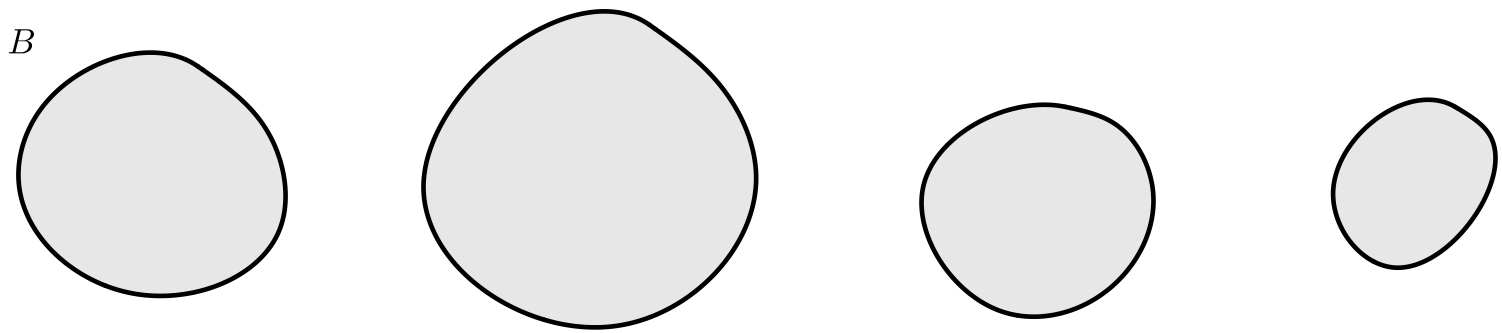

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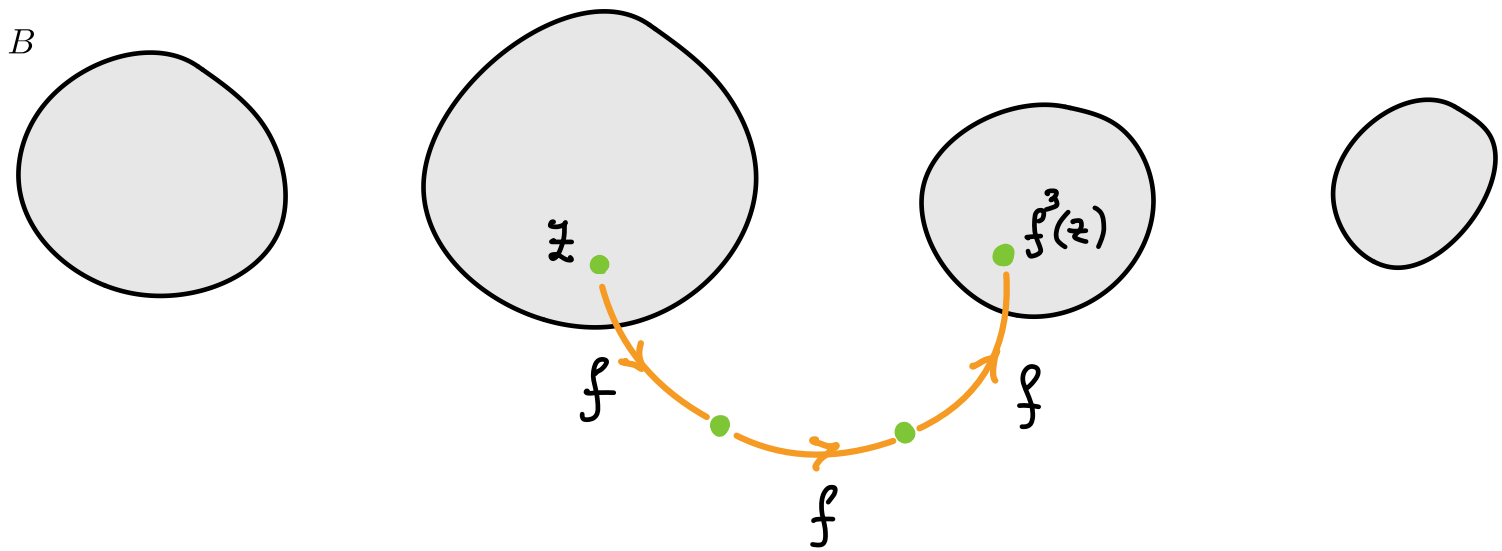

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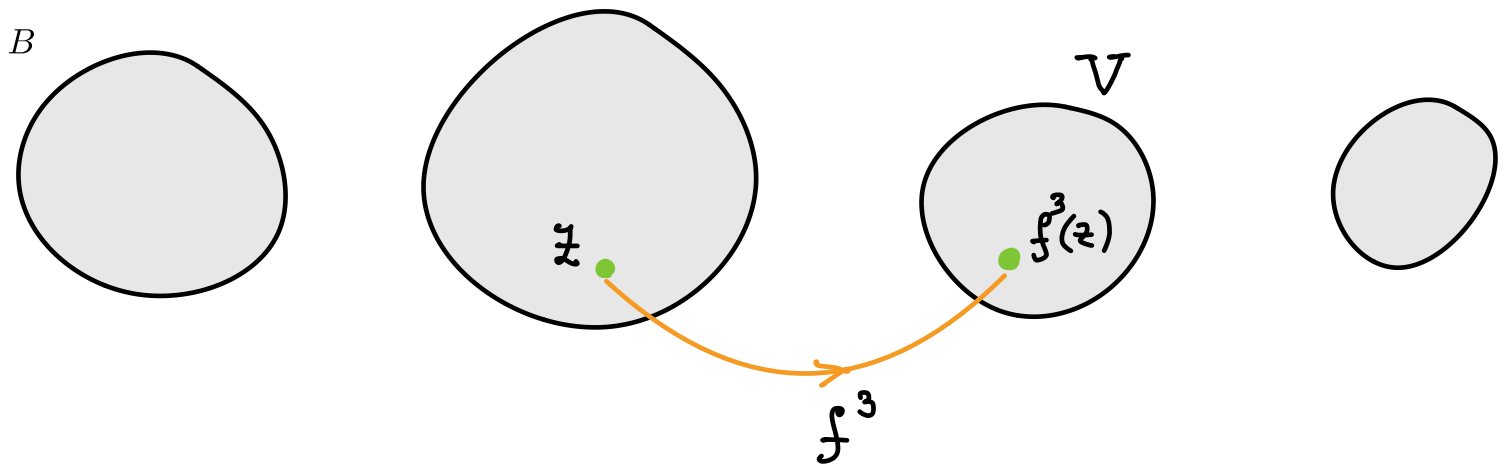

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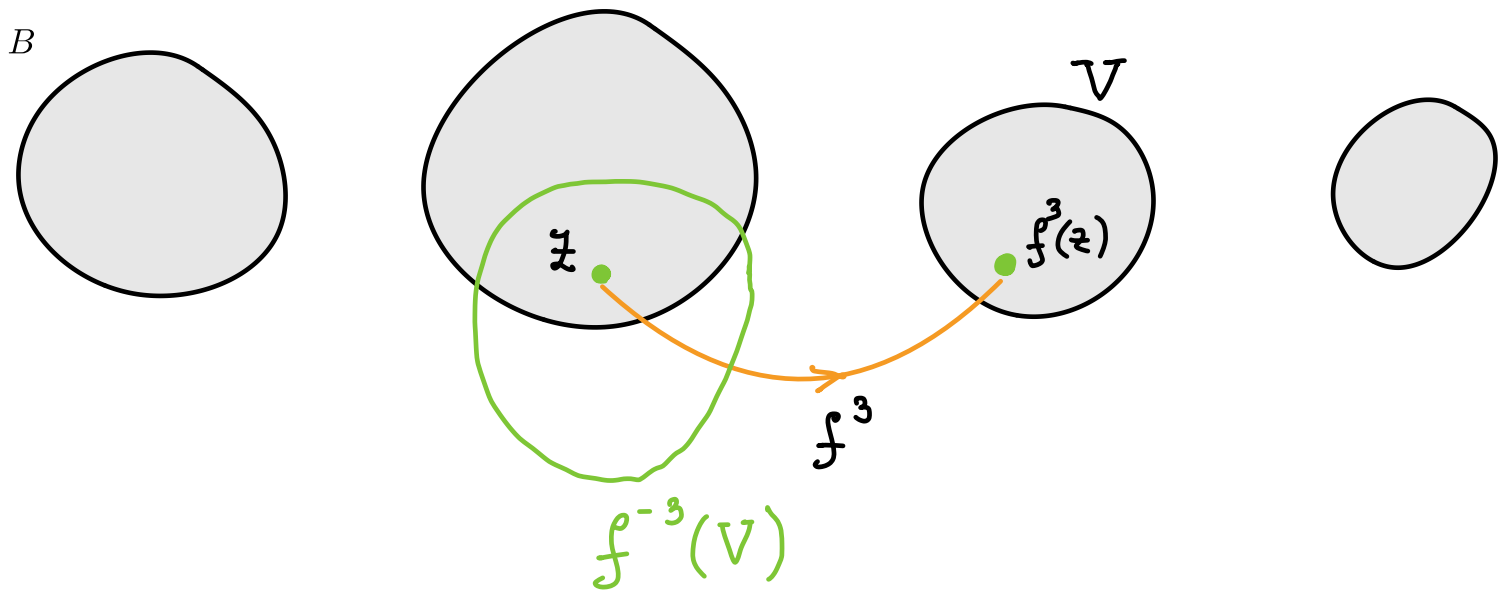

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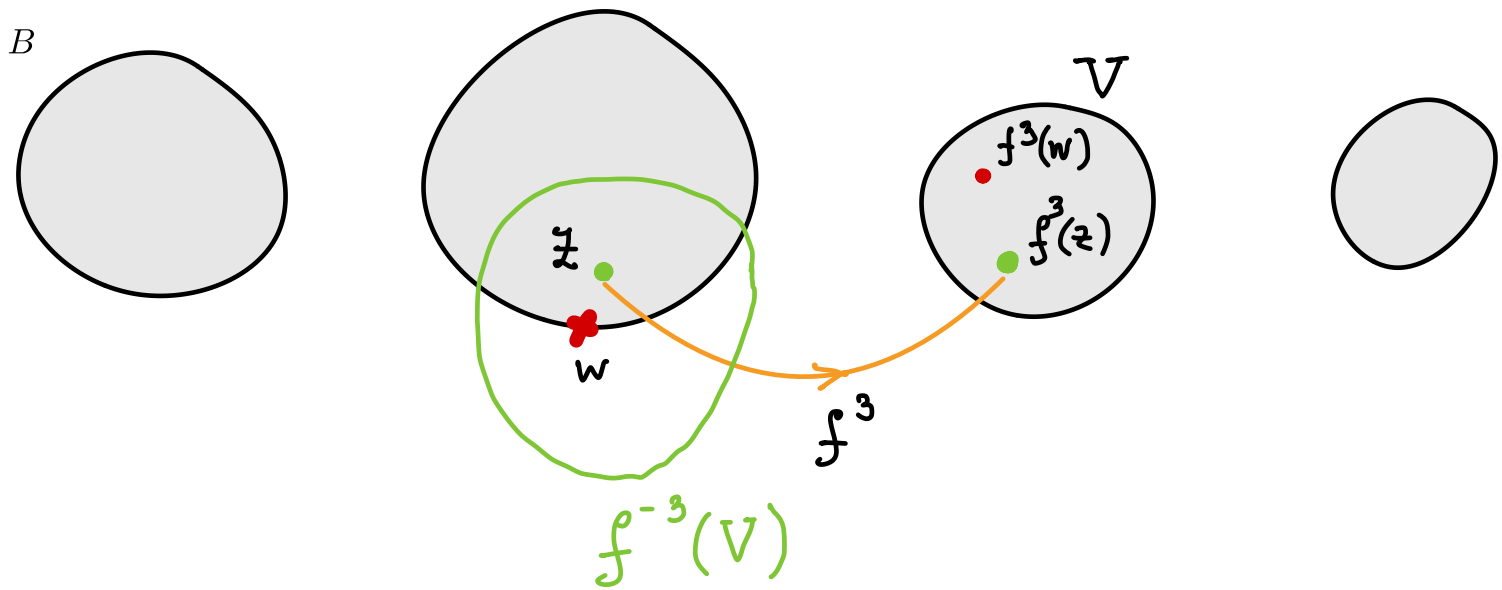

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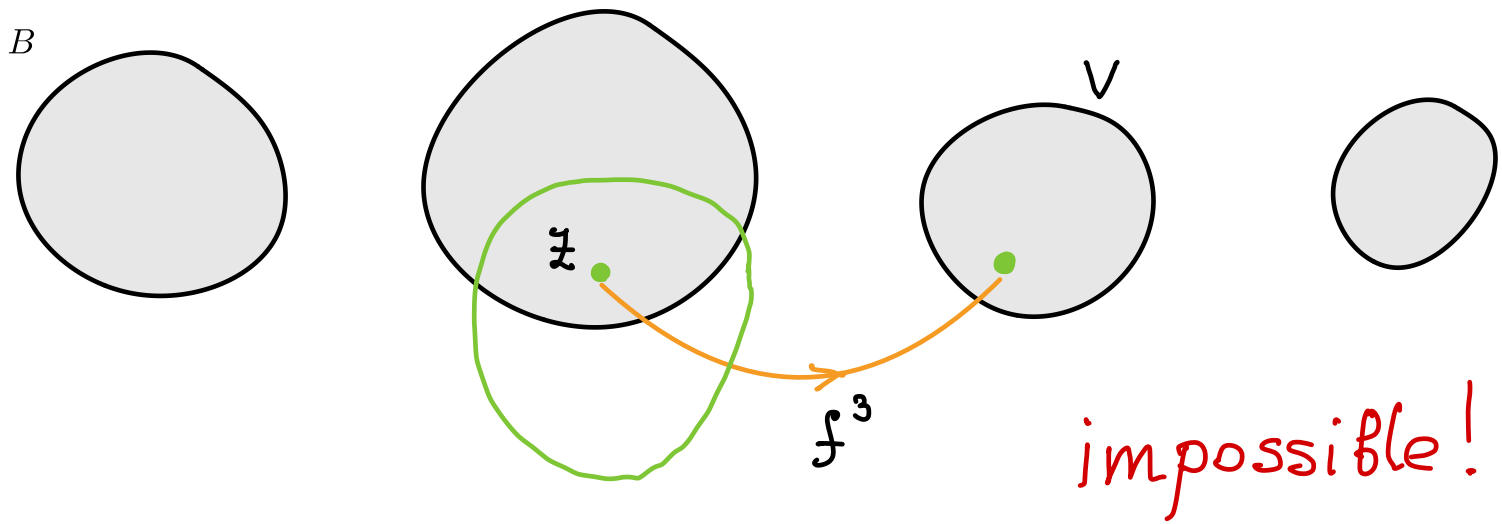

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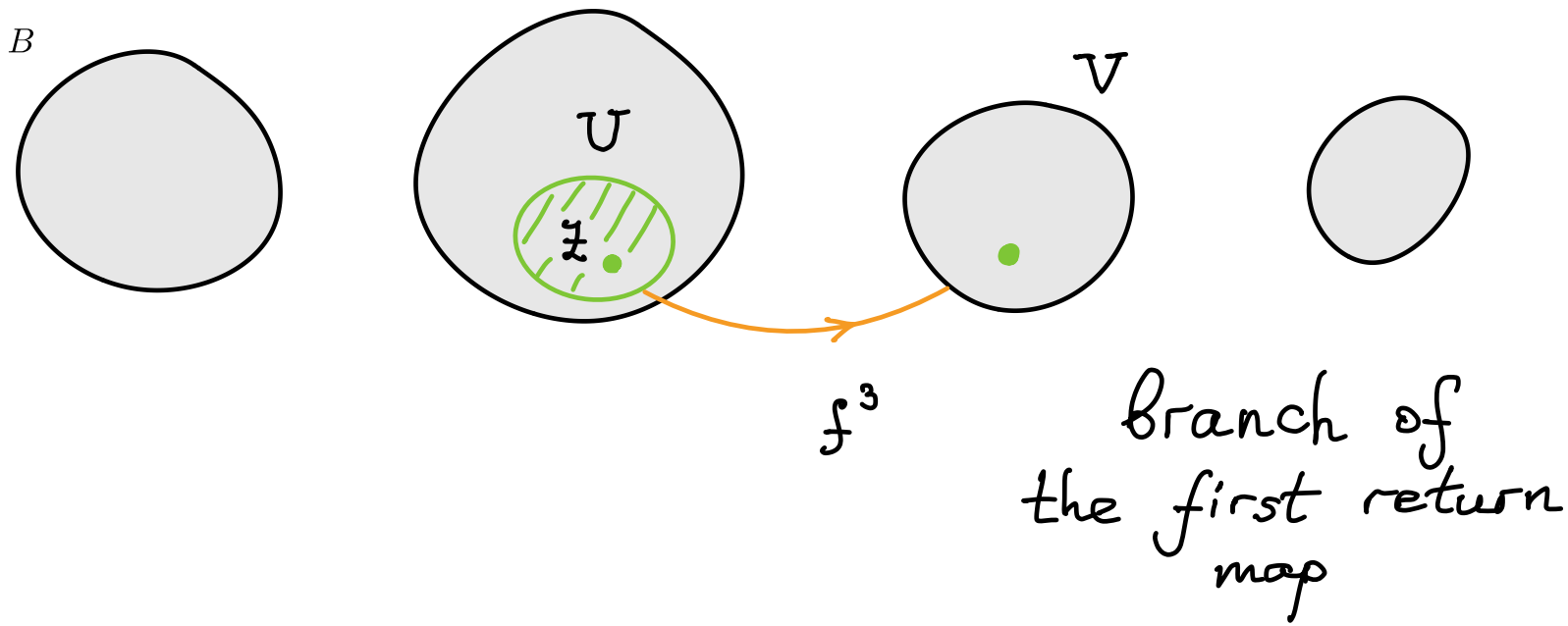

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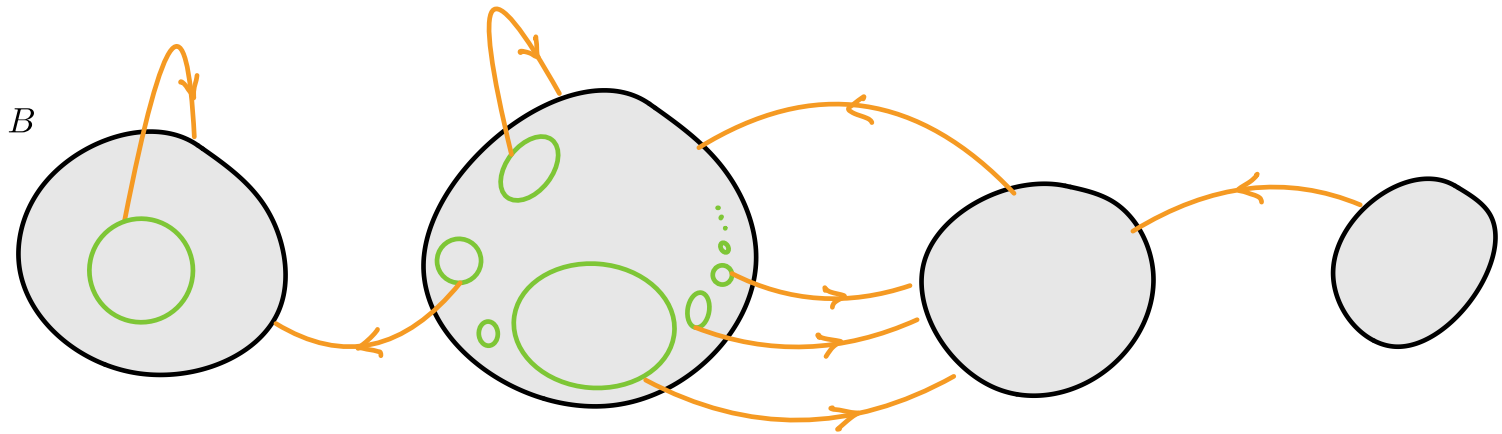

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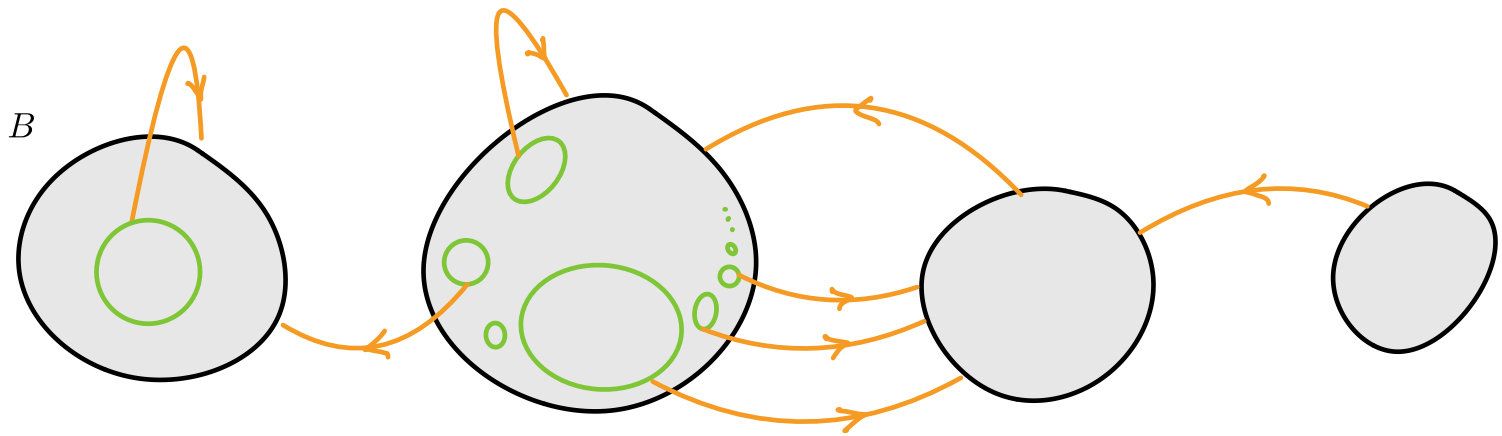

- If the f-orbit of $z \in B$ intersects B infinitely often (for example, z is recurrent), then the first return map will be defined for infinitely many iterates of z (the FRM "essentially captures" the orbit of z).
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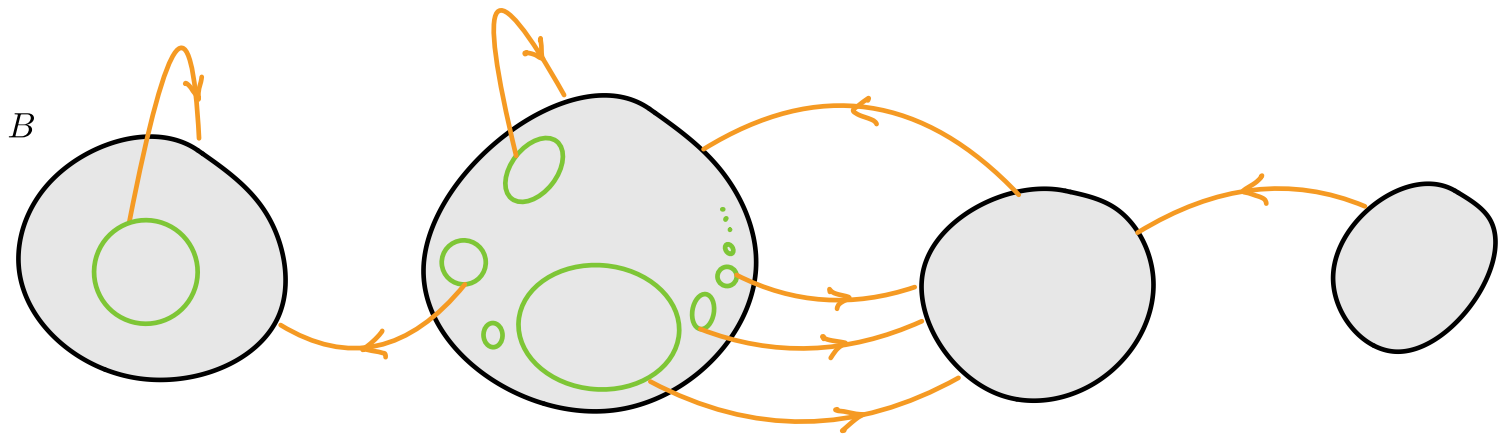

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⇝ **Structure of complex box mapping**

Box mappings

A holomorphic map $F : \mathcal{U} \to \mathcal{V}$ between open sets $\mathcal{U} \subset \mathcal{V} \subset \widehat{\mathbb{C}}$ is a **(complex) box mapping** if:

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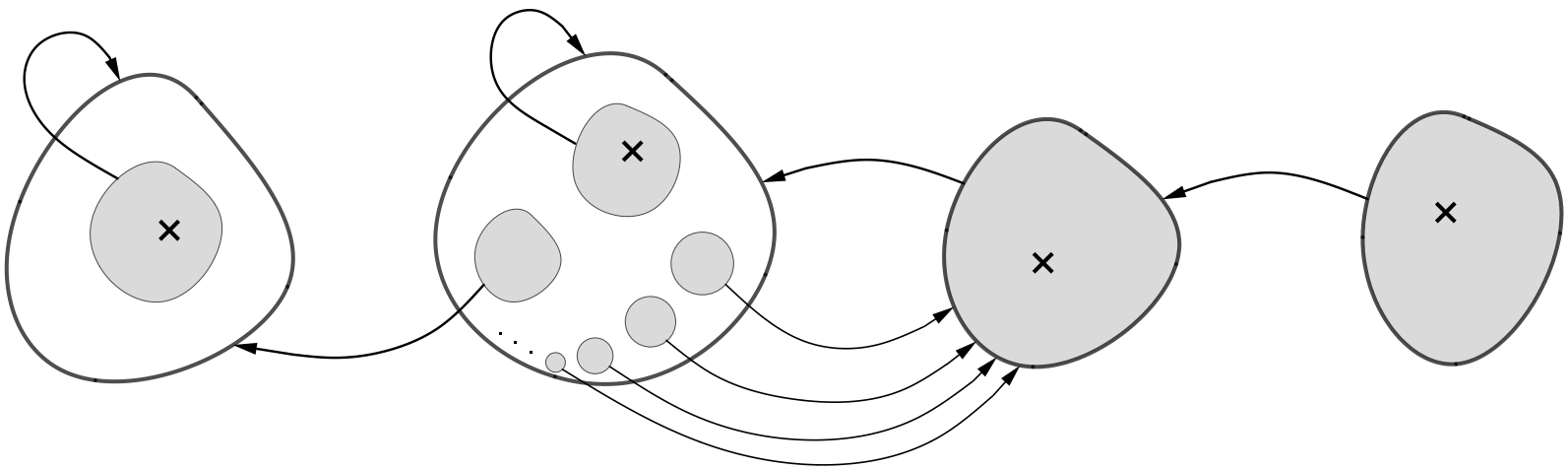

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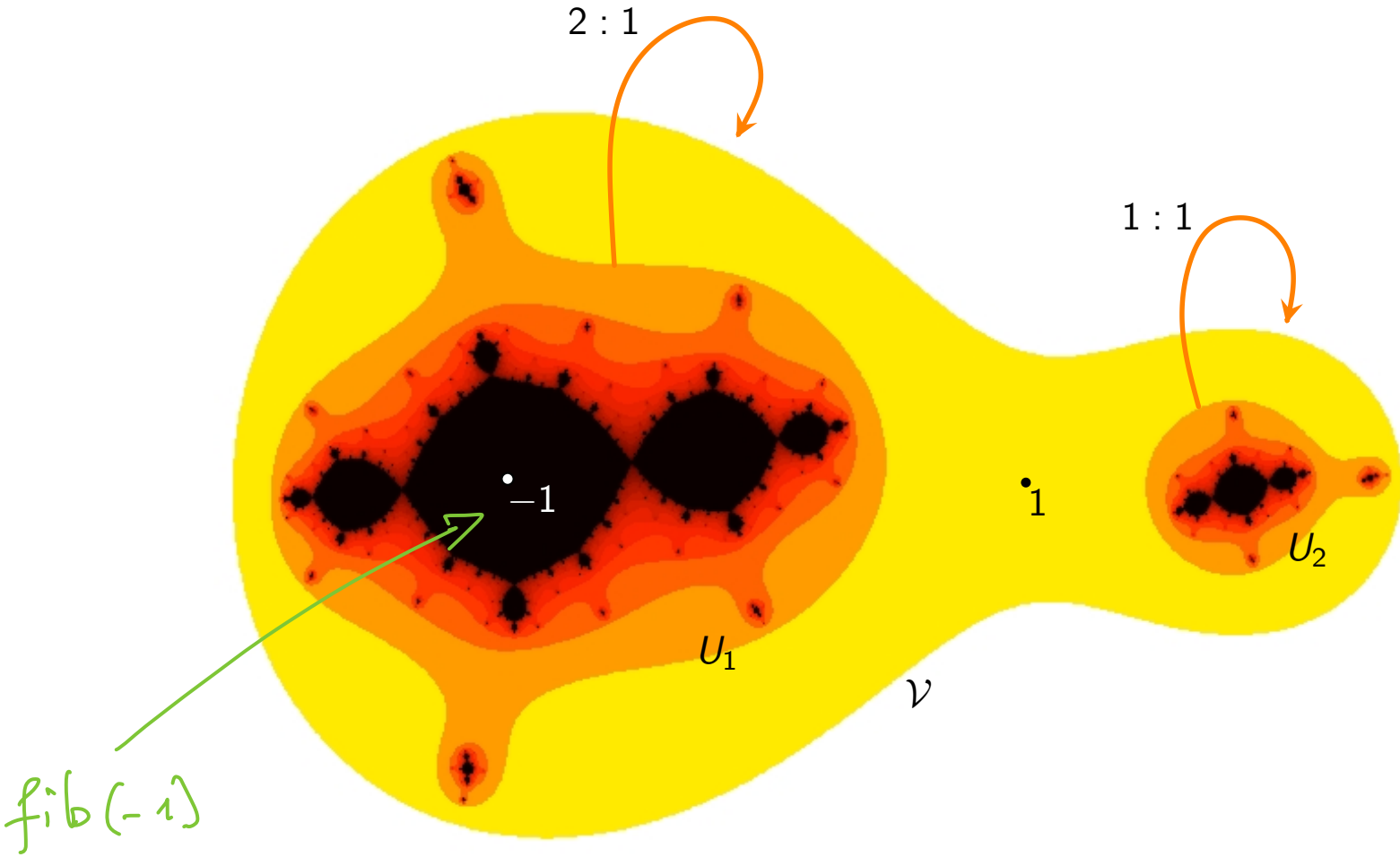
  - fiber is trivial if $\mathrm{fib}(x) = \{x\}$
  - **fib(x)** is the set of points with the **same itinerary** w.r.t. the puzzle partition
  - the fiber consists of points "traveling together" under iteration
  - if the fiber of x is trivial, then the orbit of x can be **distinguished combinatorially** among all other orbits.

An example



$$P\colon z \mapsto c - (z^3 - 3z - 2)/(c^2 - c - 2),\ c = -0.2 + 0.1i$$

$\rightsquigarrow$ restriction $F\colon U_1 \cup U_2 \to \mathcal{V}$ is a box mapping.

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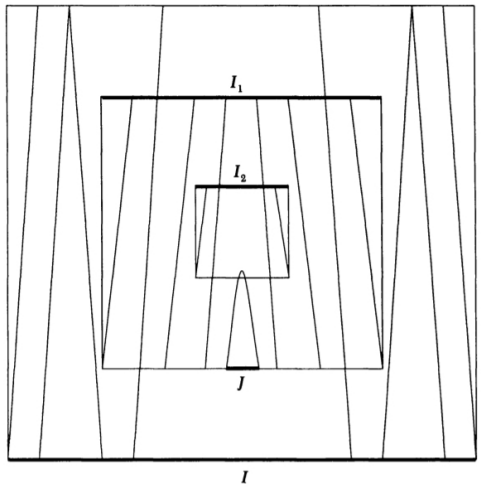
- **Real interval dynamics**: a similar construction first appears in the works of **Jakobson**, **Guckenheimer – Johnson** (who coined the term "box map"), **Graczyk–Swiatek**, **Nowicki–van Strien**, ... in the late 80's-early 90's via the idea of **inducing**.



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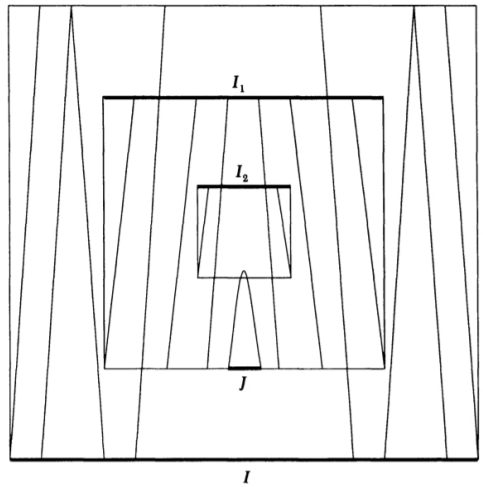

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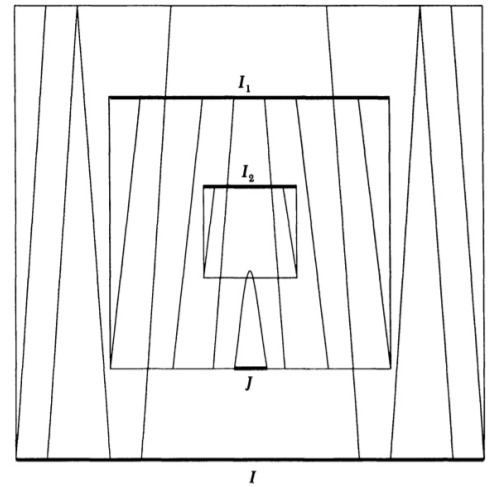

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$\rightsquigarrow$ **Our box mappings = generalized renormalizations** in the sense of Lyubich.

What can we say about general box mappings?

“Pathologies”

Pathologies of general box mappings (Clark–D–Kozlovski–van Strien[1])

There are box mappings $F_i : \mathcal{U}_i \to \mathcal{V}_i$, $i \in \{1, 2, 3, 4\}$ with the following properties:

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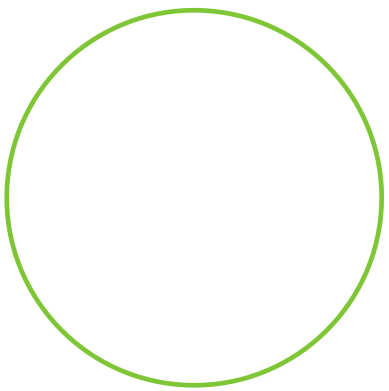

$\mathcal{U}_1 = \mathcal{V}_1 = \mathbb{D}$

$F_1 = id$

[1] The Dynamics of Complex Box Mappings. manuscript, 2021.

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[1] The Dynamics of Complex Box Mappings. manuscript, 2021.

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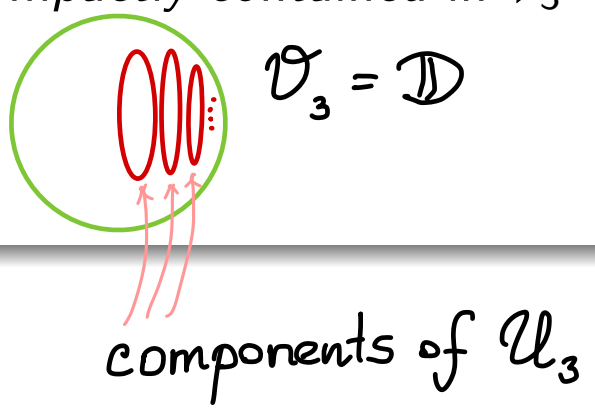

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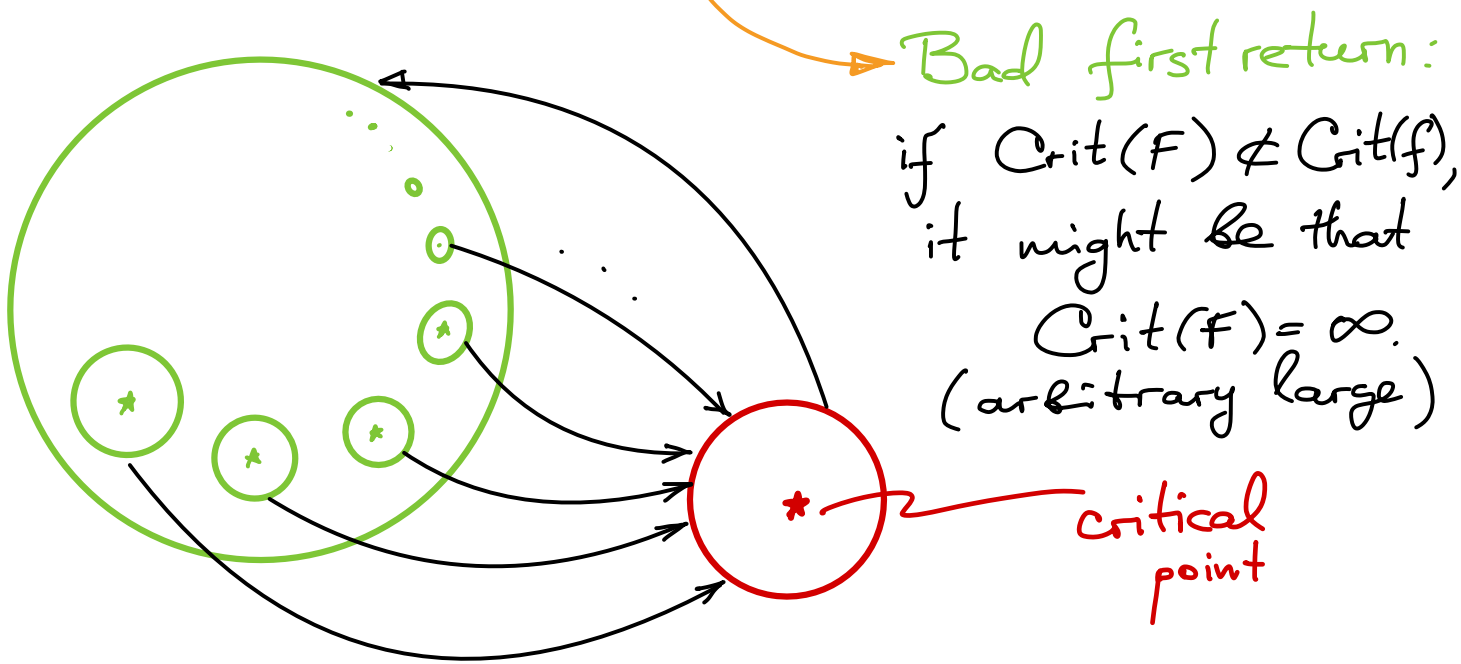
Well-inside: For each $x \in K(F)$ there exists $\delta = \delta(x)$ and infinitely many U_i and V_i (components of $\mathcal{U}$ and $\mathcal{V}$) such that $\mathrm{mod}(V_i \setminus U_i) \geqslant \delta$ and the orbit of x intersects U_i.

Box renormalization

A map f is **box-renormalizable** if there exists a nice union $\mathcal{V}$ of topological disks (in the domain of definition for f) so that the first return map to $\mathcal{V}$ under f is a dynamically natural box mapping F and $\mathrm{Crit}(F) \subset \mathrm{Crit}(f)$.



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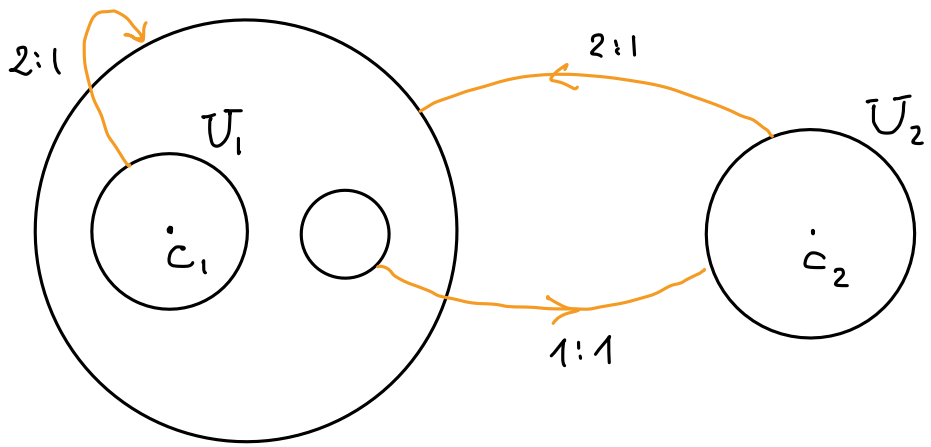

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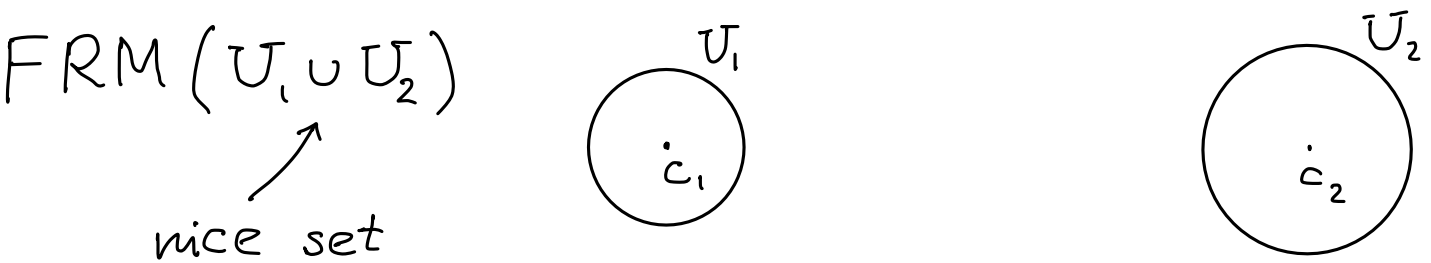

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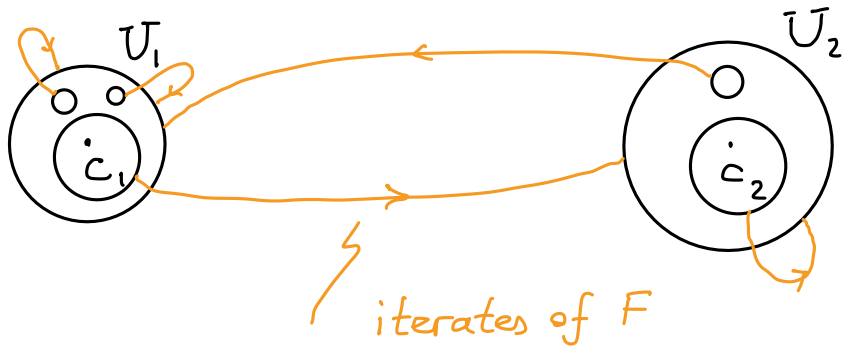

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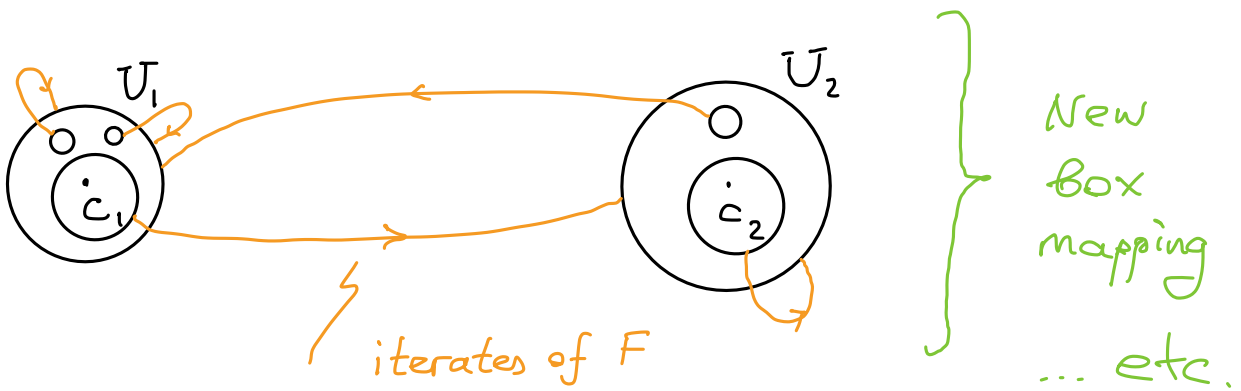

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- Starting from a globally defined f, say rational, it is not clear whether f is box-renormalizable or not; **intermediate take-away**: if it does, and the box renormalization F is not trivial (for example, $\mathrm{Crit}(F)$ contains all recurrent critical points in $\mathrm{Crit}(f) \cap J(f)$), then we are in business!

Rigidity and ergodicity results for dynamically natural box mappings

Dynamical rigidity of box mappings

Dynamical Rigidity for Box Mappings (D–Schleicher[2])

If F is a dynamically natural box mapping, and $z \in K(F)$, then either

(T) *z has **T**rivial fiber, or*

(R) *z belongs, or is mapped by some finite iterate, to the filled Julia set of **R**enormalizable dynamics (a polynomial-like restriction of F with connected filled Julia set).*

This is a generalization of the result due to **Kozlovski–van Strien**[3](the proof uses:

- the Enhanced Nest construction due to **Kozlovski–Shen–van Strien**;
- the Covering Lemma due to **Kahn–Lyubich**, who showed in their work with **Avila** and **Shen** how this lemma can be applied to prove rigidity for unicritical polynomials $z \mapsto z^d + c$.)

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In particular:

All fibers in the filled Julia set of a **non-renormalizable** dynamically natural box mapping are trivial.

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Ergodic properties

A set $E \subset K(F)$ of positive Lebesgue measure is an **ergodic component** of F if the symmetric difference $F^{-1}(E) \triangle E$ has Lebesgue measure zero.

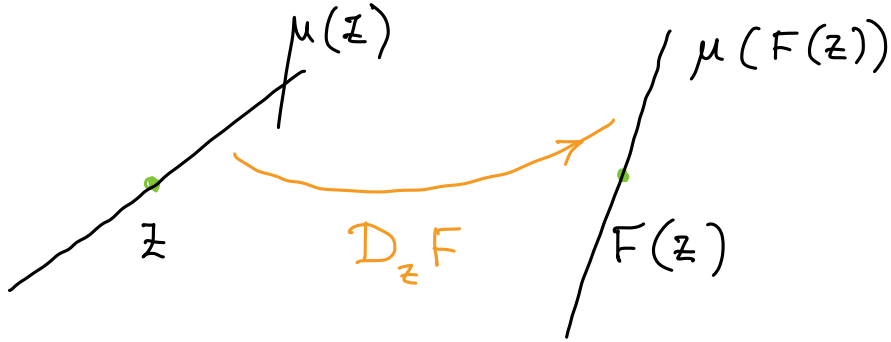
Number of ergodic components (C–D–K–vS)

If $F : \mathcal{U} \to \mathcal{V}$ is a non-renormalizable dynamically natural complex box mapping, then for each ergodic component E of F there exist one or more critical points c of F so that c is a Lebesgue density point of E.

In particular, the number of ergodic components of F is bounded above by $\#\,\mathrm{Crit}(F)$.

Line fields for box mappings

F carries an **invariant line field on its filled Julia set** if there is a measurable assignment of a line $\mu(z)$ through each $z \in K(F)$ so that $D_z F$ maps $\mu(z)$ to $\mu(F(z))$ for a.a. z; this is non-trivial only when $\mathrm{meas}(K(F)) > 0$.



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Existence of invariant line fields (C–D–K–vS)

Let F be a non-renormalizable dynamically natural box mapping.
If F carries an invariant line field on its filled Julia set $K(F)$, then

1. *there exists a puzzle piece $\mathcal{J}$ and a smooth foliation on $\mathcal{J}$ with the property that $\mathrm{meas}(K(F) \cap \mathcal{J}) = \mathrm{meas}(\mathcal{J})$ and the foliation is tangent to the invariant line field a.e. on $\mathcal{J}$;*
2. *the set $Y = \bigcup_{i \geqslant 0} F^i(\mathcal{J})$ intersects $\mathrm{Crit}(F)$ and each point in $Y \cap \mathrm{Crit}(F)$ is strictly preperiodic.*

In particular:

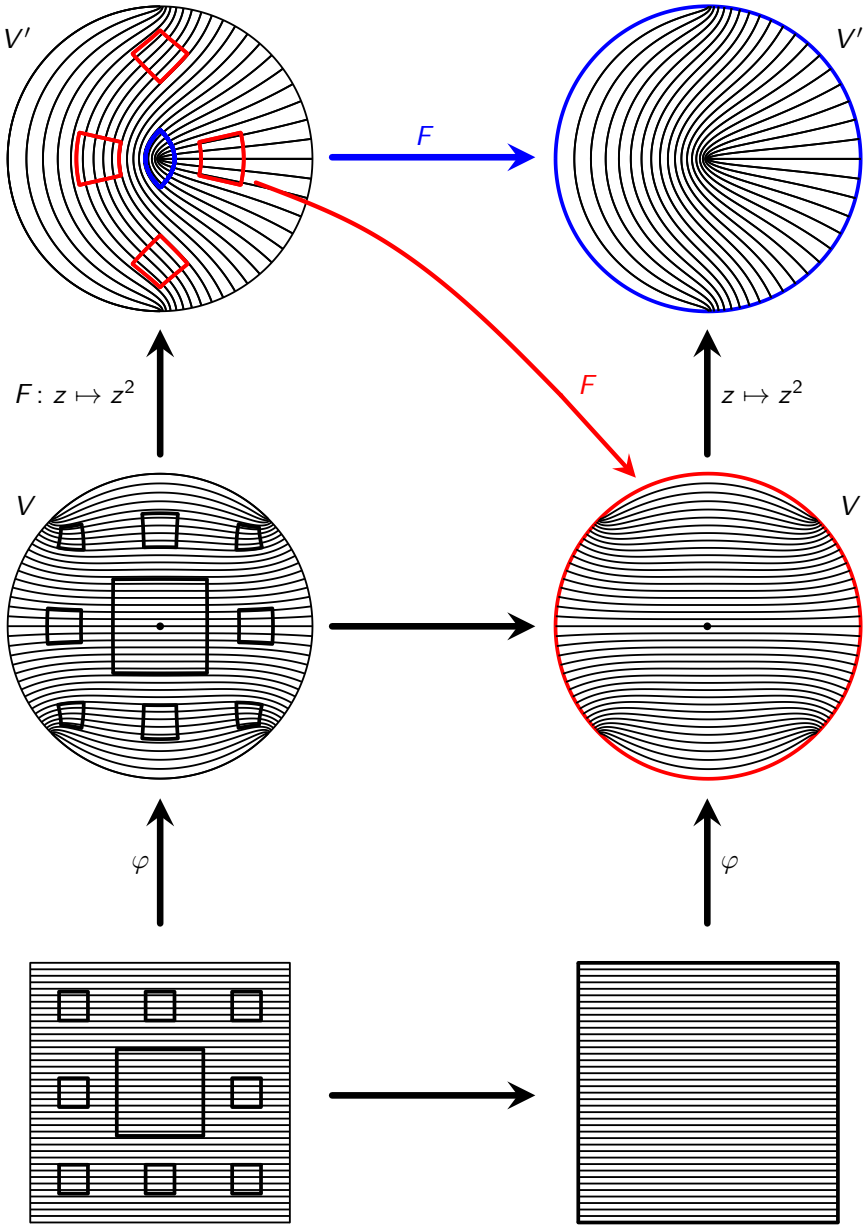
If each puzzle piece contains an open set of escaping points, then F carries no invariant line fields.

Lattès box mappings

Dynamically natural box mappings with $\mathrm{meas}(K(F)) > 0$ and which carry an invariant line field on $K(F)$ are called **Lattès box mappings**.

- They are analogues of flexible Lattès rational maps.
- In fact, Lattès rational maps are box-renormalizable to Lattès box mappings. This follows from the work of **Przytycki–Rivera-Letelier** on topological Collet–Eckmann maps and nice couples for them.

Lattès box mappings exist: example by hand



Quasiconformal rigidity of box mappings

QC rigidity of complex box mappings (K–vS, C–D–K–vS)

Let $F : \mathcal{U} \to \mathcal{V}$ and $\widetilde{F} : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{V}}$ be a pair of **non-renormalizable** *dynamically natural box mappings. Assume the boundaries of $\mathcal{U}, \widetilde{\mathcal{U}}, \mathcal{V}, \widetilde{\mathcal{V}}$ are piecewise smooth, and suppose there exists a qc homeomorphism $H : \mathcal{V} \to \widetilde{\mathcal{V}}$ so that*

1. $H(\mathcal{U}) = \widetilde{\mathcal{U}}$,
2. $\widetilde{F} \circ H = H \circ F$ *on* $\partial\mathcal{U}$,
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Then F and $\widetilde{F}$ are quasiconformally conjugate, and this conjugation agrees with H on $\mathcal{V} \setminus \mathcal{U}$.

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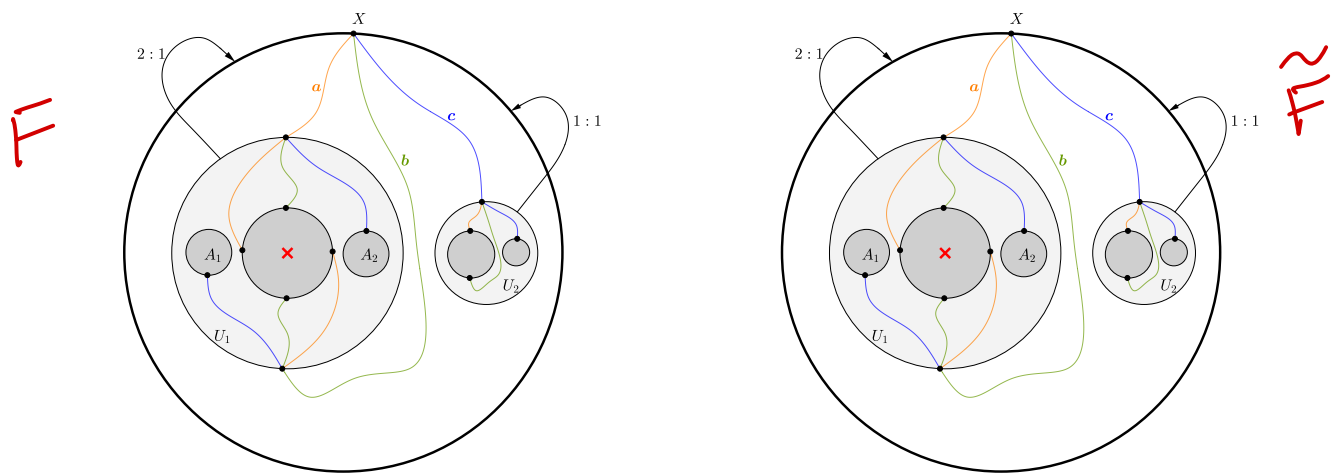

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(The proof is involved: it uses the Enhanced Nest construction and the QC-Criterion due to **Kozlovski–Shen–van Strien**, and the Covering Lemma due to **Kahn–Lyubich**.)

: How to apply?

For a dynamical system f:

1. establish that f is box-renormalizable in a non-trivial way, i.e. associate to f a box mapping F that captures **the most interesting** part of the dynamics of f:
   - in many examples, F is constructed as the first return map to a certain **nice neighborhood** of $\mathrm{Crit}(f) \cap J(f)$;
2. check that F is dynamically natural;
3. embed the existing rigidity results for such F's back into the dynamics of f.

: When can this strategy be successfully applied?

[4] Proof of the Branner-Hubbard conjecture on Cantor Julia sets. Sci. China Ser. A: Math. **52** (2009) 45–65.
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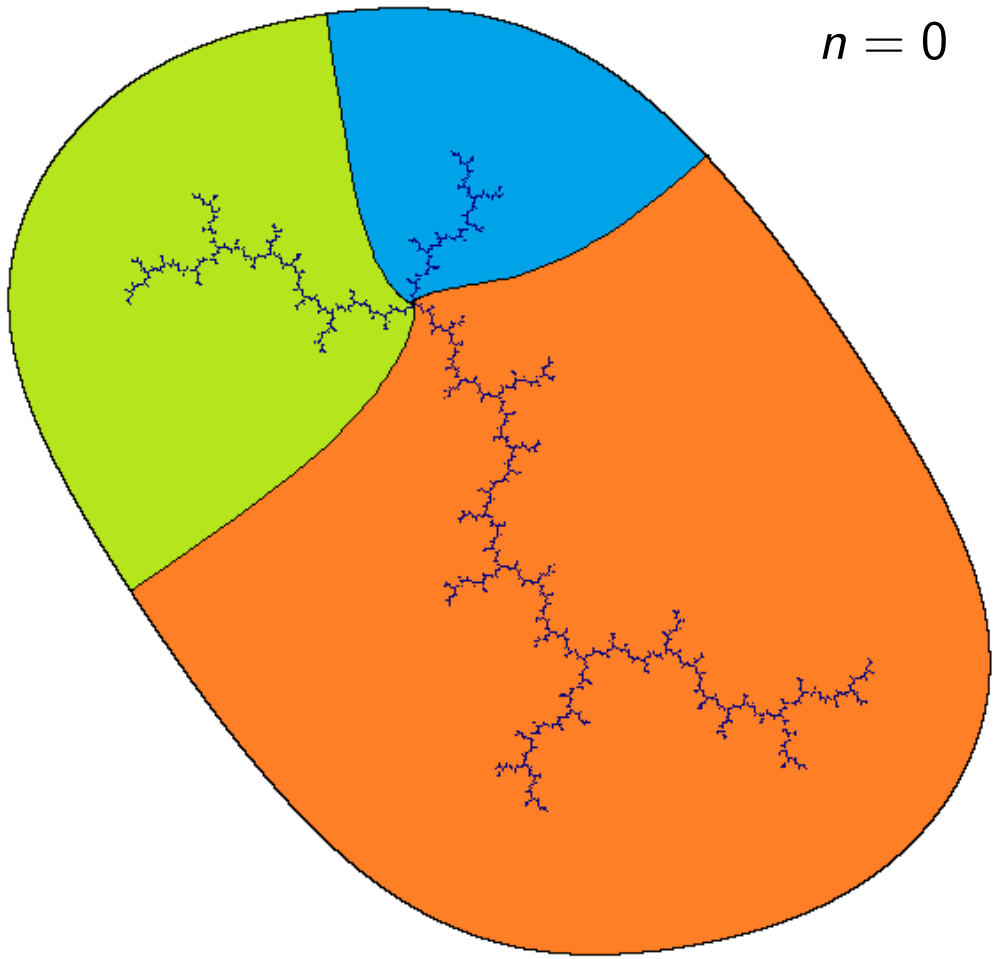

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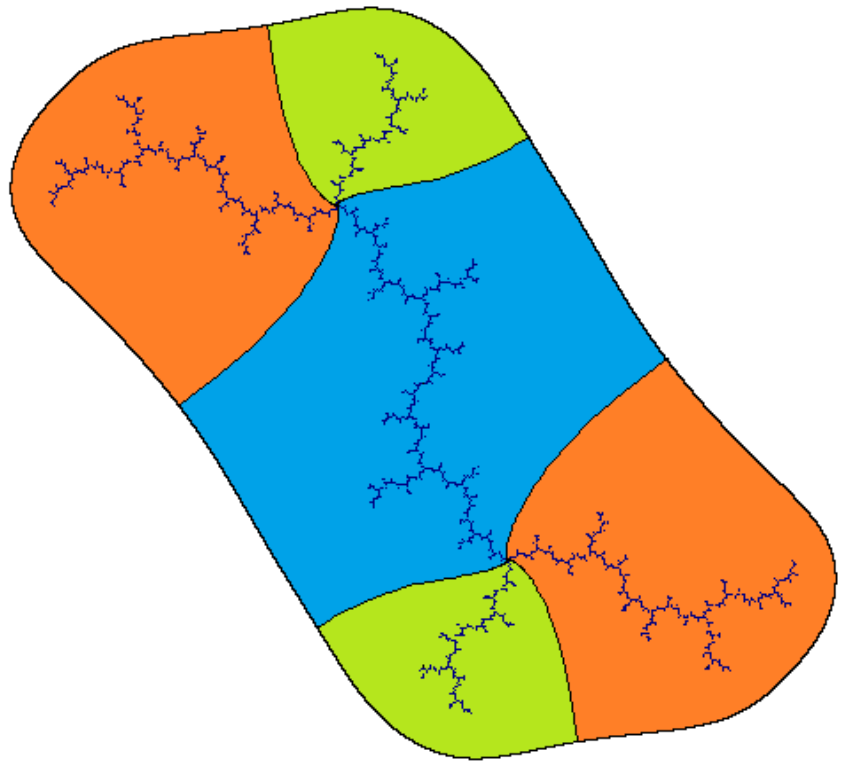

Yoccoz puzzles for polynomials

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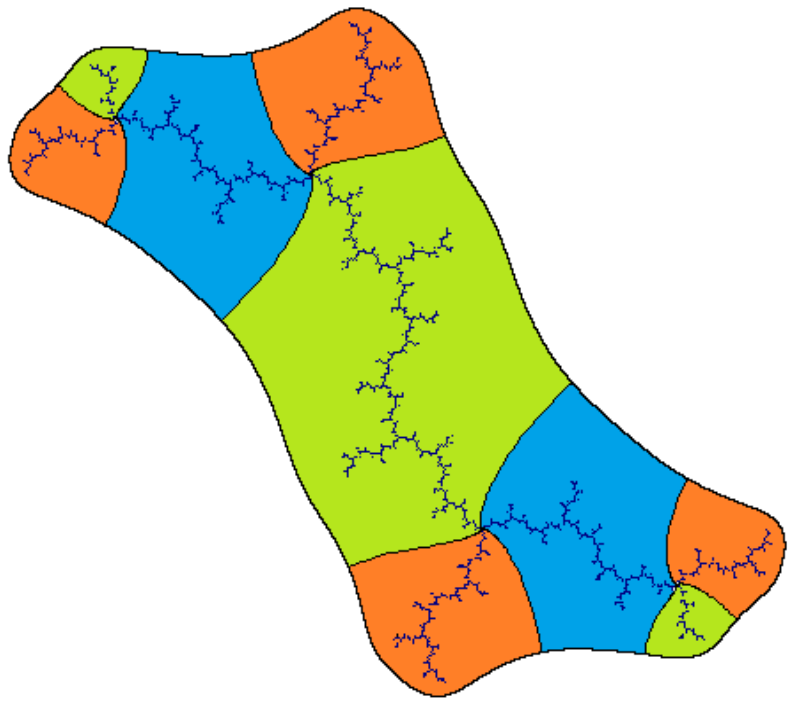

Yoccoz puzzles for polynomials

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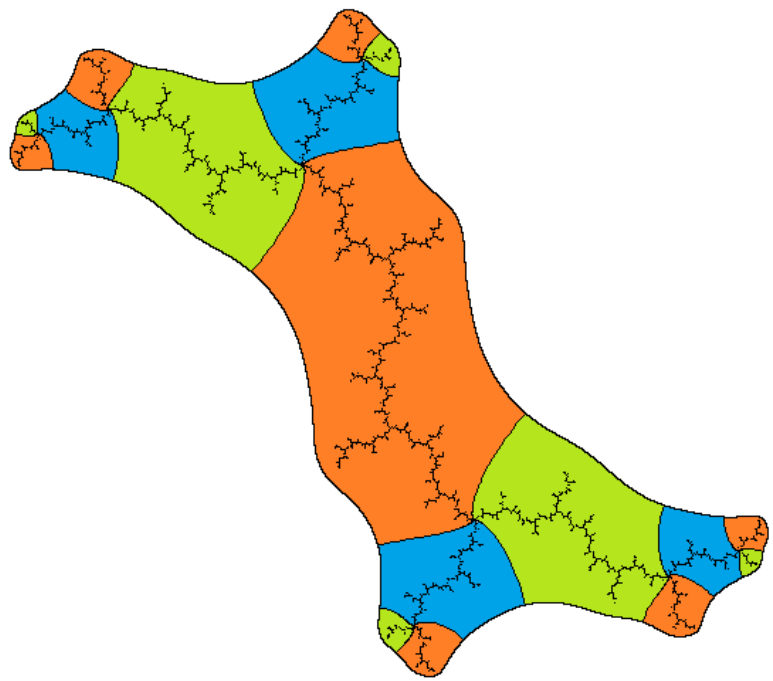

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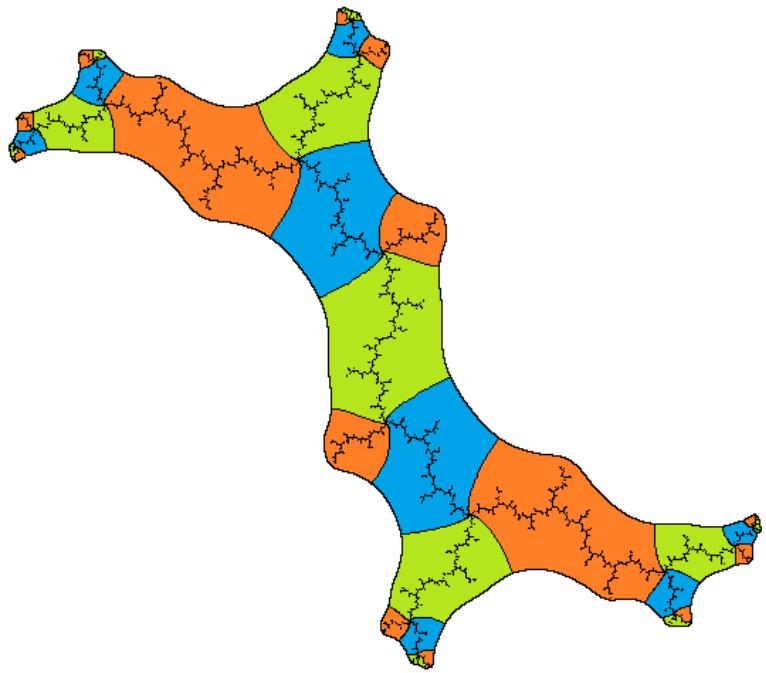

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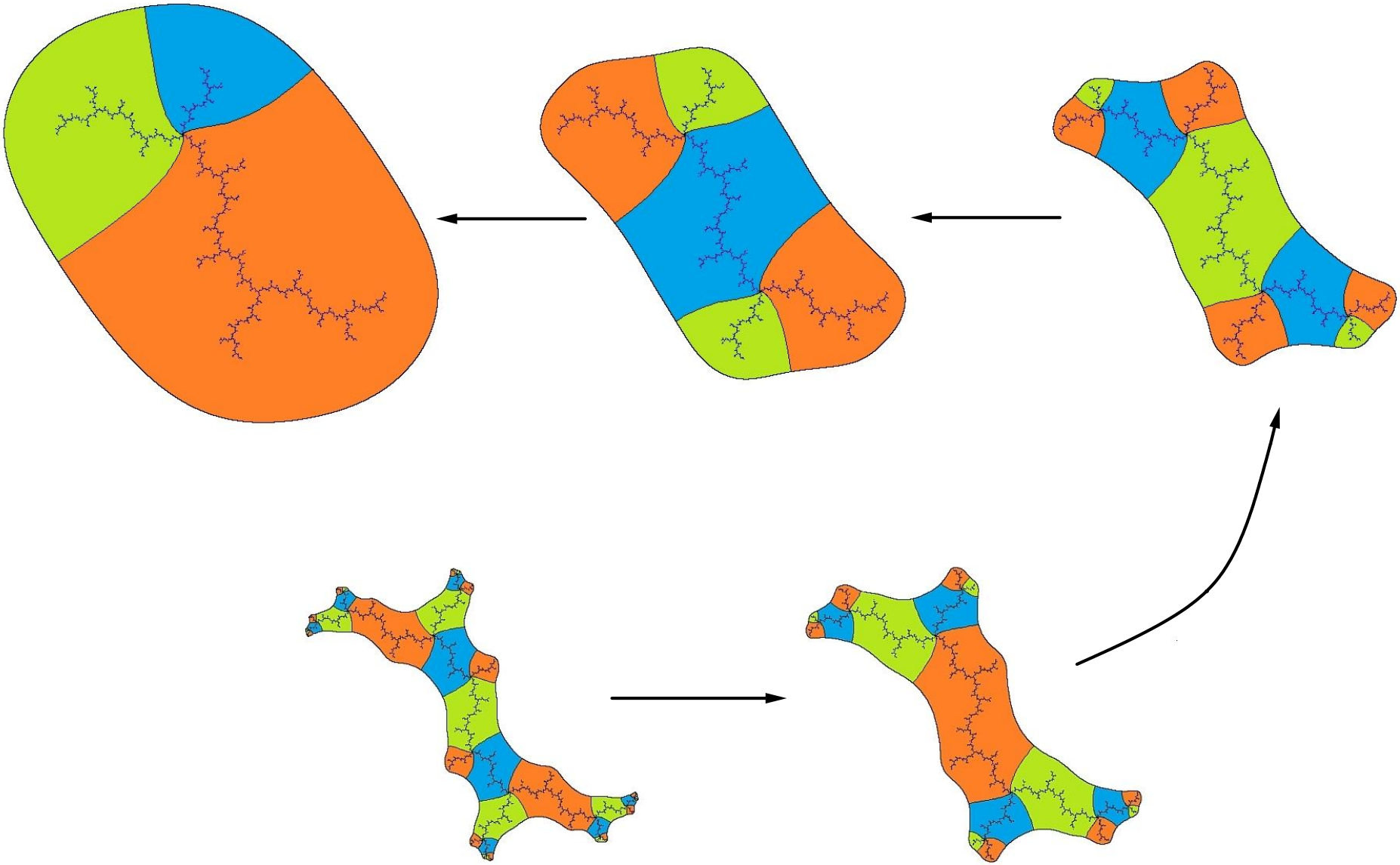

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- **McMullen's maps** $f_\lambda : z \mapsto z^n + \lambda/z^n$, $\lambda \in \mathbb{C} \setminus \{0\}$, $n \geqslant 3$.
  - $\rightsquigarrow$ local connectivity of ∂B_∞ due to **Qiu–Wang–Yin**[10].

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[10] Dynamics of McMullen maps. Adv. Math. **229** (2012) 2525–2577.

: When can this strategy be successfully applied?

- **Rational maps with totally invariant infinitely connected Fatou component**.
  - $\rightsquigarrow$ proof of the Branner–Hubbard conjecture by **Qiu–Yin**[4]; absence of invariant line fields by **Yin–Zhai**[5]; ergodicity properties by **Zhai**[6].
- **Non-renormalizable polynomials** of arbitrary degree w/o neutral periodic points.
  - $\rightsquigarrow$ quasiconformal rigidity, absence of invariant line fields, triviality of fibers due to **Avila–Kahn–Lyubich–Shen**[7] (unicritical case) and **Kozlovski–van Strien**[8] (multicritical case).
- **Bounded Fatou components of polynomials** of arbitrary degree (except Siegel).
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- **Newton maps of polynomials** of arbitrary degree.

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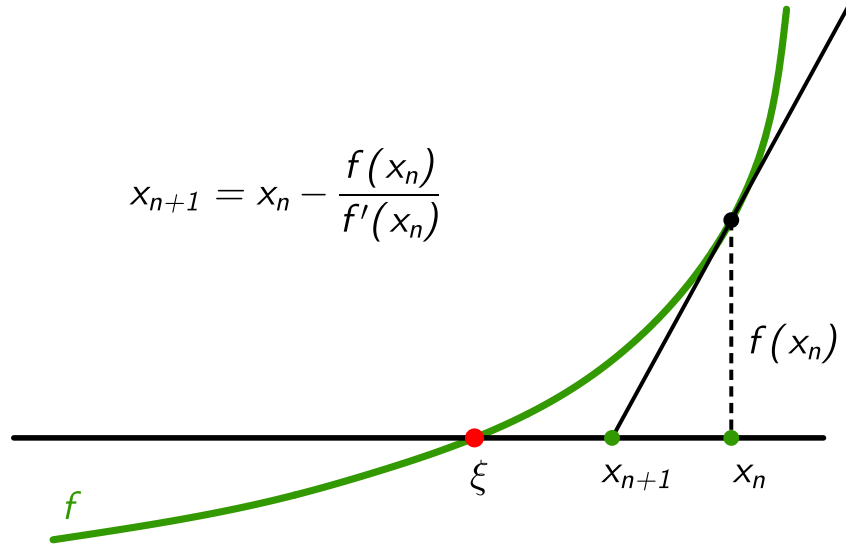
Newton maps

Newton's root-finding method:



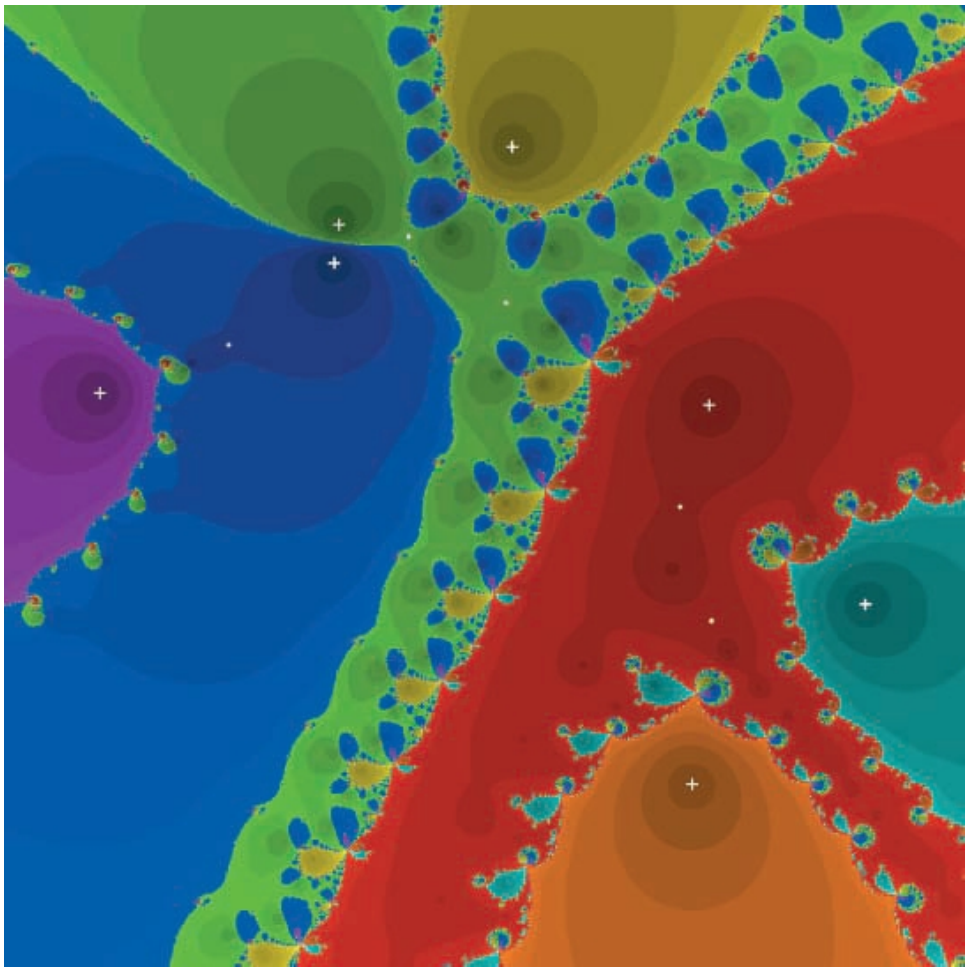
$p\colon \mathbb{C} \to \mathbb{C}$ is a complex polynomial. The **Newton map of** p is the rational map $N_p\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined as

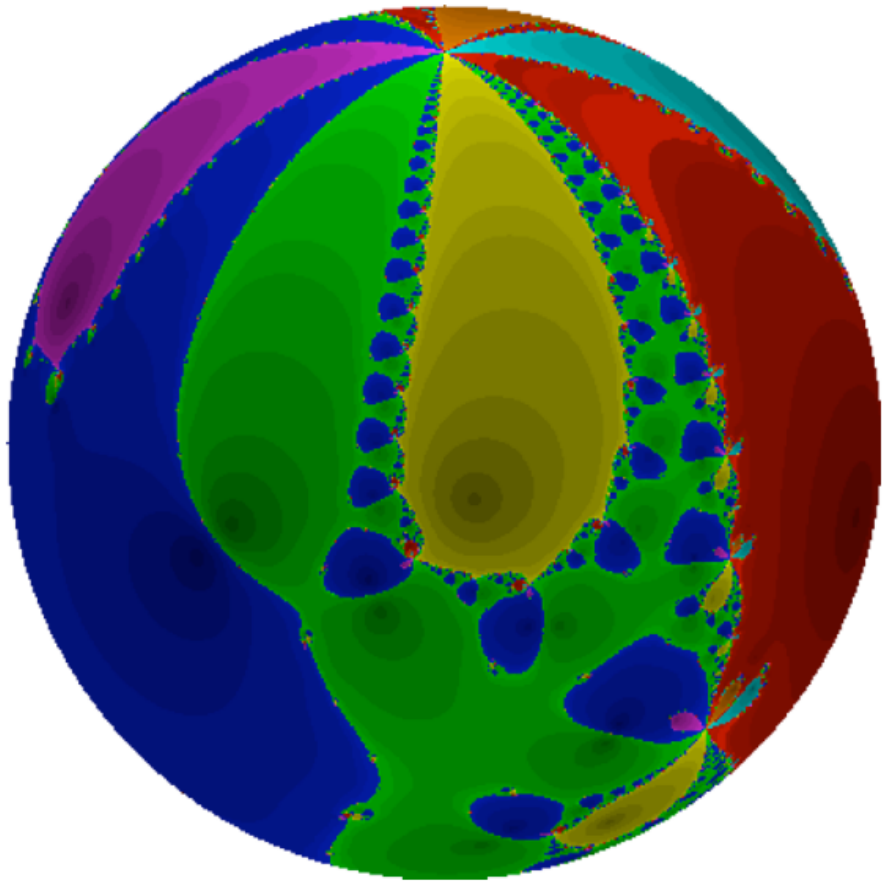
$$N_p(z) := z - \frac{p(z)}{p'(z)}.$$

Fixed points in $\widehat{\mathbb{C}}$:
$N_p(z) = z \Leftrightarrow z = \infty$ (repelling), or z is a root of p (attracting or superattracting).

By Przytycki, each component of root basins is simply connected.
By Shishikura, $J(N_p)$ is connected.

Newton dynamical plane: degree 7 example



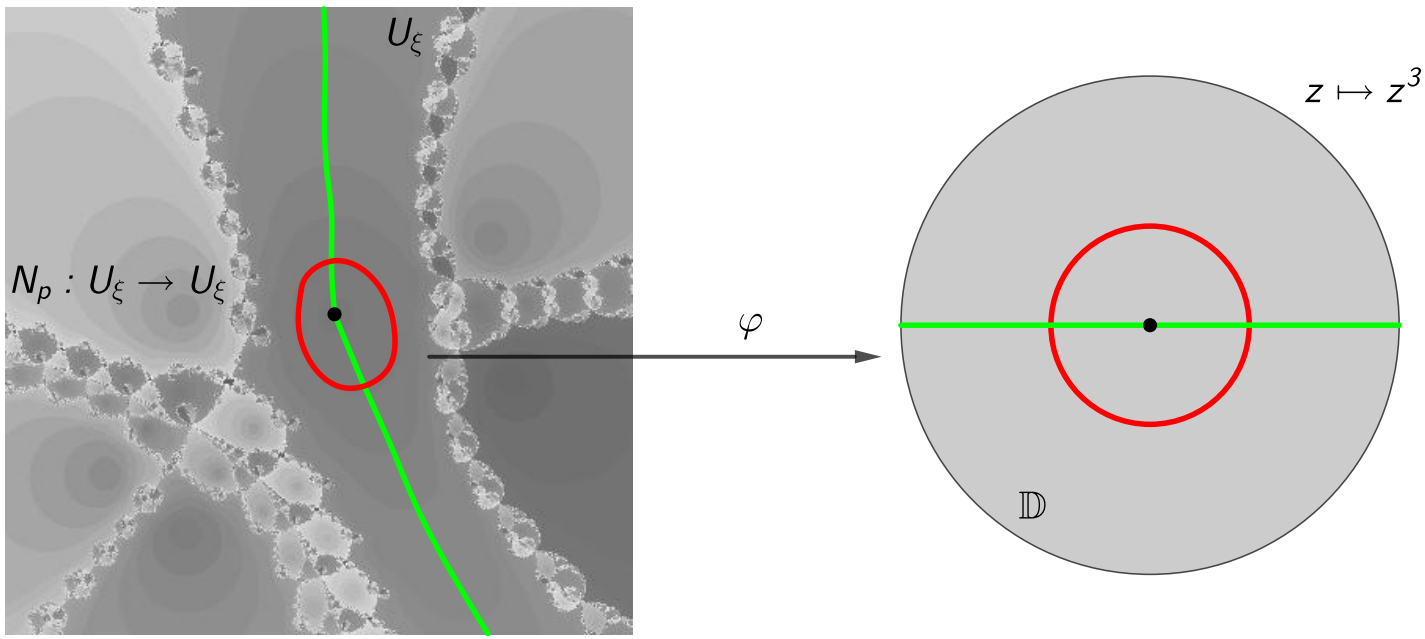


[0]The picture is courtesy of John Hubbard, Dierk Schleicher, Scott Sutherland.

Newton puzzles: outline

- Assume that N_p is *postcritically finite* in the basins of roots (wlog). Let U_ξ be the immediate basin of a root ξ.

  ξ is unique critical pt in $U_\xi \Rightarrow N_p\colon U_\xi \to U_\xi$ conjugate to $z \mapsto z^d$ on $\mathbb{D}$ $\Rightarrow$ can define equipotentials and internal rays in U_ξ.



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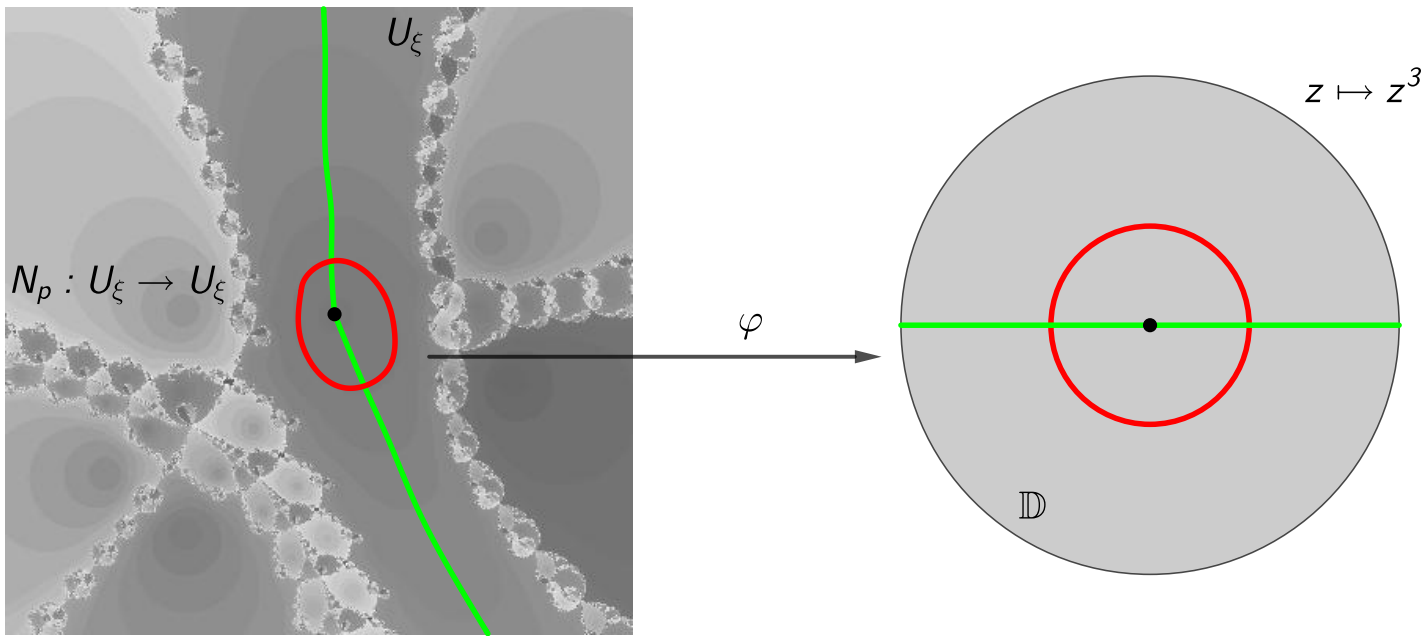
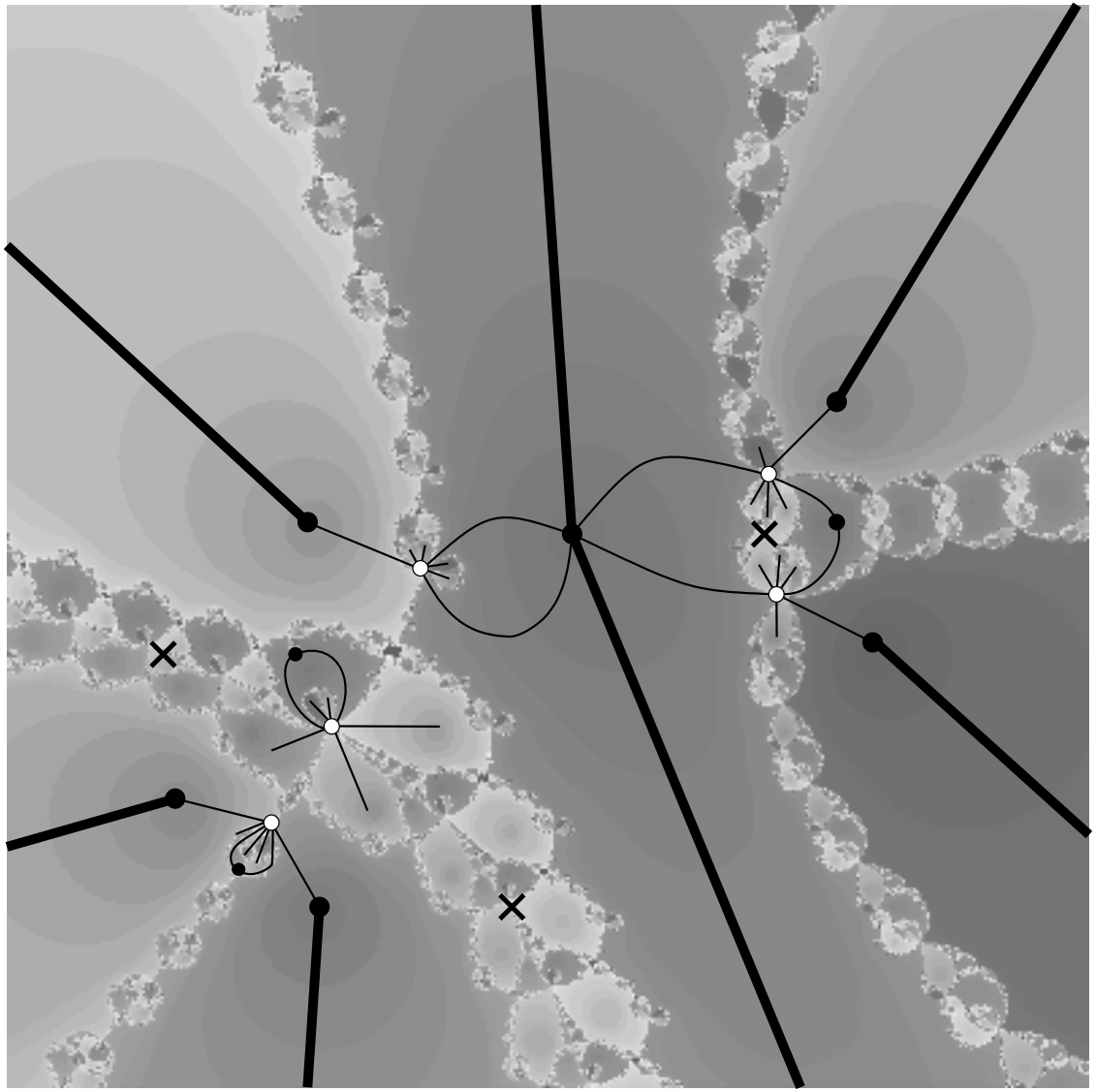


- **Channel diagram** $\Delta :=$ the union of all fixed internal rays over all U_ξ's. Δ connects all roots to ∞, $N_p(\Delta) \subseteq \Delta \Rightarrow$ components of $\widehat{\mathbb{C}} \setminus N_p^{-n}(\Delta)$ are **almost** puzzle pieces: $N_p^{-n}(\Delta)$ might be disconnected!

$N_p^{-1}(\Delta)$



Newton puzzles: outline (cont.)

A **Newton graph** (at level n) := the component Δ_n of $N_p^{-n}(\Delta)$ containing ∞.
$\Delta_n \subset \Delta_{n+1}$.

Proposition (D–Mikulich–Rückert–Schleicher[11])

There exists $N > 0$ so that Δ_N contains all the poles of N_p.

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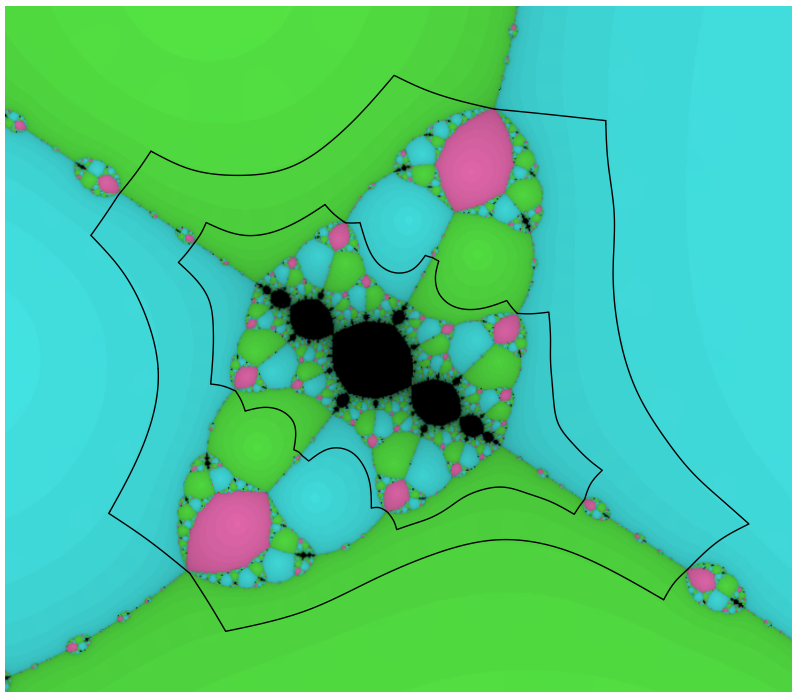
$\rightsquigarrow$ **Newton puzzle pieces**



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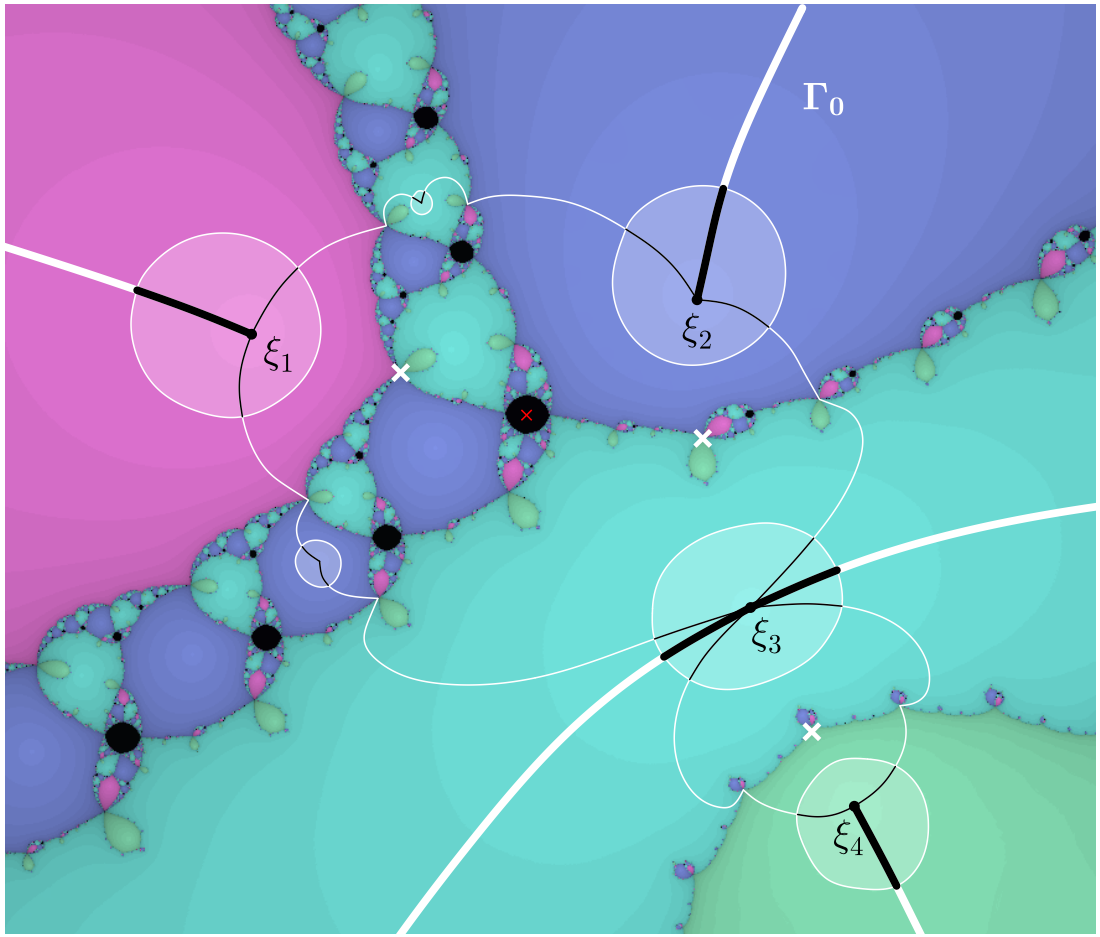
Triviality of fiber at ∞

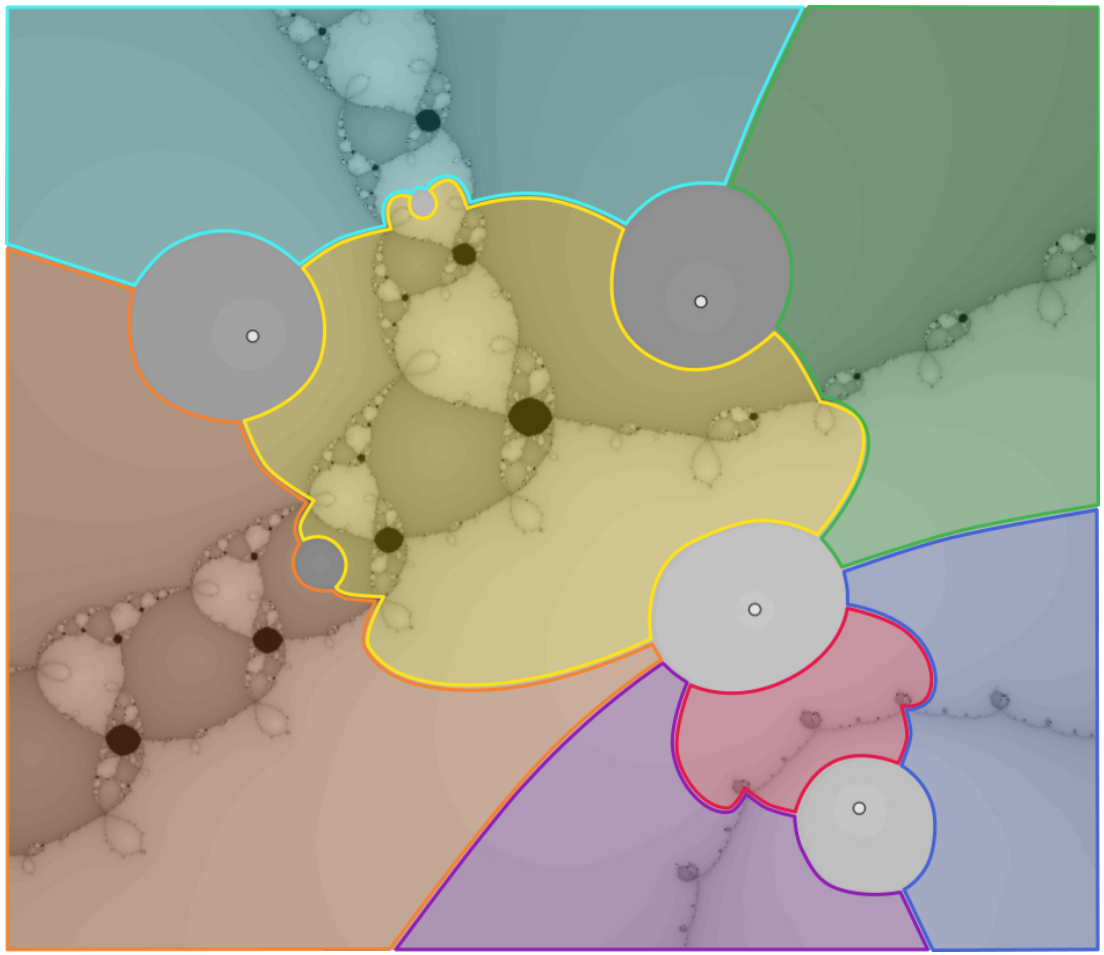
In [D–Lodge–Schleicher–Sowinski[12]] it was shown that the puzzle partition of zero depth can be chosen with good separation of critical values.



[12] Puzzles and the Fatou–Shishikura injection for rational Newton maps. Trans. AMS **374** (2021) 2753–2784.

Triviality of fiber at ∞ (cont.)

The 0-th depth Newton puzzle partition: all critical values, that are not in the basins of roots, are in the **yellow** and **magenta** puzzle pieces.



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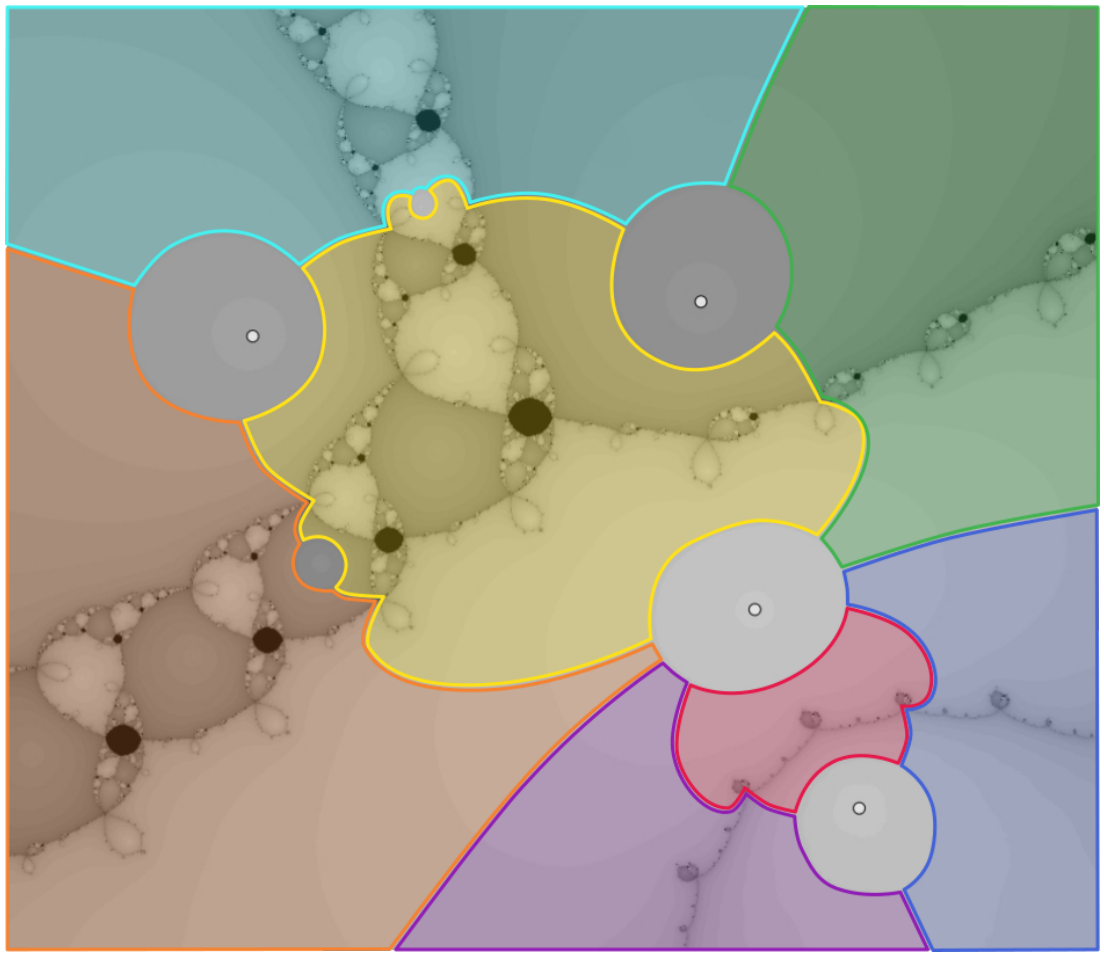

$\rightsquigarrow$ **Triviality of fibers at ∞ and its preimages** allows to prove that Newton maps are box-renormalizable.

⇒ Rigidity of Newton dynamics

Newton Dynamical Rigidity (D–Schleicher[13])

Let N_p be a polynomial Newton map of degree $d \geqslant 3$. Then for every point $z \in \widehat{\mathbb{C}}$ exactly one of the following alternatives holds true:

(B) *z belongs to the* **B***asin of attraction of a root of p;*

[13] Rigidity of Newton dynamics. arXiv:1812.11919 (31 Dec 2018).

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(R) *z belongs, or is mapped by some finite iterate, to the filled Julia set of **R**enormalizable dynamics.*

Previous work: **Roesch**[14](cubic Newton maps). **Independently:** in arbitrary degree **Wang–Yin–Zeng**[15](loc. conn. of the boundaries of the basins of roots).

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$\Rightarrow$ Rigidity of Newton dynamics (cont.)

Definition (Combinatorially equivalent Newton maps)

Two (suitably normalized) Newton maps are **combinatorially equivalent** if their Newton graphs coincide

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Moreover, if N_p and $N_{\widetilde{p}}$ are postcritically finite in the root basins, then N_p and $N_{\widetilde{p}}$ are affine conjugate.

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Parallel work: **Roesch–Yin–Zeng**[17](parameter rigidity for non-renormalizable Newton maps).

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Take-away: philosophy on rigidity

- **Dynamical rigidity:** a holomorphic map f is **rigid** if one can distinguish, *in combinatorial terms, e.g. via symbolic dynamics*, all orbits of f.

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Rational Rigidity Principle:

(dynamical rigidity) a rational map is either *rigid*, or it contains an *embedded polynomial dynamics*;

(parameter rigidity) a family of rational maps is *rigid* provided it contains no embedded polynomial dynamics, or these dynamics are embedded in "the same way"

(except some well-understood non-rigid families, e.g. flexible Lattés maps).

Thank you for your attention!