DYNAMICS ON THE BOUNDARY OF FATOU COMPONENTS

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On geometric complexity of Julia sets III
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INTRODUCTION TO HOLOMORPHIC ITERATION

$f\colon S \to S$ holomorphic, $S = \mathbb{C}$ or $S = \widehat{\mathbb{C}}$.

$$f^n = f \circ \overset{n}{\cdots} \circ f$$

Totally invariant partition of S:

Fatou set: Set of stability (normality). Open. $\mathcal{F}(f)$.

Julia set: Chaotic set. Closed. $\mathcal{J}(f) = S \setminus \mathcal{F}(f)$.

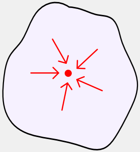
Escaping set: points which escape to ∞. $\mathcal{I}(f)$.

Fatou components: connected components of the Fatou set.

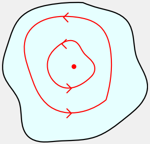
FATOU COMPONENTS

THEOREM (Fatou)

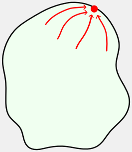
U simply-connected invariant Fatou component. Possibilities:



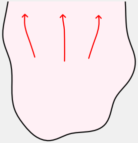
1. $f^n_{|U} \to z_0 \in U$
Attracting basin
$|f'(z_0)| < 1$



3. $f_{|U} \sim e^{2\pi i\theta} z$, $\theta \notin \mathbb{Q}$
Siegel disk

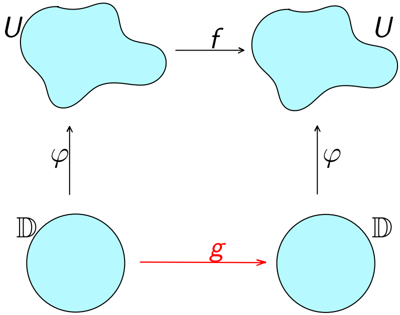


2. $f^n_{|U} \to z_0 \in \partial U$
Parabolic basin
$f'(z_0) = 1$



4. f transcendental,
$f^n_{|U} \to \infty$
Baker domain

DYNAMICS INSIDE A FATOU COMPONENT



$\varphi\colon \mathbb{D} \to U$ (Riemann map) and $f_{|U} \sim g$, where $g\colon \mathbb{D} \to \mathbb{D}$ holomorphic

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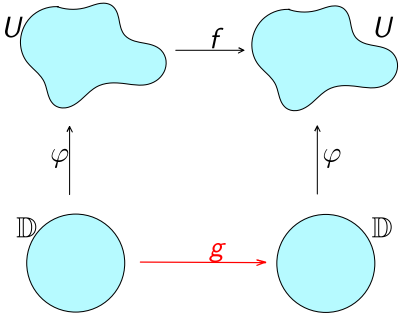


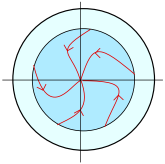
$\varphi\colon \mathbb{D} \to U$ (Riemann map) and $f_{|U} \sim g$, where $g\colon \mathbb{D} \to \mathbb{D}$ holomorphic

Tools to study the dynamics of $g\colon \mathbb{D} \to \mathbb{D}$ holomorphic:

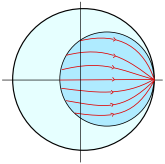
- Denjoy-Wolff Theorem
  If g is not a rotation, all orbits converge to the same point $p \in \overline{\mathbb{D}}$.
- Cowen's classification

DYNAMICS OF $g: \mathbb{D} \to \mathbb{D}$. Cowen's classification

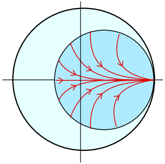
Assume g is holomorphic and not conjugate to a rotation.
Then, there exists an **absorbing domain** where g is conjugate to $\phi\colon \Omega \to \Omega$ (Möbius).



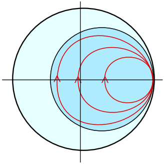
1. $\Omega = \mathbb{C}$
$\phi(z) = \lambda z$, $|\lambda| < 1$.
(elliptic)



3. $\Omega = \mathbb{H}$
$\phi(z) = \lambda z$, $\lambda > 1$.
(hyperbolic)

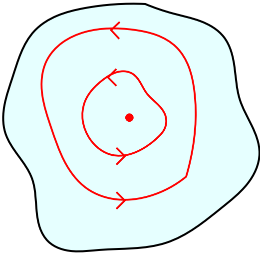


2. $\Omega = \mathbb{C}$
$\phi(z) = z + 1$.
(doubly-parabolic)

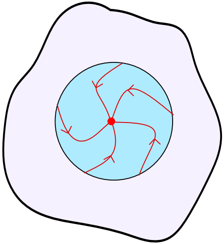


4. $\Omega = \mathbb{H}$
$\phi(z) = z \pm 1$.
(simply-parabolic)

DYNAMICS INSIDE A FATOU COMPONENT



(a) Siegel disk
(irrational rotation)



(b) Attracting basin
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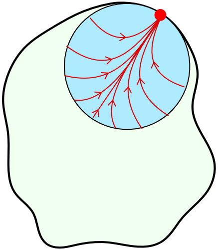

(c) Parabolic basin
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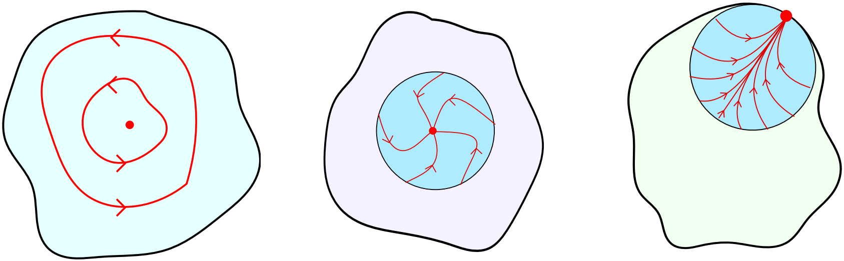

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(b) Attracting basin (elliptic)

(c) Parabolic basin (doubly-parabolic)

For Baker domains, doubly-parabolic, hyperbolic and simply-parabolic types are possible

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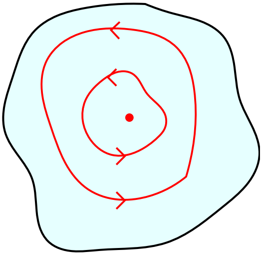

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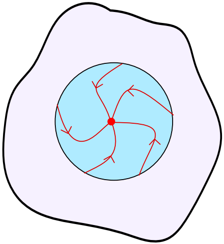

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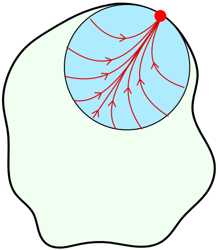

(c) Parabolic basin
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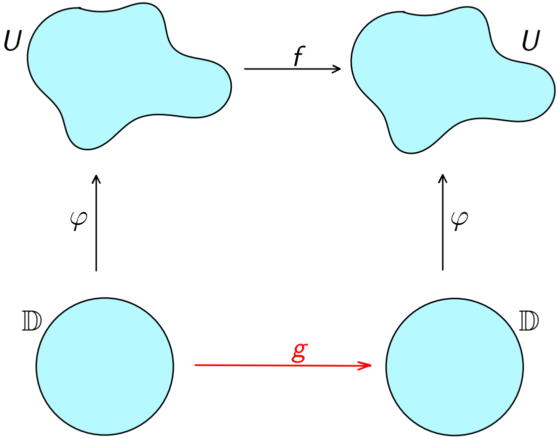
For Baker domains, doubly-parabolic, hyperbolic and simply-parabolic types are possible $\rightsquigarrow$ classification of Baker domains

QUESTION: Dynamics on ∂U?



Intuitive idea: study $g_{|\partial\mathbb{D}}$.

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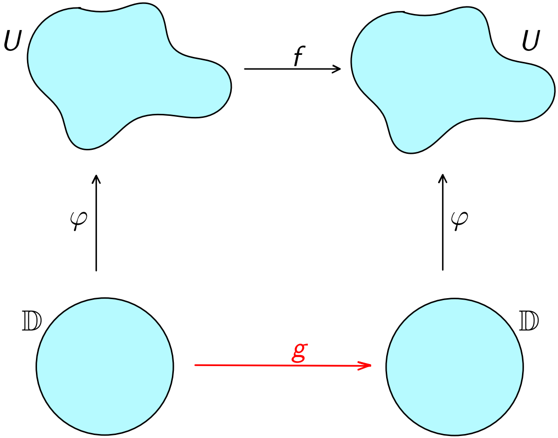

Intuitive idea: study $g_{|\partial\mathbb{D}}$.

But g and φ may not be defined on $\partial\mathbb{D}$...

INNER FUNCTIONS

DEFINITION: Radial limit

Let $g\colon \mathbb{D} \to \mathbb{D}$ holomorphic, $e^{i\theta} \in \partial\mathbb{D}$. The **radial limit** of g at $e^{i\theta}$ is:

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g^* induces a dynamical system almost everywhere on $\partial\mathbb{D}$.

ERGODICITY AND RECURRENCE

Ergodic properties of measurable maps

Let $(X, \mathcal{A}, \mu)$ be a measure space and $T\colon X \to X$ measurable. Then we say that T is:

- **ergodic**, if for every $A \in \mathcal{A}$ such that $T^{-1}(A) = A$, there holds $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.
- **recurrent**, if for every $A \in \mathcal{A}$ and μ-almost every $x \in A$, $T^n(x) \in A$ for infinitely many n's.

[1] General result in ergodic theory. A proof can be found in Aaronson. *Introduction to Infinite Ergodic Theory.*

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THEOREM[1]

If T is ergodic and recurrent with respect to the Lebesgue measure, then Lebesgue-almost every point has a dense orbit.

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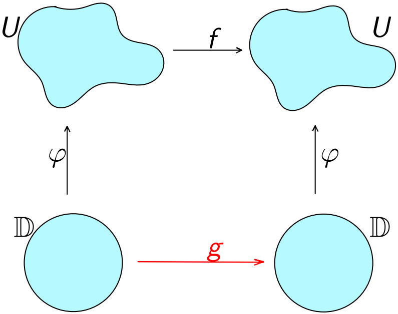
QUESTION: Dynamics on ∂U?

Measure on ∂U. The harmonic measure



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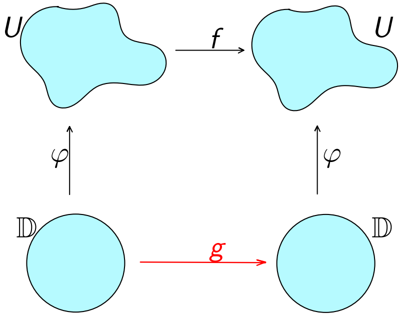

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DEFINITION: Harmonic measure

Let $U \subset \widehat{\mathbb{C}}$ be simply-connected and let $\varphi \colon \mathbb{D} \to U$ be a Riemann map, such that $\varphi(0) = z \in U$. The **harmonic measure** ω of ∂U with base point z is the image under φ of the normalized Lebesgue measure of $\partial\mathbb{D}$.

With this measure, we only need to study $g^* \colon \partial\mathbb{D} \to \partial\mathbb{D}$.

ERGODIC PROPERTIES OF INNER FUNCTIONS

INNER FUNCTION	FATOU COMPONENT	Ergodicity	Recurrence
Rational rotation		✗	✓
Irrational rotation	Siegel disk	✓	✓
Elliptic *	Attracting basin	✓	✓
Doubly-parabolic *	Parabolic b./Baker d.	✓	?
Hyperbolic	Baker domain	✗	✗
Simply-parabolic	Baker domain	✗	✗

* In case of degree $d < \infty$, the boundary map is conjugate to $x \mapsto dx \mod 1$.

Summary of different results in:

Aaronson. *Ergodic theory for inner functions of the upper half plane.*
Aaronson. *A remark on the exactness of inner functions.*
Barański, Fagella, Jarque, Karpińska. *Escaping points in the boundaries of Baker domains.*
Bourdon, Matache, Shapiro. *On the convergence to the Denjoy-Wolff point.*
Doering, Mañé. *The dynamics of inner functions.*
Hamilton. *Absolutely continuous conjugacies of Blaschke products.*
Shub, Sullivan. *Expanding endomorphisms of the circle revisited.*

THE EXAMPLE: $f(z) = z + e^{-z}$

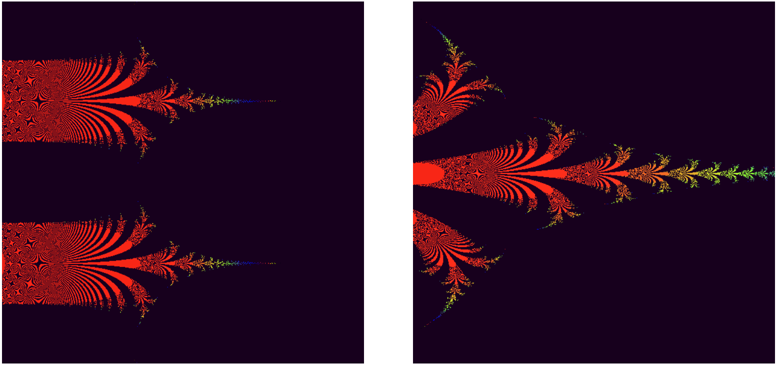

Figure: On the left, the dynamical plane of $f(z) = z + e^{-z}$. On the right, a zoom of it.

Previously studied in:

Baker, Domínguez. *Boundaries of unbounded Fatou components of entire functions.*
Fagella, Henriksen. *Deformation of entire functions with Baker domains.*
Barański, Fagella, Jarque, Karpińska. *Escaping points in the boundaries of Baker domains.*

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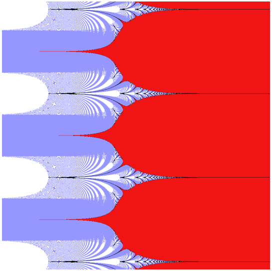
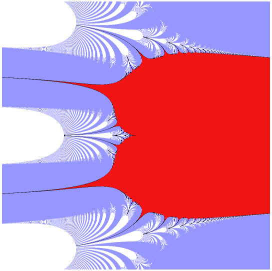
Semiconjugacy to $h(w) = we^{-w}$



$\xrightarrow{w=e^{-z}}$



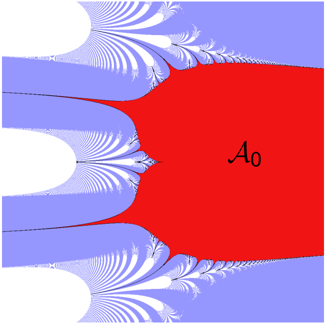
$z \mapsto f(z) = z + e^{-z}$ $\qquad$ $w \mapsto h(w) = we^{-w}$

$$\begin{array}{ccc} z & \xrightarrow{\quad f \quad} & f(z) = z + e^{-z} \\ {\scriptstyle w=e^{-z}} \big\downarrow & & \big\downarrow {\scriptstyle w=e^{-z}} \\ w & \xrightarrow{\quad h \quad} & h(w) = we^{-w} \end{array}$$

* Figures courtesy of Christian Henriksen

THE EXAMPLE: $f(z) = z + e^{-z}$

The parabolic basin of $h(w) = we^{-w}$



- 0 is a parabolic fixed point for h
- Singular values: 0, $\frac{1}{e}$
- $h^n(\frac{1}{e}) \to 0$, as $n \to \infty$
- $\mathcal{F}(h) = \mathcal{A}$, parabolic basin of 0
- $\mathcal{A}_0$, immediate parabolic basin

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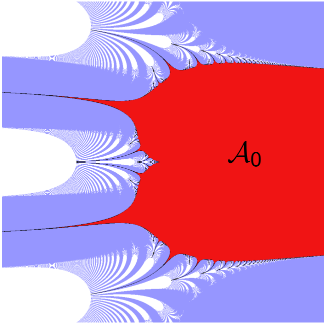
The parabolic basin of $h(w) = we^{-w}$



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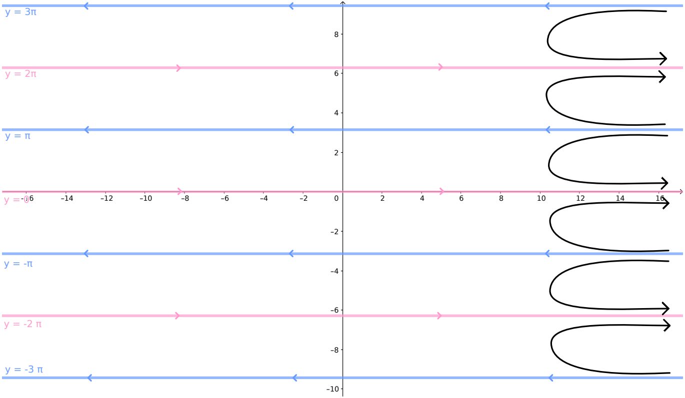
THEOREM (Baker-Domínguez, Fagella-Henriksen)

- $\mathbb{R}_+ \subset \mathcal{A}_0$, so $\mathcal{A}_0$ is unbounded
- $\mathbb{R}_- \subset \mathcal{J}(h)$
- h has degree 2 on $\mathcal{A}_0$ and $h_{|\mathcal{A}_0} \sim g(z) = \frac{3z^2+1}{z^2+3}$ (**doubly-parabolic**)

THE EXAMPLE: $f(z) = z + e^{-z}$

The dynamical plane of f



- $f(z + 2k\pi i) = f(z) + 2k\pi i$, for all $z \in \mathbb{C}$
- The lines $\{\mathrm{Im}\ z = k\pi\}_{k\in\mathbb{Z}}$ are invariant
- In each strip $\{(2k-1)\pi < \mathrm{Im}\ z < (2k+1)\pi\}_{k\in\mathbb{Z}}$, there is one preimage of $\mathcal{A}_0$, which is a **doubly-parabolic Baker domain** U_k

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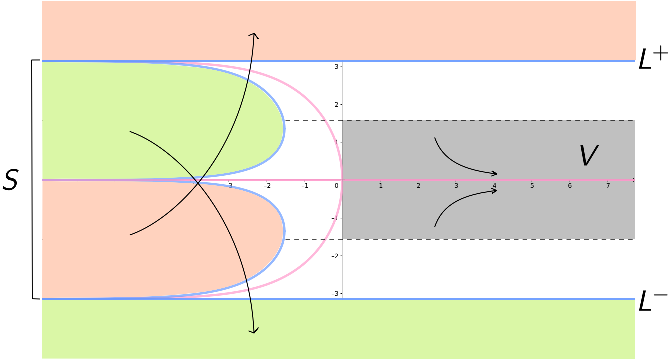
The dynamical plane of f



- $S := \{z \in \mathbb{C} \colon -\pi \leq \operatorname{Im} z \leq \pi\}$
- $f \colon f^{-1}(S) \cap S \to S$ proper map of degree 2
- $U := U_0 \subset S$, doubly-parabolic invariant Baker domain
- ω-almost every orbit is dense and $\mathcal{I}(f) \cap \partial U$ has zero measure

Goal: Study the boundary of the Baker domain U and its dynamics

THE EXAMPLE: $f(z) = z + e^{-z}$

Accesses to infinity from U

DEF: Accessible points and accesses

A point $v \in \partial U$ is **accessible** if there exists a curve $\gamma \subset U$ such that $\gamma(t) \to v$. A homotopy class (with fixed endpoints) of such curves is called an **access**.

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Accesses from U to ∞ are defined by the preimages of $\mathbb{R}_+$ under f.

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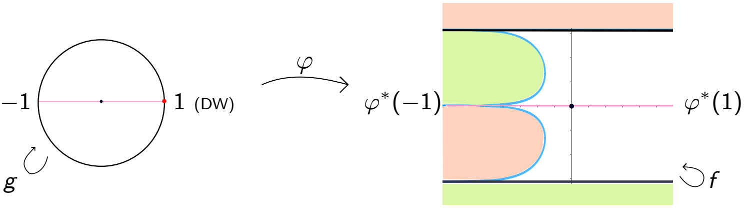
Idea of the proof: Fix $\varphi\colon \mathbb{D} \to U$ (Riemann) s.t. $\varphi(0) = 0$ and $\varphi(\mathbb{R} \cap \mathbb{D}) = \mathbb{R}$.



$$\{e^{i\theta} \in \partial\mathbb{D}\colon \varphi^*(e^{i\theta}) = \infty\} = \{e^{i\theta} \in \partial\mathbb{D}\colon g^n(e^{i\theta}) = 1\}$$

THE EXAMPLE: $f(z) = z + e^{-z}$

THEOREM

Let $z_0 \in \partial U$ be periodic under f, i.e. $f^p(z_0) = z_0$, for some p. Then z_0 is accessible.

THE EXAMPLE: $f(z) = z + e^{-z}$

Accessibility of periodic points

THEOREM

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THEOREM

Let $e^{i\theta} \in \partial \mathbb{D}$ be periodic under g, i.e. $g^p(e^{i\theta}) = e^{i\theta}$ for some $p > 1$. Then, $\varphi^*(e^{i\theta})$ exists and it is a periodic point of period p.

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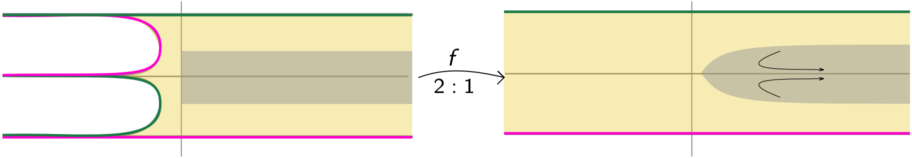
Let $e^{i\theta} \in \partial \mathbb{D}$ be periodic under g, i.e. $g^p(e^{i\theta}) = e^{i\theta}$ for some $p > 1$. Then, $\varphi^*(e^{i\theta})$ exists and it is a periodic point of period p.

Consequence: Characterization of periodic points in ∂U.
A point $z \in \partial U$ satisfies $f^p(z) = z$ for some $p \geq 1$ if, and only if, $z = \varphi^*(e^{i\theta})$ for some $e^{i\theta} \in \partial \mathbb{D}$ satisfying $g^p(e^{i\theta}) = e^{i\theta}$.

THE EXAMPLE: $f(z) = z + e^{-z}$

The escaping set in ∂U

$$S \coloneqq \{z \in \mathbb{C} \colon -\pi \leq \mathrm{Im}\ z \leq \pi\}$$



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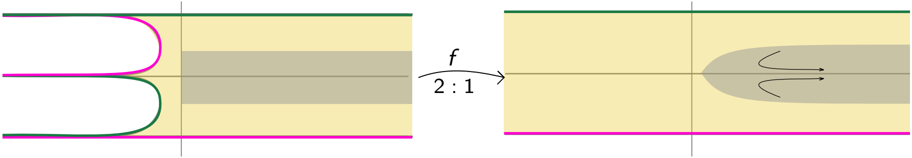

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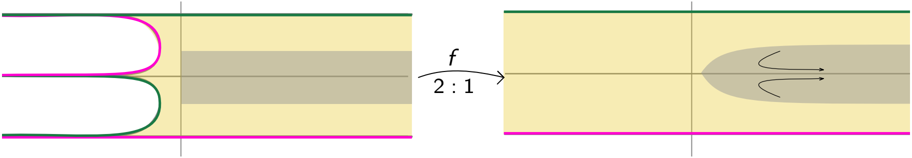

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- $U \subset \widehat{S}$ and $f_{|U}$ has degree 2

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The escaping set in ∂U

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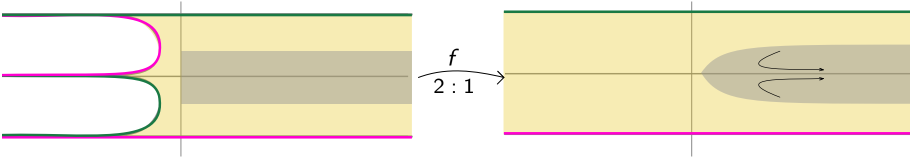

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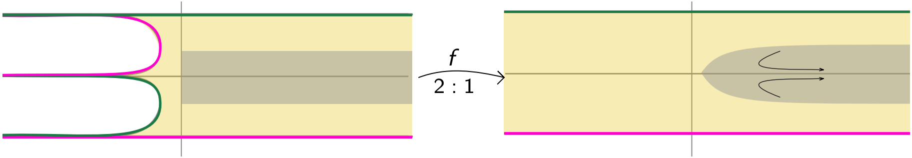

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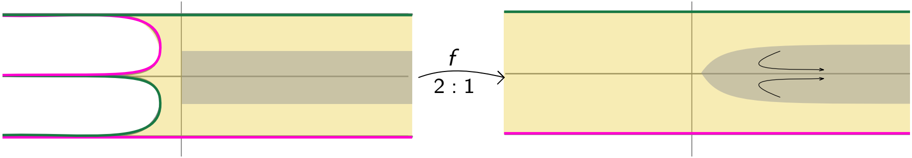

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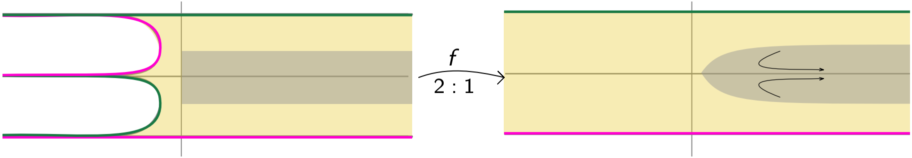

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Two ways of escaping to ∞ in $\widehat{S}$

$$\mathcal{I}_S^+ := \left\{ z \in \mathcal{I}(f) \cap \widehat{S} \colon \; \exists \{n_k\}_k \text{ s.t. } \mathrm{Re}\, f^{n_k}(z) \to +\infty \right\}$$

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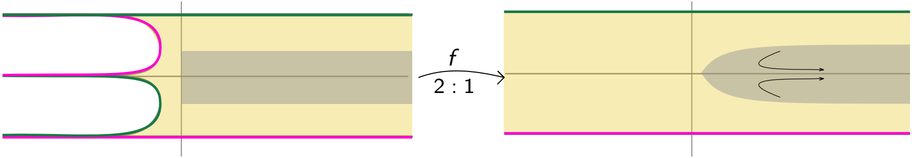

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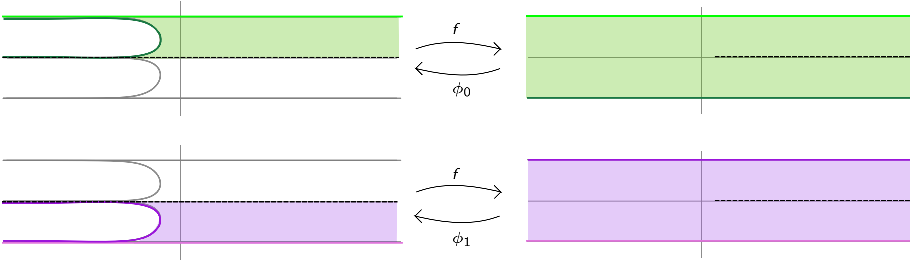

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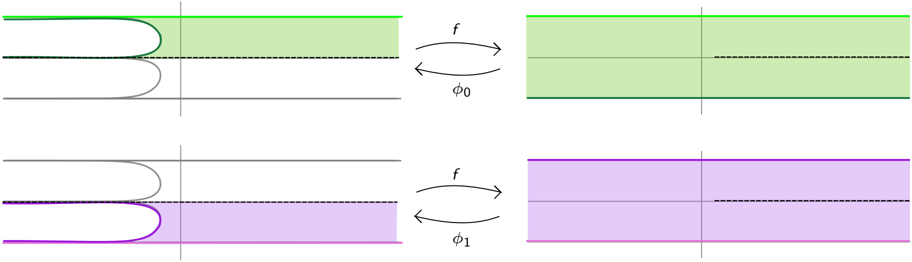

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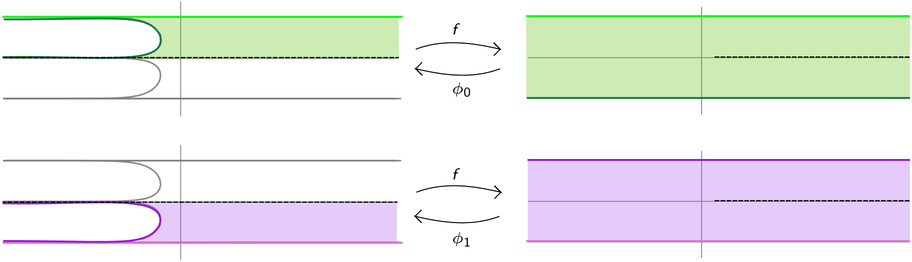

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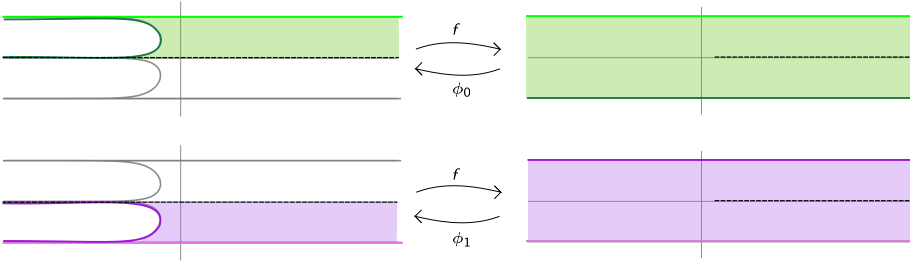

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THEOREM

For every sequence $k = \{k_j\}_j$, $k_j \in \{0, 1\}$, there exists a curve $\gamma_k \subset S$ whose points belong to $\mathcal{I}_S^-$, with itinerary k and $\gamma_k \subset \partial U$.

- Describing the **topology** of ∂U

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Further questions

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Thank you for your attention!!!