Quasisymmetric Uniformization, quasi-visual approximations, and Thurston maps

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joint with Mario Bonk and Mikhail Hlushchanka

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Quasiconformal geometry

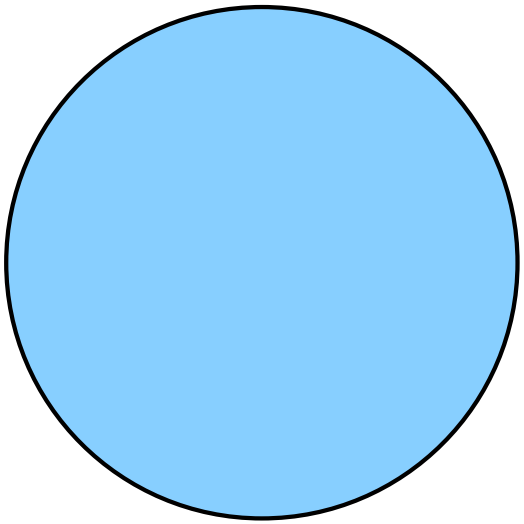
Quasiconformal geometry:

- study geometric properties invariant under scalings
- involve relative distances, i.e., ratios of distances
- appropriate for self-similar geometric objects (fractals in dynamics)

Doubling

Definition

A metric space X is doubling if there is N, s.t. any ball of radius r can be covered by N balls of radius $r/2$.



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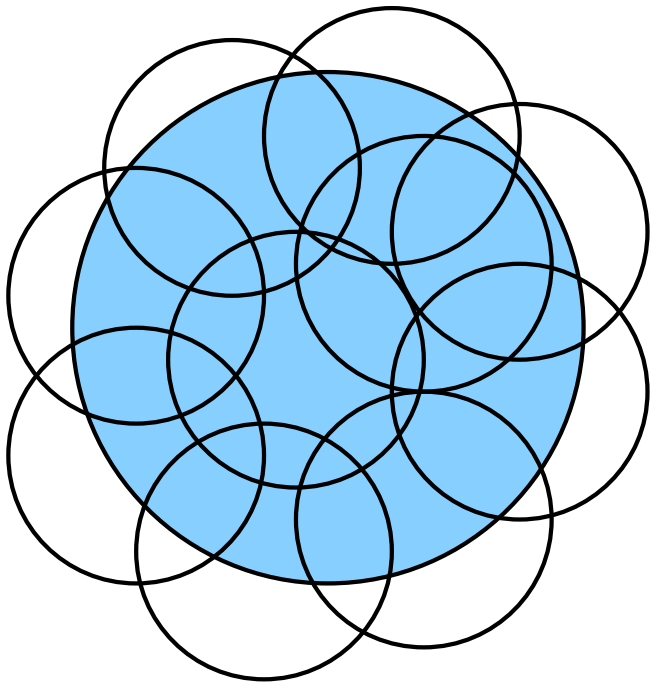

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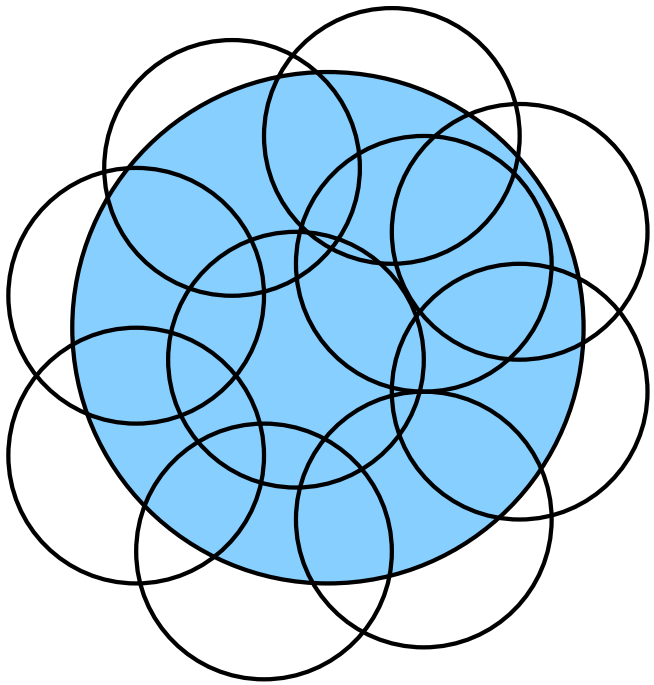
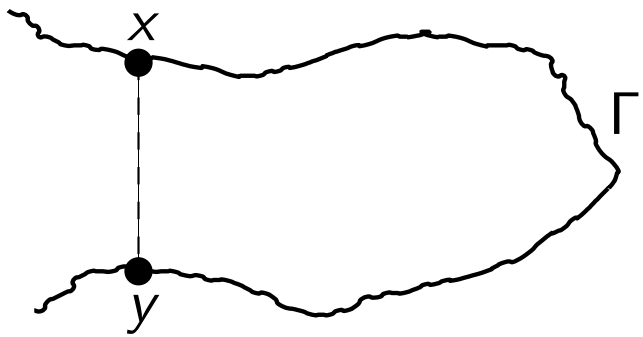


Examples: $\mathbb{R}^n$, $A \subset \mathbb{R}^n$.
Non-examples: infinite dimensional Banach space, hyperbolic space.
X doubling iff "Assouad dimension" finite.

Bounded turning

Metric space X bounded turning, if there is a constant $K \geq 1$, s.t. $\forall x, y \in X \ \exists$ continuum Γ with $x, y \in \Gamma$ and

$$\operatorname{diam} \Gamma \leq K|x - y|.$$



Quasisymmetry

Definition

$\varphi: X \to Y$ is a quasisymmetry if there is a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that

$$\frac{|x - z|}{|x - y|} \leq t \Rightarrow \frac{|x' - z'|}{|x' - y'|} \leq \eta(t),$$

for distinct $x, y, z \in X$, where $x' = \varphi(x), y' = \varphi(y), z' = \varphi(z)$.

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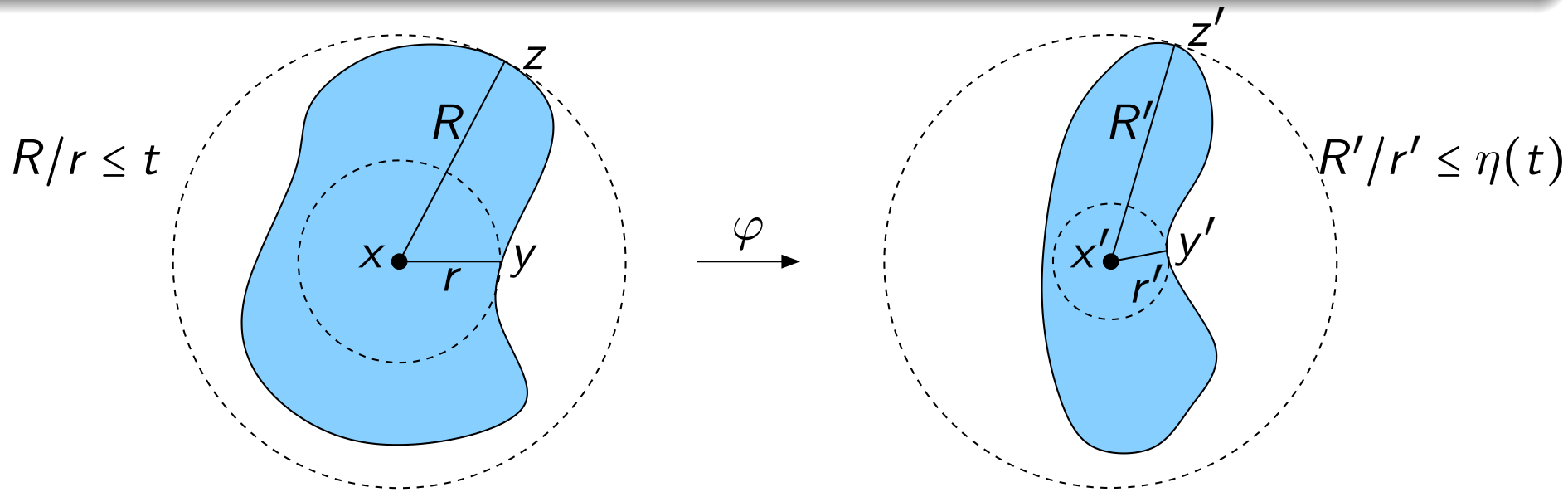

Quasisymmetry vs quasiconformality

quasisymmetry $\Rightarrow$ quasiconformality.
quasisymmetry $\Leftrightarrow$ quasiconformality in $\mathbb{R}^n$.
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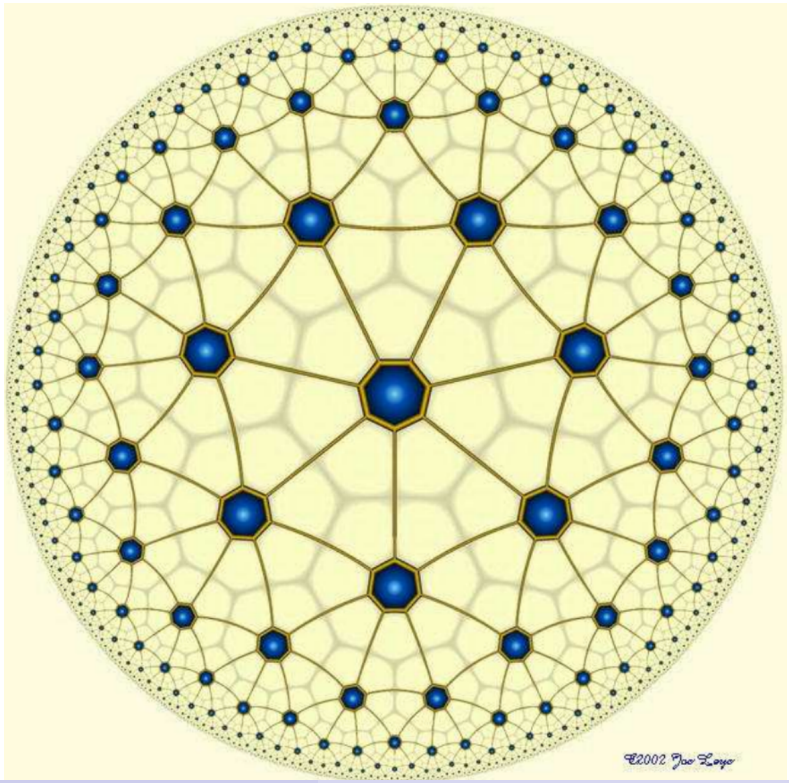
Appears in geometric group theory (boundaries of Gromov hyperbolic spaces/groups).
A quasi-isometry between proper and geodesic Gromov hyperbolic spaces induces a quasisymmetry on the boundary.



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When is a Sierpiński carpet qs to a round carpet?
Closely related to the Kapovic-Kleiner conjecture.
Bonk, Merenkov, Kleiner.

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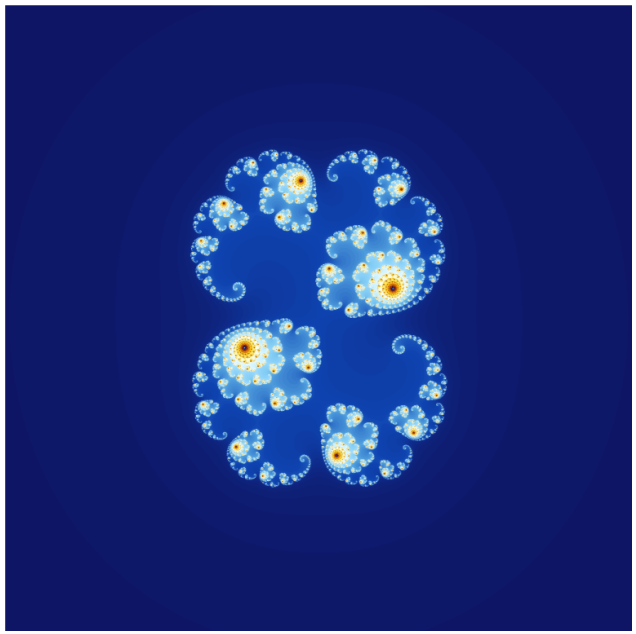
C qs to middle-third Cantor set (quasi-Cantor set) iff C compact, doubling, uniformly perfect, uniformly disconnected (David-Semmes '97).



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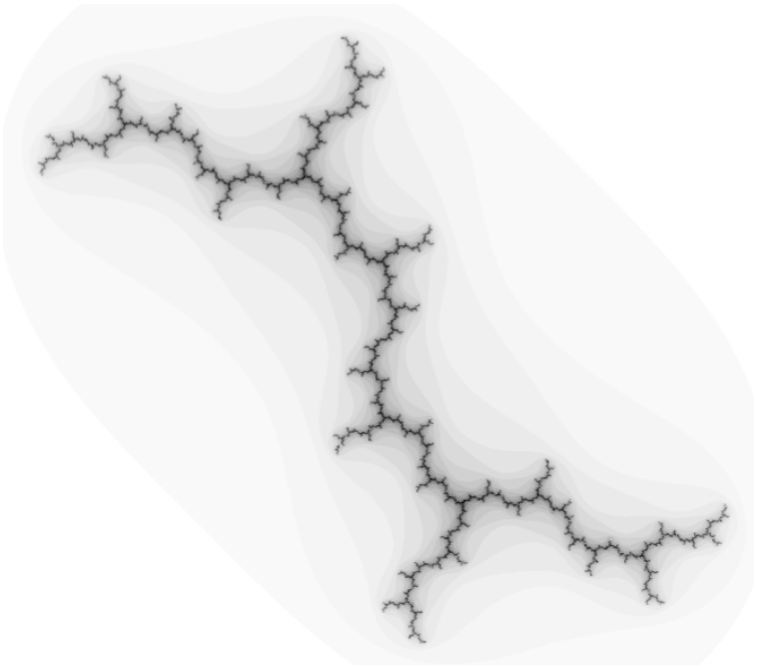

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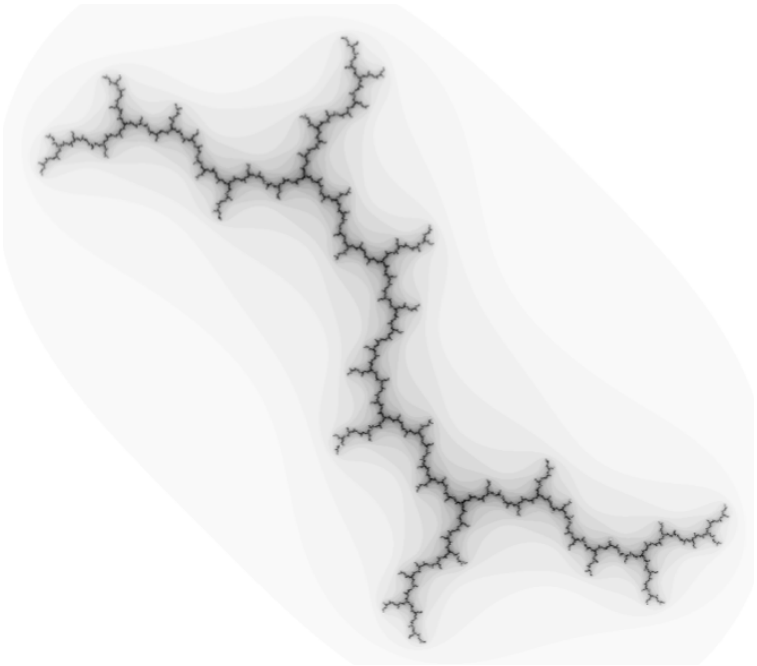

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Examples: Julia sets of certain polynomials.
Attractors of certain iterated function systems.

The continuum random tree is a random metric tree constructed from Brownian motion, closely related to the "Brownian map".

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Can "straighten" every arc in T simulaneously.

Note: arc that is doubling and bounded turning is quasi-tree.
Theorem generalization of Tukia-Väisälä.

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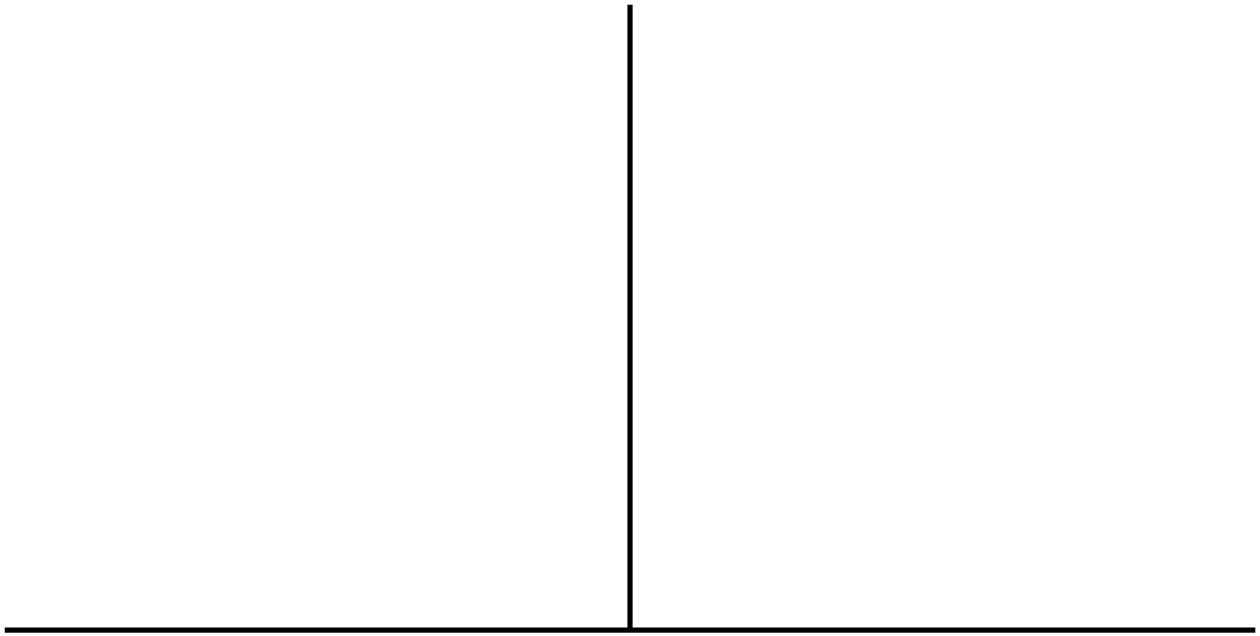

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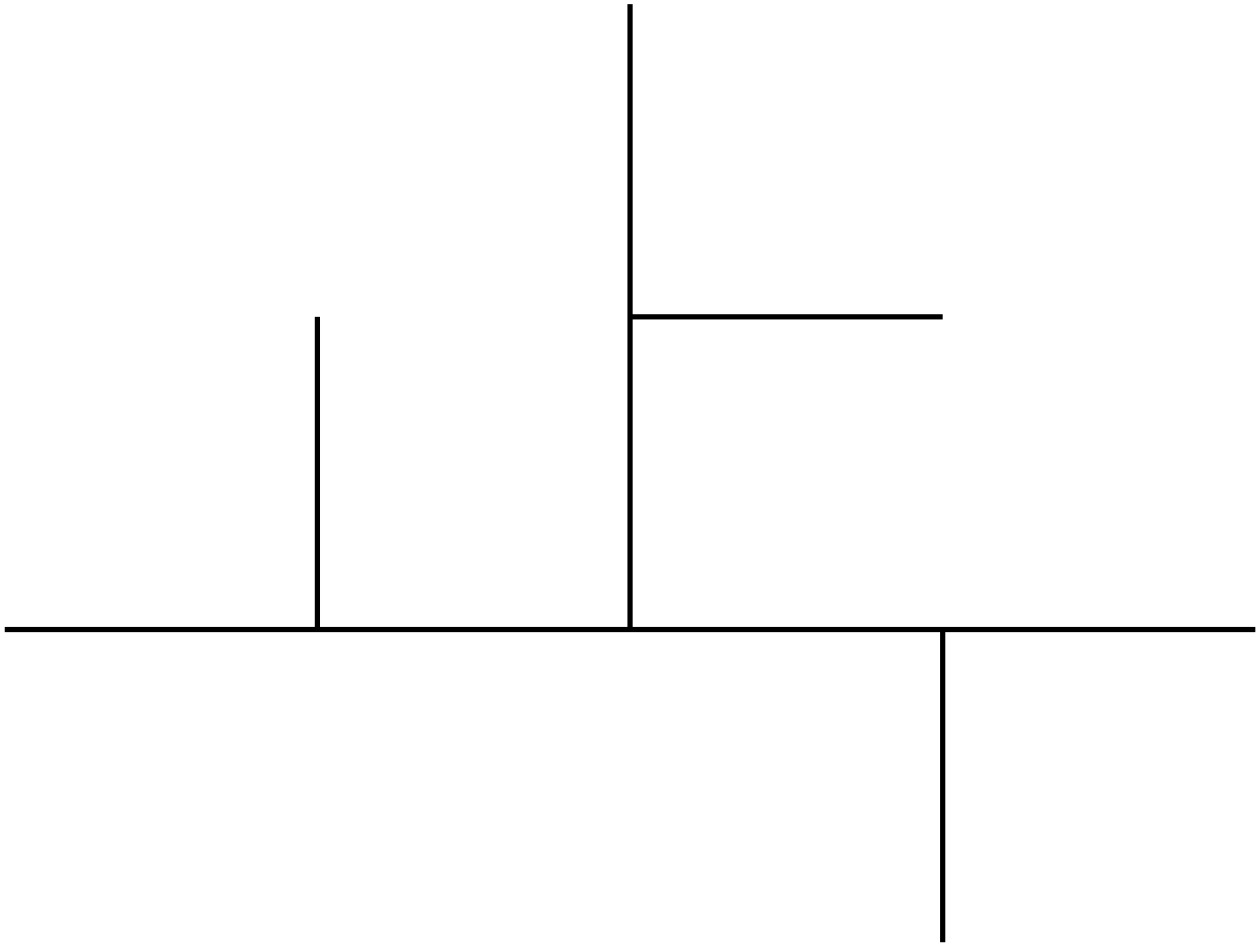

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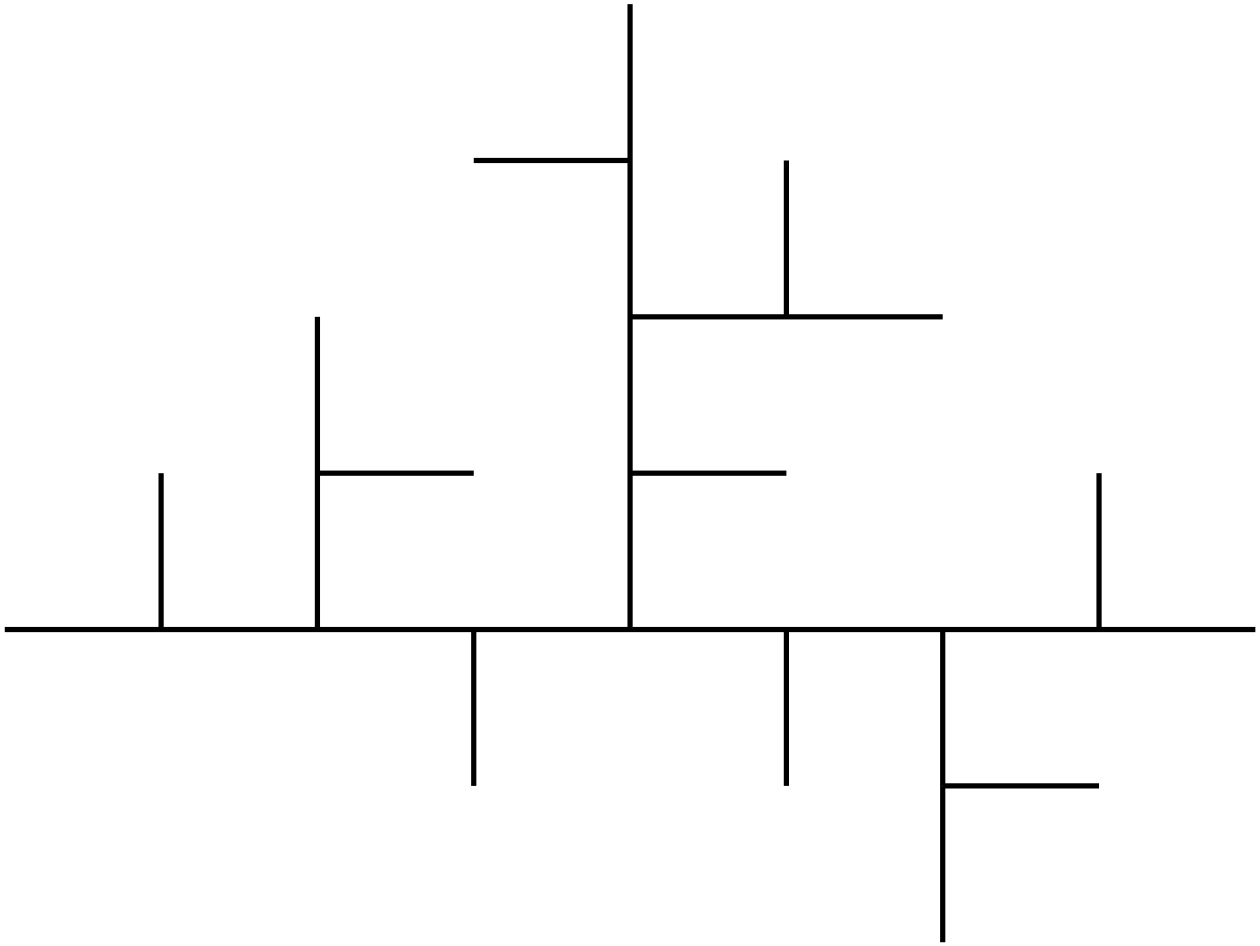

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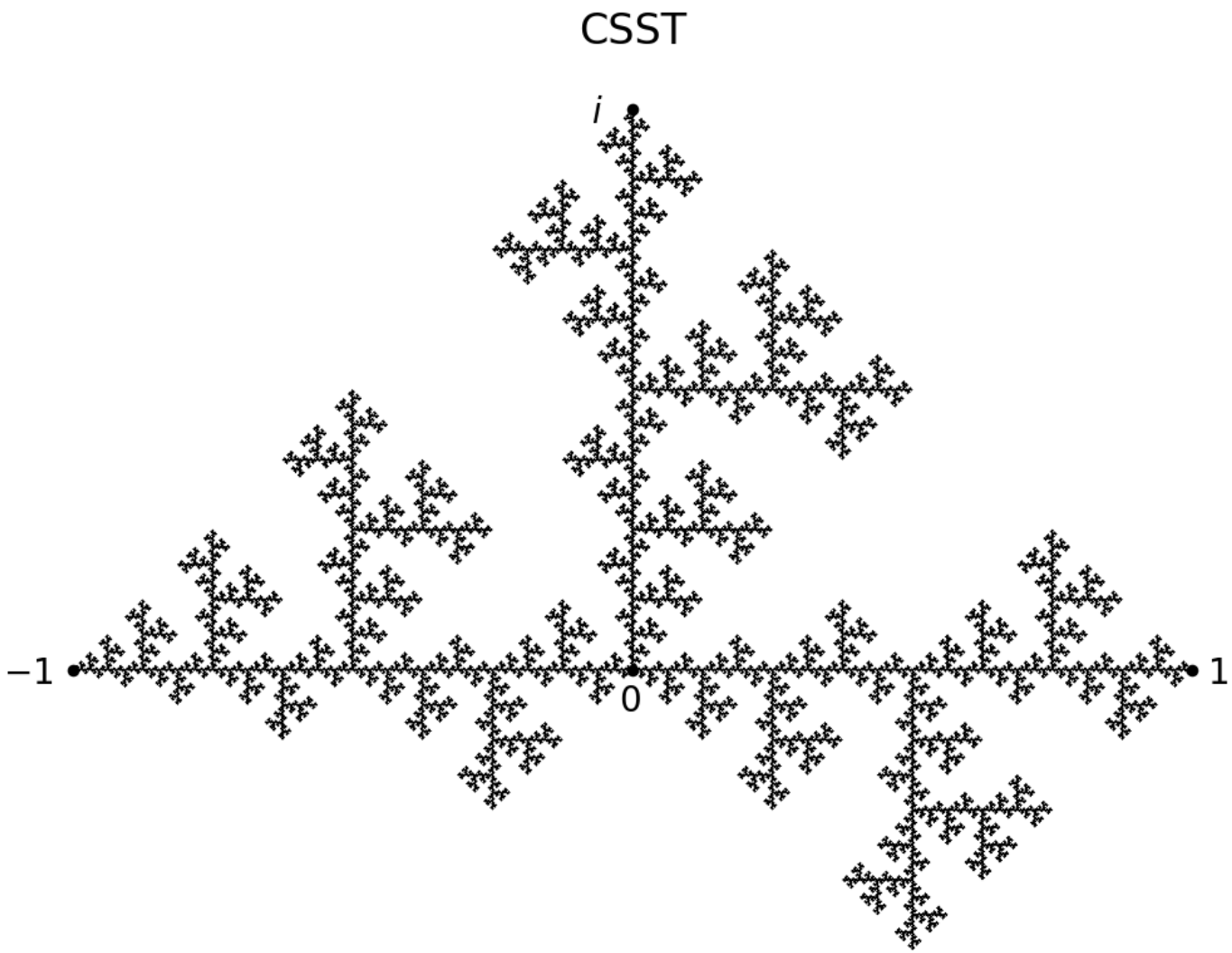

CSST $\mathbb{T}$

The CSST $\mathbb{T}$ is:

- the attractor of the following iterated function system (as a subset of the plane). Define

$$g_1(z) = \frac{1}{2}z - \frac{1}{2}, \quad g_2(z) = \frac{1}{2}\bar{z} + \frac{1}{2}, \quad g_3(z) = \frac{i}{2}\bar{z} + \frac{i}{2}.$$

  Then $\mathbb{T} \subset \mathbb{C}$ is the unique non-empty compact set satisfying

$$\mathbb{T} = g_1(\mathbb{T}) \cup g_2(\mathbb{T}) \cup g_3(\mathbb{T}).$$

  So $\mathbb{T}$ is the attractor of the iterated function system $\{g_1, g_2, g_3\}$ in the plane.

- a quasiconformal tree, bi-Lipschitz to a geodesic tree.

Quasisymmetric equivalence to CSST

$p \in T$ branch point, with components of $T \setminus \{p\}$:

$$\operatorname{diam}(U_1) \geq \operatorname{diam}(U_2) \geq \operatorname{diam}(U_3) \geq \dots .$$

The height of $p \in T$ is

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Both preserved under quasisymmetry.

T uniformly branching $:\Leftrightarrow$
uniformly relatively separated and uniformly dense.

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Theorem (Bonk-M. '21)

A metric space T is quasisymmetrically equivalent to the CSST if and only if
T is a uniformly branching quasiconformal tree where each branch point has degree 3.

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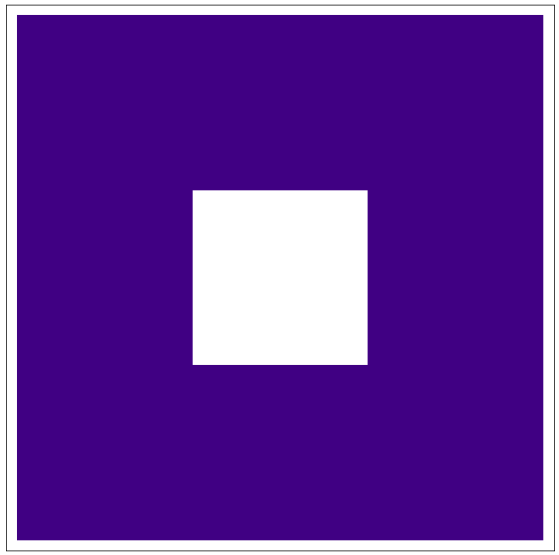
Corollary

T, T' uniformly branching quasiconformal trees, each branch point has degree 3. Then T, T' are quasisymmetrically equivalent.

Visual approximations

Many self-similar fractals S have natural decompositions by sets of size Λ^{-n}, for some fixed $\Lambda > 1$.



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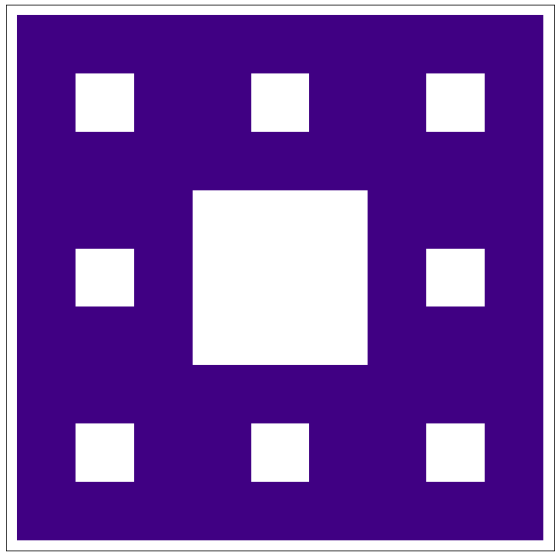

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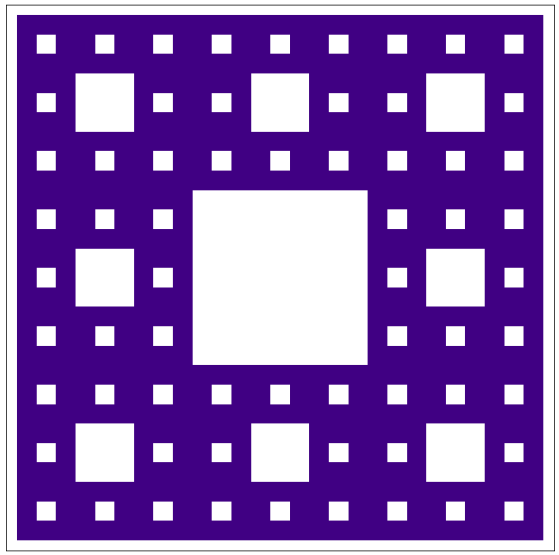

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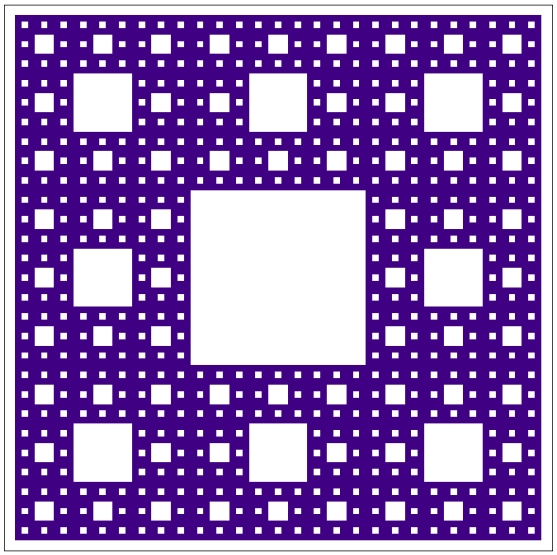

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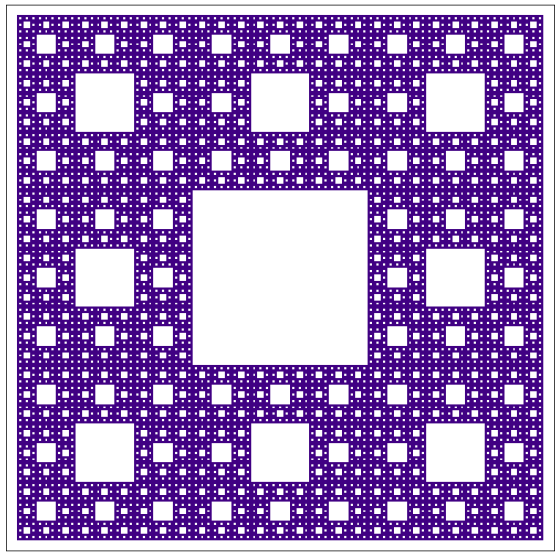

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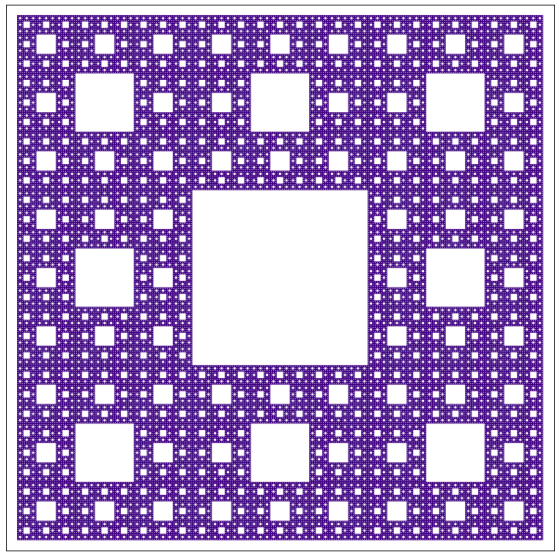

Have $\{X^n\}_{n\in\mathbb{N}_0}$, where X^n is a finite cover of S, and

1. $\operatorname{diam}(X) \asymp \Lambda^{-n}$ for all $X \in X^n$
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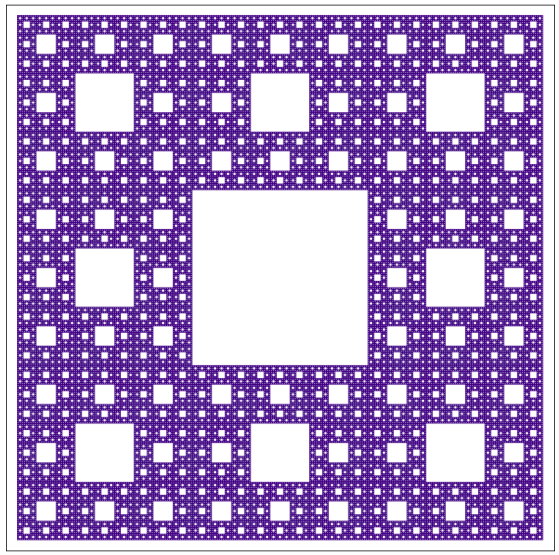

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$\{X^n\}$ visual approximation of S.

Every doubling, connected metric space has a visual approximation.

(Mackay '08, Bonk-Hlushchanka-M '21)

Expanding Thurston Maps

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A Thurston map $f: S^2 \to S^2$ is a branched cover of S^2, locally

$$z \mapsto z^d,$$

after *homeomorphic* coordinate changes, that is postcritically finite ($\#\,\mathrm{post}(f) < \infty$).

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$\mathcal{X}^0$ finite cover of S^2, consider preimage-covers $\mathcal{X}^n$ by f^n,

$$f^n(\mathcal{X}^n) = \mathcal{X}^0.$$

If $\max_{X \in \mathcal{X}^n} \mathrm{diam}(X) \to 0$ then f expanding.

Metric d on S^2 is visual if $\{\mathcal{X}^n\}$ is visual approximation with respect to d.

Existence of visual metrics

Visual metrics defined for expanding Thurston maps as before.

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Distinct visual metrics d, d' are snowflake equivalent

$$d(x, y) \asymp d'(x, y)^{\alpha},$$

for some $C, \alpha > 0$.

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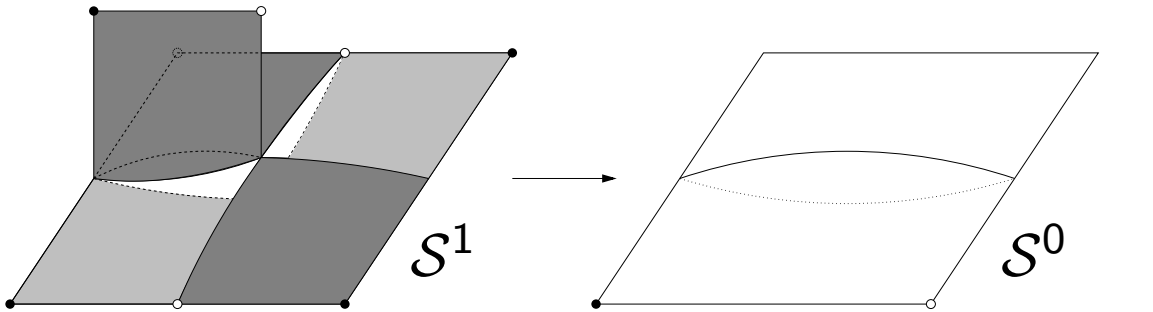
  $\Leftrightarrow f$ is a Lattès map (M. '09).
- (S^2, d) is doubling $\Leftrightarrow f$ has *no periodic critical points* (Bonk-M. '17).

  In fact then (S^2, d) is Ahlfors regular, i.e.,

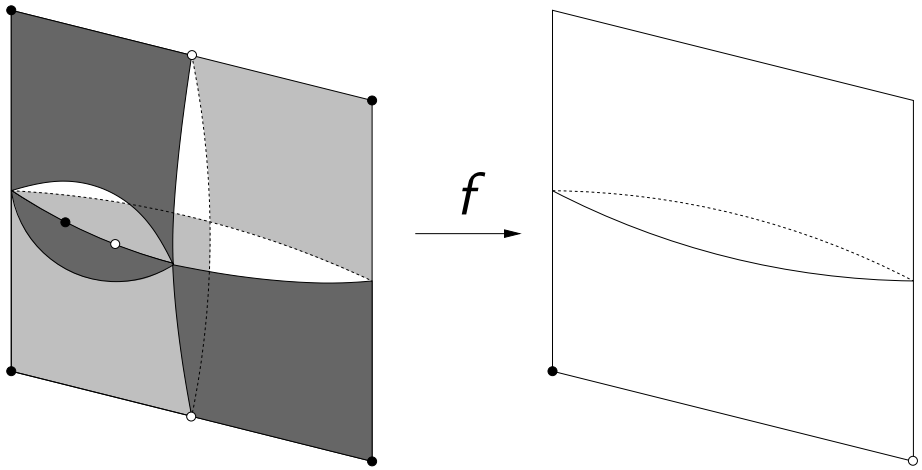
$$\mathcal{H}^Q(B_r) \asymp r^Q,$$

  $Q = \log(\deg(f))/\log(\Lambda) = \dim(S^2, d)$.

An example: Thurston map generates fractal sphere



$X^0 = \{\text{top, bottom}\}$



An obstructed Thurston map.

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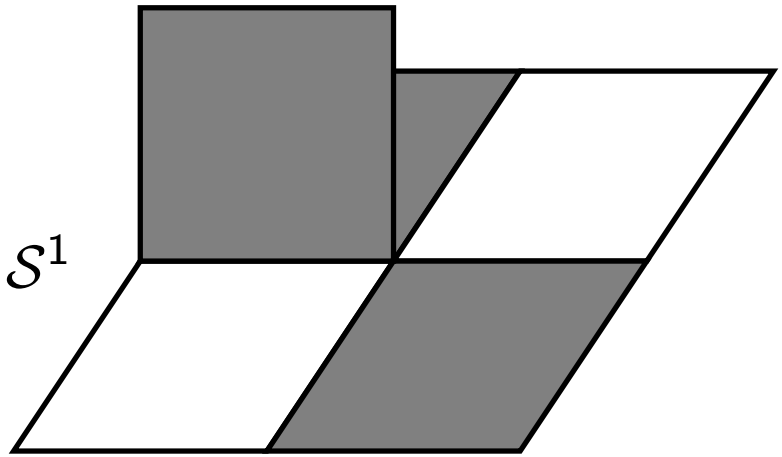

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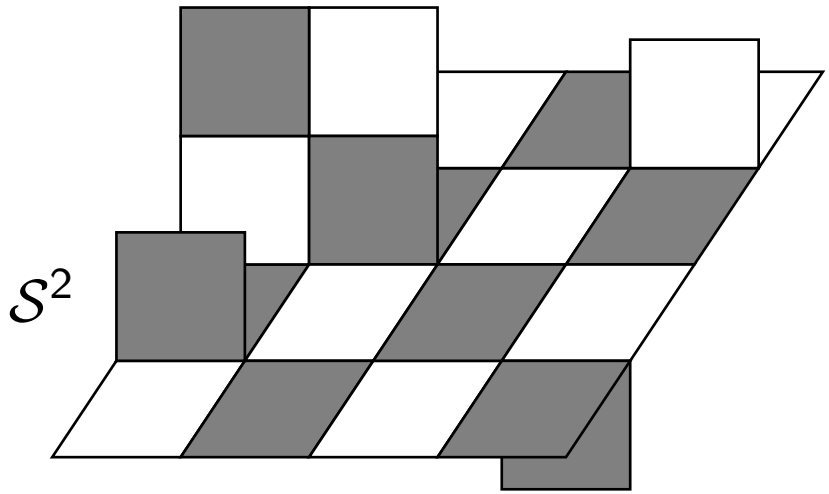

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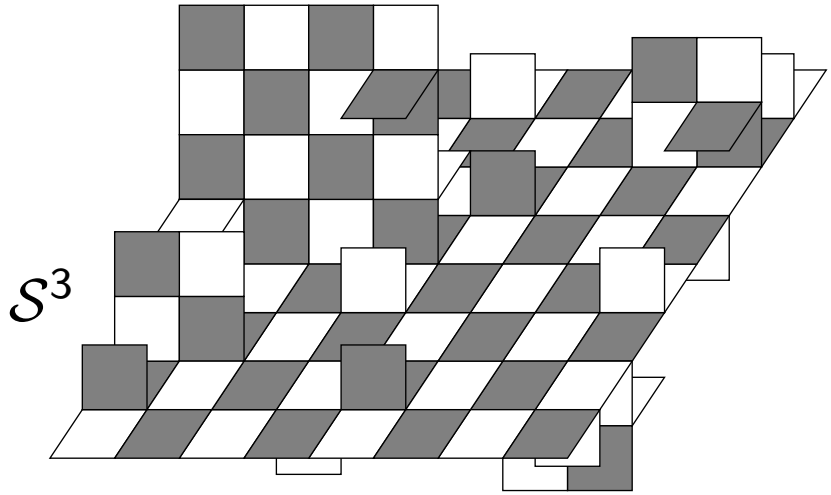

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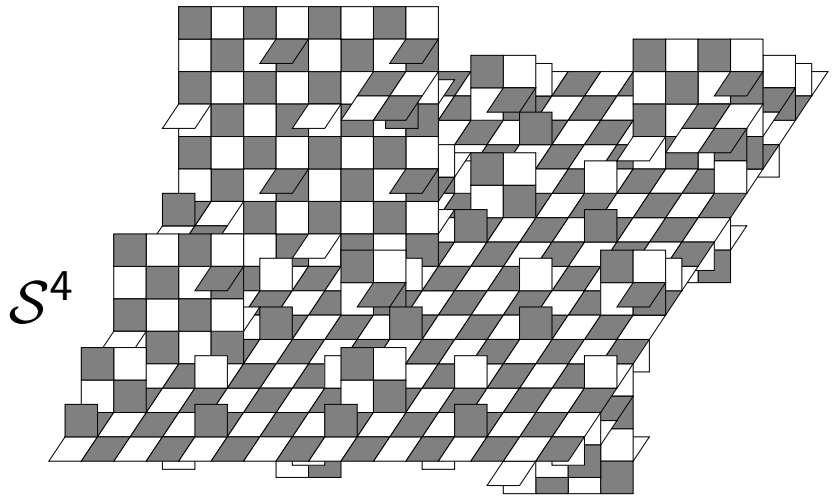

d_n path-metric on polyhedral surface $\mathcal{S}^n$. Then $(\mathcal{S}^n, d_n) \to (S^2, d)$ (Gromov-Hausdorff).

f "scales locally by factor 2".

$f\colon (S^2, d) \to (S^2, d)$ "is conformal".

(S^2, d) quasisphere $\Rightarrow$ f rational

Let $\varphi\colon (S^2, d) \to \widehat{\mathbb{C}}$ quasisymmetric. Let $g = \varphi \circ f \circ \varphi^{-1}$.

$$\begin{array}{ccc} (S^2, d) & \xrightarrow{f} & (S^2, d) \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \varphi} \\ \widehat{\mathbb{C}} & \xrightarrow{g} & \widehat{\mathbb{C}}. \end{array}$$

g is uniformly quasiregular $\Rightarrow$ conjugate to rational map. $\square$

Quasi-visual approximations

Definition (Quasi-visual approximations)

(S, d) metric space. A quasi-visual approximation is a sequence $\{X^n\}_{n\in\mathbb{N}_0}$ of finite covers X^n of S.

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$$\operatorname{diam}(X') \asymp \operatorname{diam}(X).$$
4. For $X' \in \mathcal{X}^{n+k_0}$ with $X' \cap X \neq \emptyset$ we have
$$\operatorname{diam}(X') \leq \lambda \operatorname{diam}(X).$$

Quasi-visual approximations and Quasisymmetry

Proposition (Bonk-M '21)

d_1, d_2 metrics on S,
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Then

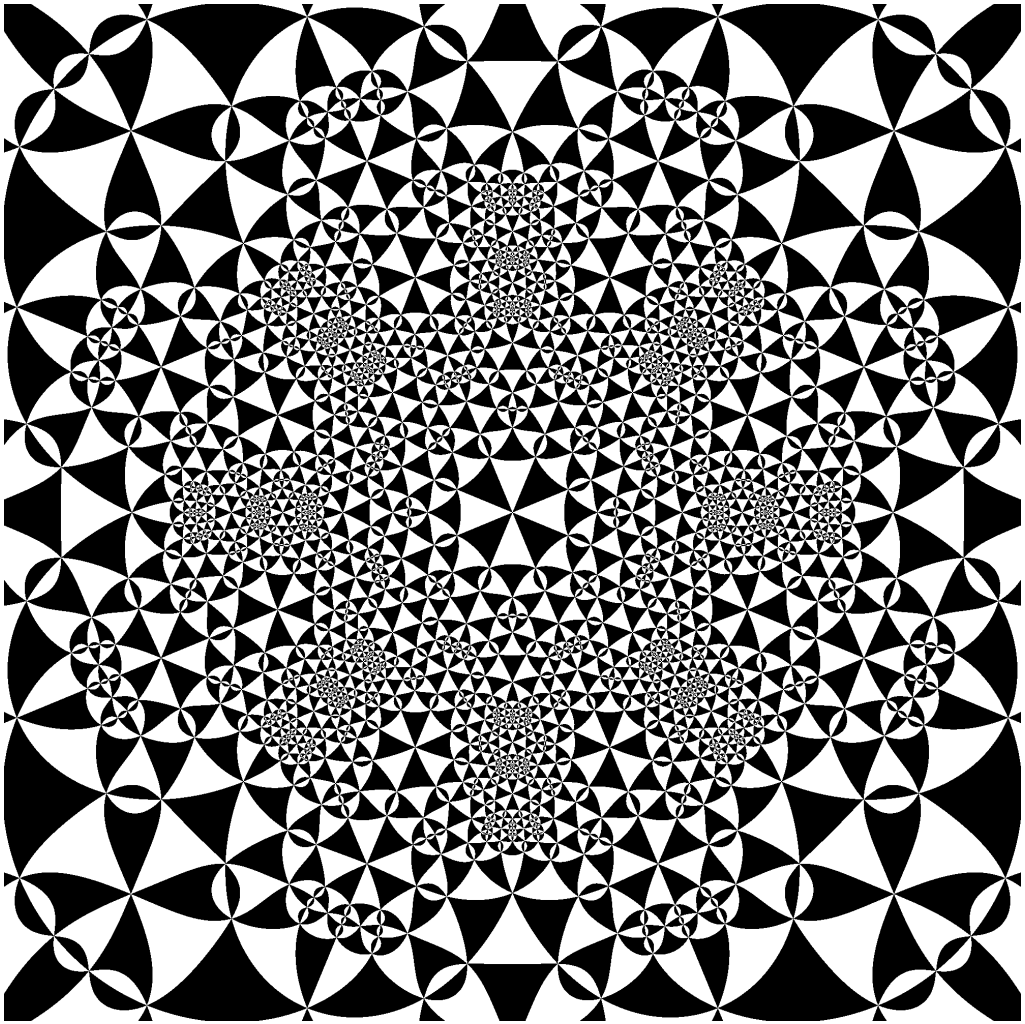
$$\mathrm{id}\colon (S, d_1) \to (S, d_2) \text{ is quasisymmetry}$$

if and only if

$$\{X^n\}_{n \in \mathbb{N}_0} \text{ is quasi-visual for } (S, d_2).$$

See also Carrasco '13, Kigami '20.

f rational $\Rightarrow$ (S^2, d) quasisphere



Jordan curve $\mathcal{C} \subset \widehat{\mathbb{C}}$ with $\mathrm{post}(f) \subset \mathcal{C}$.

$X^n = \{\text{components of } \widehat{\mathbb{C}} \setminus f^{-n}(\mathcal{C})$

These form conformal tilings (see Bowers-Stephenson '17).

Combinatorial equivalence $\Rightarrow$ conformal equivalence.

Standard estimates (Koebe distortion + Schwarz) $\Rightarrow \{X^n\}$ is quasi-visual.

$\Rightarrow (S^2, d)$ quasisphere.

Conformal dimension

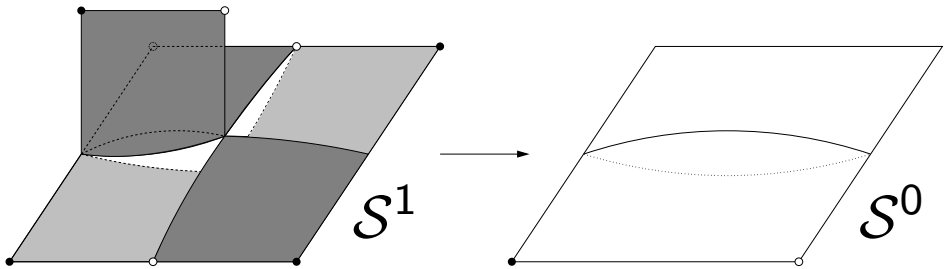
(S^2, d) not quasisphere $\Leftrightarrow$ f obstructed.

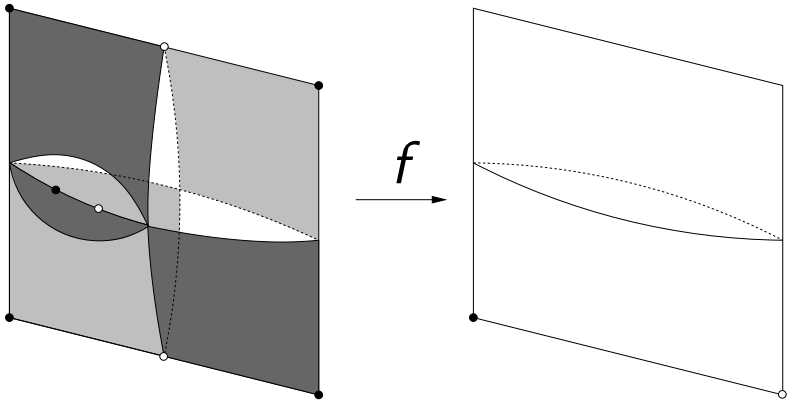
How far is (S^2, d) from being a quasisphere?
The conformal dimension is

$$\text{confdim}(S^2, d) = \inf\{\dim(Y) : Y \text{ qs to } (S^2, d)\}.$$

How far can we lower the dimension by a quasisymmetry?
(Note: trivial to increase dimension arbitrarily by quasisymmetry).

Example again





confdim$(S^2, d) = 2$ (Bonk-Hlushchanka-M '21)

M_n = modulus of n-th approximation of front $\sim n$.

Thurston obstruction = 1.

Bonk-Geyer-Pilgrim conjecture

Multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ pairwise disjoint non-peripheral Jordan curves in $S^2 \setminus \mathrm{post}(f)$.

stable: each non-peripheral component of $f^{-1}(\gamma_i)$ is isotopic to some γ_j rel. $\mathrm{post}(f)$.

$\gamma_{i,j,1}, \ldots, \gamma_{i,j,N}$ all such components
$d_{i,j,\ell} = \deg(f : \gamma_{i,j,\ell} \to \gamma_i)$

$$a_{ij} = \sum_{\ell} \frac{1}{d_{i,j,\ell}}, \quad M = M(\Gamma) = (a_{ij}) \text{ Thurston matrix}$$

Spectral radius $\varrho(M) \geq 1$, then Γ is an obstruction.

Bonk-Geyer-Pilrim conjecture

For $Q \geq 2$ consider the modified Thurston matrix

$$a_{ij} = \sum_{\ell} \frac{1}{d_{i,j,\ell}^{Q-1}}, \quad M = M(\Gamma, Q) = (a_{ij})$$

Let $Q_0 = \inf\{ Q \geq 2 : \varrho(M(\Gamma, Q) = 1, \Gamma \text{ stable } \}$.

Conjecture

$f: S^2 \to S^2$ expanding Thurston map then

$$\mathrm{ARconf}(S^2, d) = Q_0.$$

Have $\mathrm{ARconf}(S^2, d) \geq Q_0$ (Haïssinsky-Pilgrim '08).