Orbits and bungee sets

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Part 1: Orbits

- Usually, given $f\colon X \to X$ we study the *orbits* $(f^n(x))_{n\geq 0}$ for $x \in X$.
- Now reverse this: Given a sequence, is it an orbit for some f?

Part 2: Bungee sets

- The *bungee set* of f is the set of points for which the orbit has both bounded and unbounded subsequences.
- We'll consider quasiregular $f\colon \mathbb{R}^d \to \mathbb{R}^d$.

Part 1: Which sequences are orbits?

Start with a sequence $(z_n)_{n\geq 0}$ in $\mathbb{C}$ (or $(x_n)_{n\geq 0}$ in $\mathbb{R}^d$).
Questions:

- Is there a function $f : \mathbb{C} \to \mathbb{C}$ that *realises* the sequence? That is,

$$f^n(z_0) = z_n$$

  so that (z_n) is the orbit of z_0 under iteration of f.
- If so, is f unique?

Note: some sequences are not orbits. For example, $1, 2, 1, 3, \ldots$ ($f(1) =?$).

Answers depend on which class of functions we consider:

- continuous;
- entire (polynomial or transcendental);
- quasiconformal or quasiregular.

Continuous functions

Definition

A sequence $(x_n)_{n\geq 0}$ in $\mathbb{R}^d$ (or $\mathbb{C}$) is a *candidate orbit* if the following holds: suppose $x \in \mathbb{R}^d$ and that (n_j) is a sequence of integers such that $x_{n_j} \to x$ as $j \to \infty$. Then there exists $x' \in \mathbb{R}^d$ depending only on x such that $x_{n_j+1} \to x'$ as $j \to \infty$.

Note: it follows that, for a candidate orbit, $x_p = x_q$ implies $x_{p+1} = x_{q+1}$. (So $1, 2, 1, 3, \ldots$ is not a candidate orbit.)

Theorem (N., Sixsmith)

A sequence $(x_n)_{n\geq 0}$ in $\mathbb{R}^d$ is a candidate orbit if and only if there exists a continuous $f : \mathbb{R}^d \to \mathbb{R}^d$ that realises (x_n) (i.e. $f^n(x_0) = x_n$).

Moreover, f is unique if and only if $\{x_n : n \geq 0\}$ is dense in $\mathbb{R}^d$.

Types of sequence

Some simple terminology is helpful.

A sequence (z_n) in $\mathbb{C}$ (or $\mathbb{R}^d$) is called ...

- *bounded* if there is $L > 0$ such that $|z_n| \leq L$;
- *escaping* if $z_n \to \infty$ as $n \to \infty$;
- *bungee* if it is not bounded and not escaping;
- *periodic* if there exist $n \neq m$ such that $z_{n+k} = z_{m+k}$ for $k \geq 0$ (so this includes "pre-periodic" or "eventually periodic" sequences).

Which sequences are orbits under entire functions?

Theorem (N., Sixsmith)

Let (z_n) be a candidate orbit. Then exactly one of the following holds:

(a) (z_n) is periodic, and is realised by infinitely many transcendental entire functions and infinitely many polynomials.

(b) (z_n) is escaping, and is realised by infinitely many transcendental entire functions and at most one polynomial.

(c) (z_n) is bungee, and is realised by at most one entire function and no polynomials.

(d) (z_n) is bounded and not periodic, and is realised by at most one entire function.

Note 'uniqueness' is settled, but 'existence' question open for polynomials in cases (b) & (d) and for tefs in cases (c) & (d).

The sequence has a finite accumulation point in cases (c) & (d). There are very strong necessary conditions for such a sequence to be the orbit of an entire function.

Examples

From now on, consider only sequences $z_n \to 0$.
The following candidate orbits cannot be realised by any entire function.

1. $(z_n) = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$

   Here $z_{n+1} = {z_n}^2$ for $n \geq 2$ so this could only be realised by $f(z) = z^2$. But this fails at the first step: $1^2 \neq \frac{1}{2}$.

2. $(z_n) = \frac{1}{2} + \varepsilon_1, \frac{1}{4} + \varepsilon_2, \frac{1}{16} + \varepsilon_3, \frac{1}{256} + \varepsilon_4, \ldots$ where $\varepsilon_n \searrow 0$ fast.

   Again, can show the "only possibility" is $f(z) = z^2$. But this fails at every step when $\varepsilon_{n+1} << \varepsilon_n$.

3. Take $q > 1$, $q \notin \mathbb{N}$ and $z_n = \left(\frac{1}{2}\right)^{q^n}$.

   If this were realised by entire f with Taylor series $f(z) = a_p z^p + \ldots$ then we'd find $p = q$ (not an integer).

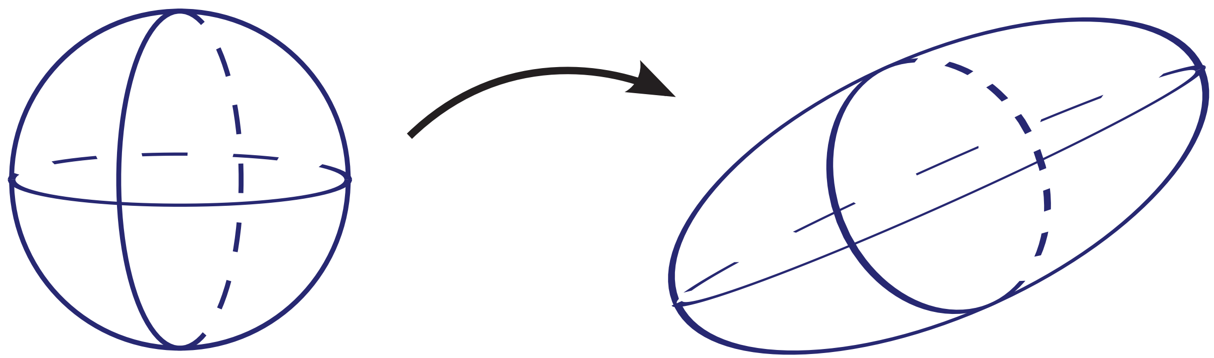
The moral is:

> *"In general, analytic functions are too rigid to realise sequences with accumulation points."*

Can we realise more sequences if we consider instead quasiconformal or quasiregular maps?

Quasiregular maps

Informally, a *quasiregular map* $f : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous, sense-preserving map that sends infinitesimal spheres to ellipsoids of bounded eccentricity.



- Quasiregular (qr) maps generalise analytic maps on $\mathbb{C}$.
- An injective quasiregular map is called *quasiconformal*.
- On the plane, any qr map can be factorised as a composition (analytic) $\circ$ (quasiconformal).

Next, we will state conditions for a sequence $z_n \to 0$ to be realised by a quasiregular map — one necessary, then one sufficient.

Realising sequences $z_n \to 0$ by quasiregular maps

Theorem (N., Sixsmith) — Necessary condition

A sequence $z_n \to 0$ in $\mathbb{R}^d$ is realised by a qr map only if there exist $\mu, \nu, C > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\frac{1}{C^2}\left(\frac{|z_n|}{|z_{n+1}|}\right)^{\mu} \leq \frac{|z_{n+1}|}{|z_{n+2}|} \leq C^2\left(\frac{|z_n|}{|z_{n+1}|}\right)^{\nu} \quad \text{whenever } |z_{n+1}| \leq |z_n| \qquad (1)$$

and

$$\frac{1}{C^2}\left(\frac{|z_{n+1}|}{|z_n|}\right)^{\mu} \leq \frac{|z_{n+2}|}{|z_{n+1}|} \leq C^2\left(\frac{|z_{n+1}|}{|z_n|}\right)^{\nu} \quad \text{whenever } |z_{n+1}| \geq |z_n|. \qquad (2)$$

Theorem (N., Sixsmith) — Sufficient condition

Let $z_n \to 0$ in $\mathbb{C}$. Suppose there exist μ, ν, C, n_0 such that (1) holds and $0 < D < 1$ such that

$$|z_{n+1}| \leq D|z_n| \quad \text{for } n \geq 0.$$

Then (z_n) is realised by a quasiconformal map on $\mathbb{C}$.

Two remarks on the sufficient condition

- It follows that the examples $z_n \to 0$ we saw earlier, that could not be realised by entire functions, *can* all be realised by quasiconformal maps.

- The theorem fails if we try to replace

  $$\text{"there exists } 0 < D < 1 \text{ such that } |z_{n+1}| \leq D|z_n|\text{"}$$

  by simply

  $$\text{"}|z_{n+1}| < |z_n|.\text{"}$$

PART 2: Bungee sets

We now return to the usual direction of study. We fix $f : \mathbb{C} \to \mathbb{C}$ or $f : \mathbb{R}^d \to \mathbb{R}^d$ and study the orbits. We can partition space based on the behaviour of orbits as follows:

- the *escaping set* $I(f) = \{z : f^n(z) \to \infty\}$;
- the *bounded orbit set* $K(f) = BO(f) = \{z : (f^n(z))_{n \geq 0} \text{ is bounded}\}$;
- the *bungee set* $BU(f)$ — everything else!

For a trans entire function f on $\mathbb{C}$, the bungee set is always non-empty, and these sets are related to the Julia set by

$$J(f) = \partial BU(f) = \partial I(f) = \partial BO(f).$$

(Osborne and Sixsmith, Eremenko)

Some definitions for quasiregular maps

A qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ is of *transcendental type* if it has an essential singularity at ∞; that is, $\lim_{x\to\infty} f(x)$ does not exist.

Recall that for entire functions on $\mathbb{C}$ the Julia set is the set of points with the blowing-up property

$$J(f) = \Big\{ z : \text{for all nhds } U \text{ of } z,\ \ \mathbb{C} \setminus \bigcup_{n\geq 1} f^n(U) \text{ is finite} \Big\}. \tag{3}$$

For a qr map on $\mathbb{R}^d$ of trans type, we *define* the Julia set $J(f)$ as

$$\Big\{ x : \text{for all nhds } U \text{ of } x,\ \ \mathbb{R}^d \setminus \bigcup_{n\geq 1} f^n(U) \text{ has zero conf. capacity} \Big\}. \tag{4}$$

Then $J(f)$ is non-empty and completely invariant. Moreover, when $d = 2$, (3) and (4) agree and $\operatorname{cap} J(f) > 0$ (Bergweiler, N.).

Conjecture For any $d \geq 2$, (3) and (4) agree and $\operatorname{cap} J(f) > 0$.

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a transcendental type qr map.

- Siebert: f has infinitely many periodic points (so $BO(f) \neq \emptyset$).
- Bergweiler, Fletcher, Langley, Meyer: $I(f) \neq \emptyset$.
- Bergweiler, N.: $J(f) \subset \partial I(f) \cap \partial BO(f)$.
  Examples show inclusion can be strict.

What about the bungee set?

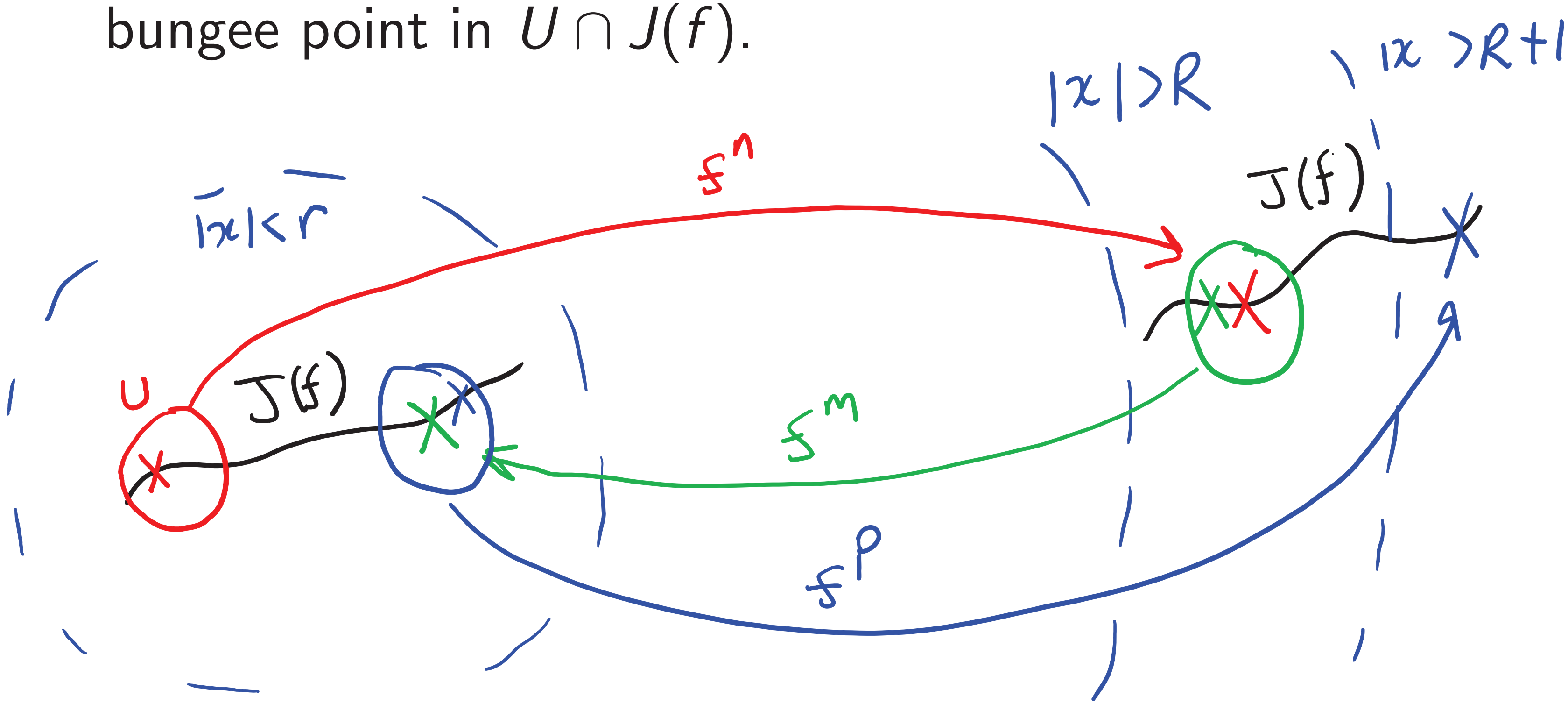
Theorem (N., Sixsmith)

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be qr of transcendental type.

- $BU(f) \cap J(f)$ is non-empty.
- If $\operatorname{cap} J(f) > 0$, then $J(f) \subset \partial BU(f)$.

Sketch proof that $J(f) \subset \partial BU(f)$ when cap $J(f) > 0$

- Show that for large $r, R > 0$ neither $J(f) \cap \{|x| < r\}$ nor $J(f) \cap \{|x| > R\}$ has zero capacity.
- Take U meeting $J(f)$ and aim to use blowing-up property to find a bungee point in $U \cap J(f)$.



Can we show $J(f) = \partial BU(f)$ for qr maps? No...

Theorem (N., Sixsmith)

There is a trans type qr map $f\colon \mathbb{C} \to \mathbb{C}$ such that $J(f) \neq \partial BU(f)$.

The construction relies on the following (surprising?) result.

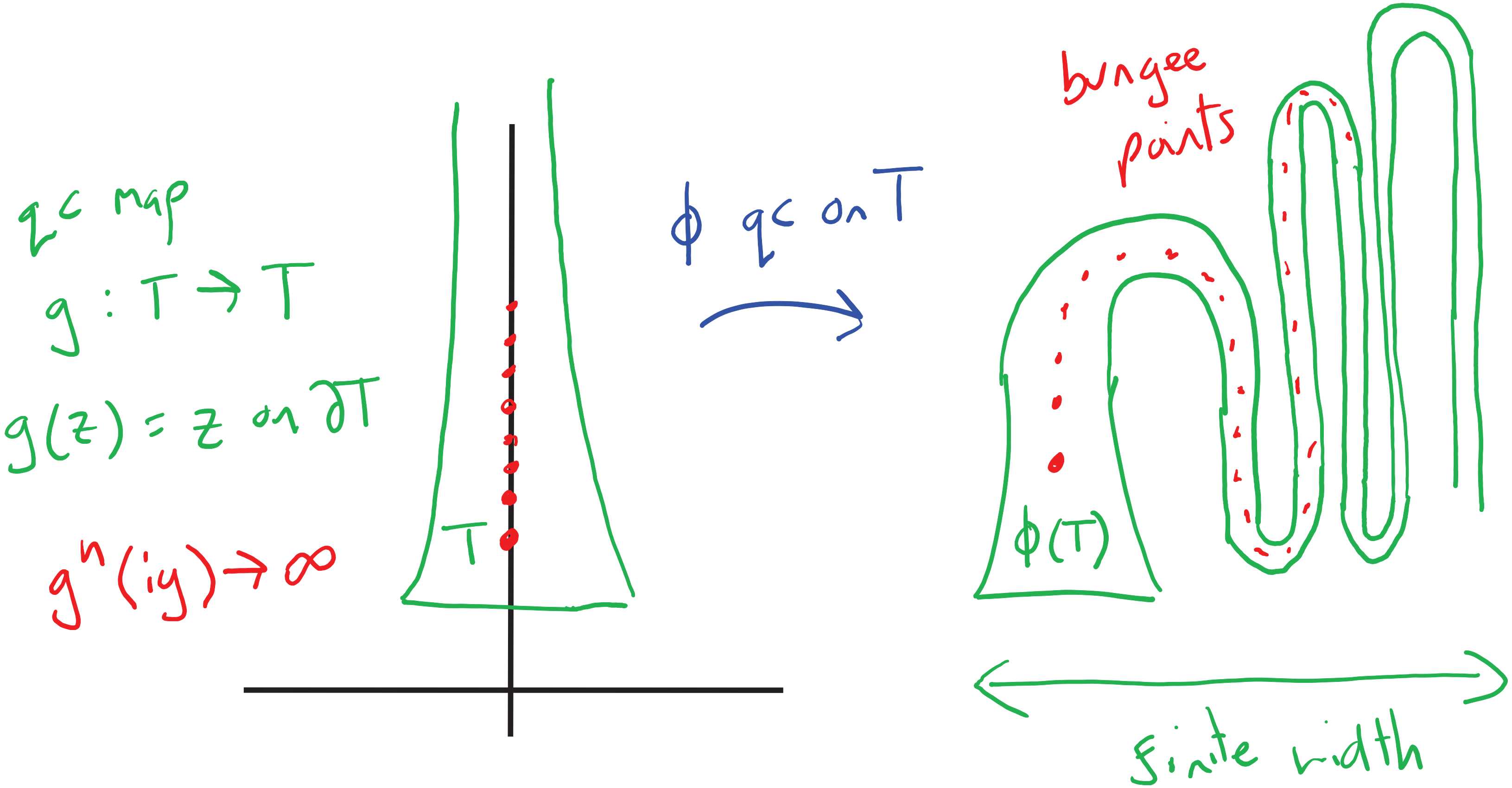
Theorem (N., Sixsmith)

There is a quasiconformal map $F\colon \mathbb{C} \to \mathbb{C}$ such that $BU(F)$ is non-empty.

Note the contrast to conformal maps $\mathbb{C} \to \mathbb{C}$ (i.e. $z \mapsto az + b$), which have uninteresting dynamics — certainly no bungee points!

We'll next sketch the idea for the qc map F and then show how it yields the qr map f in the first theorem.

A qc map F with bungee points

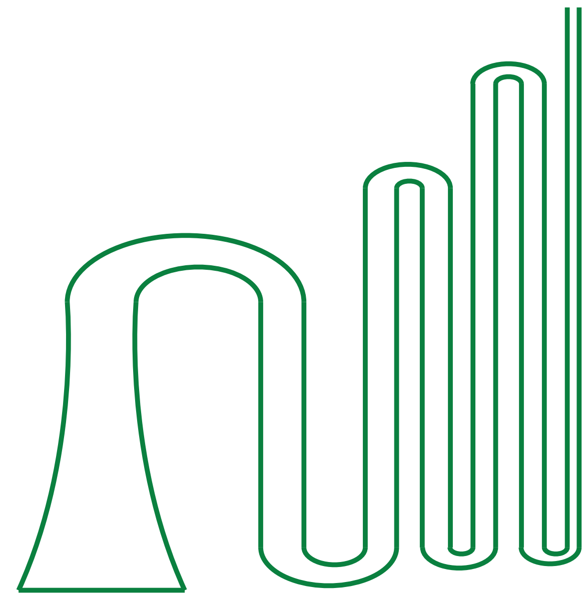


$$F = \begin{cases} \phi \circ g \circ \phi^{-1} & \text{on } \phi(T) \\ \text{id} & \text{elsewhere} \end{cases}$$

A trans type qr map f with $J(f) \neq \partial BU(f)$

$\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ **"snakes on a half-plane"**

for $z \in \mathbb{H}$, $f(z) = F(z)$



for $z \in \mathbb{R}$, $f(z) = z$

interpolate in strip

for $\operatorname{Im} z \leq -1$, $f(z) = z + \delta \exp(-z^2)$ (small $\delta > 0$)

- $f(\mathbb{H}) \subset \mathbb{H}$, so no "blowing up" in $\mathbb{H}$, so no points of $J(f)$ in $\mathbb{H}$, but certainly $\partial BU(f)$ intersects $\mathbb{H}$.
- Therefore, $J(f) \neq \partial BU(f)$.