Note Title 27/09/2021

Boundary dynamics of wandering domains: Sufficient conditions for uniform behaviour 1

Joint work with:

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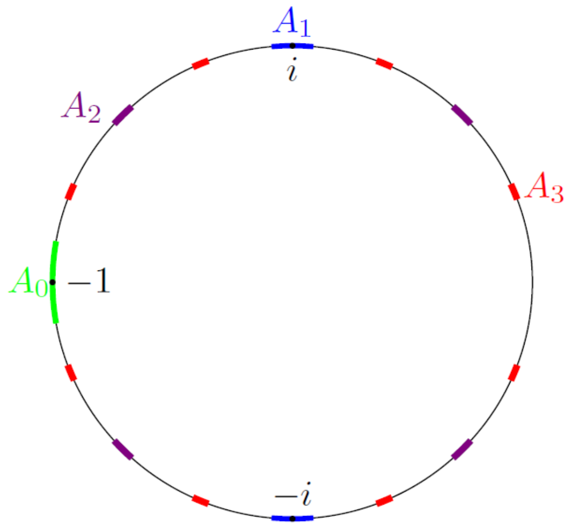


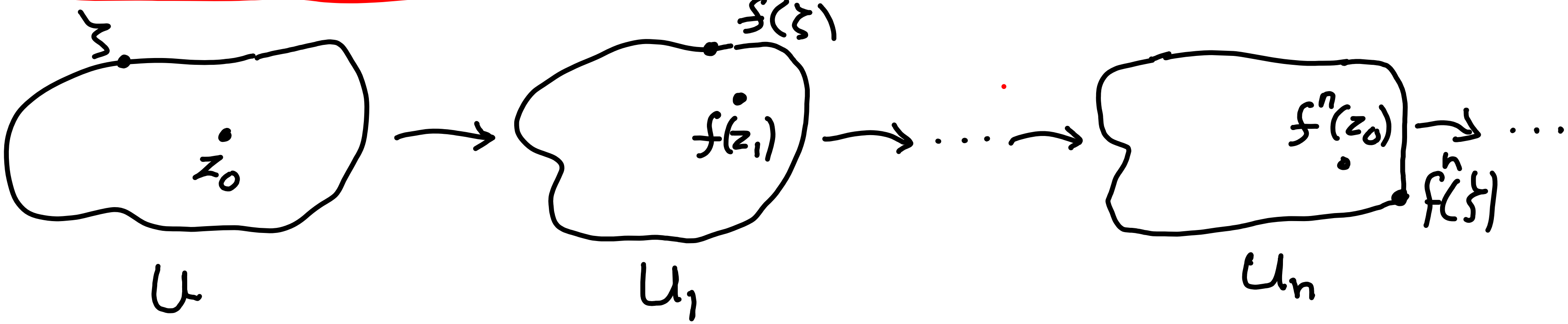
Motivation For a simply connected wandering domain of a transcendental entire function: relate the internal dynamics to the boundary dynamics.

Theorem Let U be a SCWD of f, a TEF, with orbit (U_n). If $\exists\, z_0 \in U$ s.t

$$\sum_{n=1}^{\infty} \operatorname{dist}\left(f^n(z_0), \partial U_n\right)^{\frac{1}{2}} < \infty,$$

then, for almost every $\zeta \in \partial U$, w.r.t. harmonic measure,

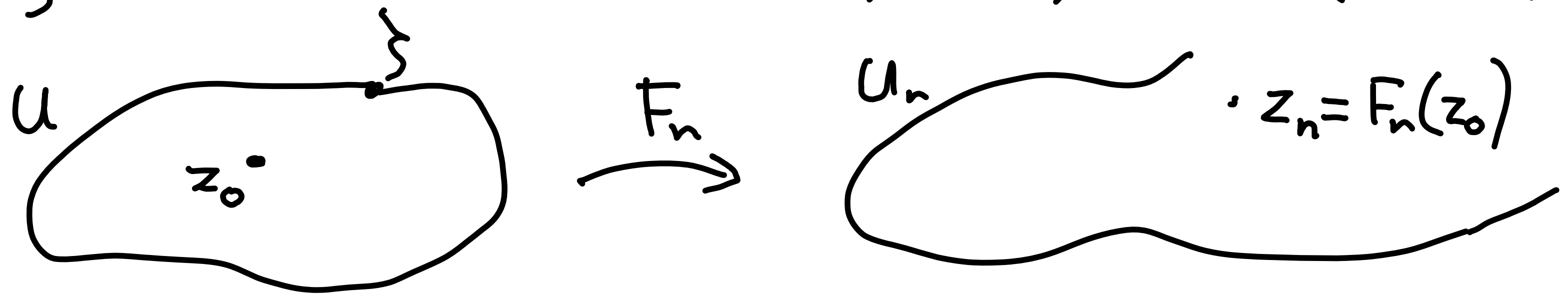
$$|f^n(\zeta) - f^n(z_0)| \to 0 \text{ as } n \to \infty.$$



General setting

Holomorphic functions: f, f_n, F_n, $n \geq 1$.

Simply connected domains: U, U_n, $n \geq 1$ $(\neq \mathbb{C})$.



Question Relate behaviour of $F_n(\zeta)$, $\zeta \in \partial U$, to $F_n(z_0)$.

Forward compositions: $F_n = f_n \circ \cdots \circ f_1$, $\quad f_n : U_{n-1} \to U_n$.

orbits

Wandering domains: $F_n = f^n$, $\quad U$ a Fatou component

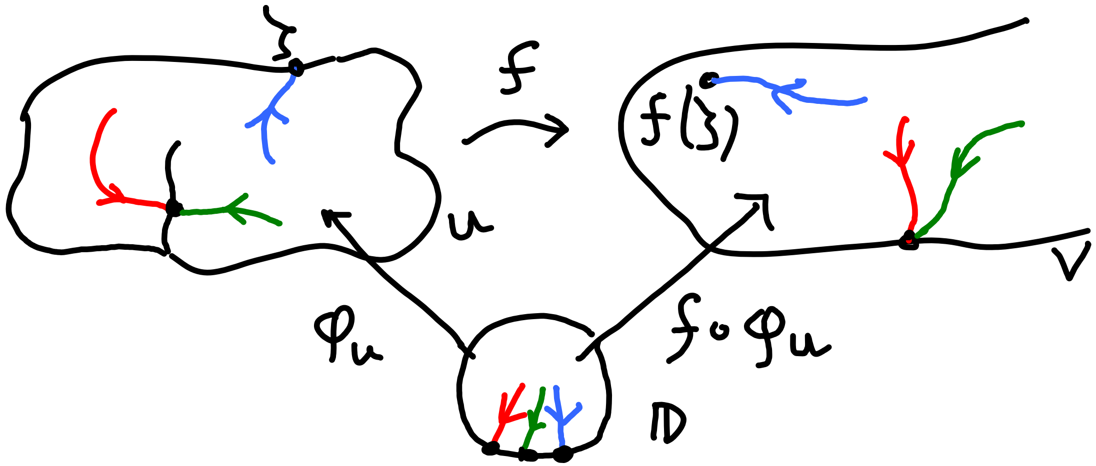
Self-maps of $\mathbb{D}$: $F_n = f^n$, $\quad f : \mathbb{D} \to \mathbb{D}$ e.g. inner fns.

Denjoy-Wolff set

BEFRS, Ferreira

Theorem A $F_n : U \to U_n$ holomorphic, $z_0 \in U$. If
$$\operatorname{dist}(F_n(z_0), \partial U_n) \to 0 \quad \text{as } n \to \infty,$$
then
$$|F_n(z) - F_n(z_0)| \to 0 \quad \text{as } n \to \infty, \qquad \text{for all } z \in U.$$

Radial extension of f to accessible points of ∂U with 'consistent' extension.



D-W set of $F_n : U \to U_n$: the set of $\zeta \in \partial U$ s.t.
$$|F_n(z) - F_n(\zeta)| \to 0 \quad \text{as } n \to \infty, \quad \text{for all } z \in U.$$

Aaronson (1978), Doering & Mañé (1991) dichotomy

Theorem (ADM) Let $f : \mathbb{D} \to \mathbb{D}$ be an inner function.

(a) If $\sum_{n=1}^{\infty} (1 - |f^n(0)|) < \infty$, then

$f^n(\zeta) \to p$ as $n \to \infty$, for a.e. $\zeta \in \partial\mathbb{D}$.

p is Denjoy-Wolff point of f

(b) If $\sum_{n=1}^{\infty} (1 - |f^n(0)|) = \infty$, then

$(f^n(\zeta))$ is dense in $\partial\mathbb{D}$, for a.e. $\zeta \in \partial\mathbb{D}$.

'chaotic'

So, the D-W set has (a) full measure or (b) zero measure.

Qn Does the ADM dichotomy hold in our general setting?

ADM(a) works in our general setting

Theorem B Let $F_n : U \to U_n$, $n \geq 1$, with full radial ext^n.
If $\exists z_0 \in U$ s.t.

$$\sum_{n=1}^{\infty} \operatorname{dist}(F_n(z_0), \partial U_n)^{\alpha} < \infty,$$

where $\alpha = \frac{1}{2}$ in general, $\alpha = 1$ if ∂U_n 'smooth',
then, for almost every $\zeta \in \partial U$,

$$F_n(\zeta) - F_n(z_0) \to 0 \quad \text{as } n \to \infty.$$

- An example where F_n are self-maps of a cardioid region shows that $\alpha = 1/2$ cannot be improved in general.
- Regions with 'corners': can have $\alpha \in (\frac{1}{2}, 1)$.

Qn Can we generalise ADM(b)? For example: suppose

$$F_n : \mathbb{D} \to \mathbb{D} \quad \text{and} \quad \sum_{n=1}^{\infty} \left(1 - |F_n(0)|\right) = \infty.$$

- Is F_n 'chaotic' on $\partial\mathbb{D}$?
- What if $F_n = f_n \circ \cdots \circ f_1$, and f_n inner?

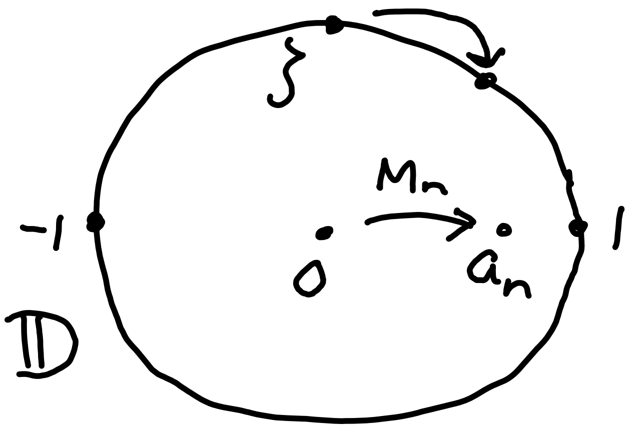
Möbius example Take $a_n \uparrow 1$, $a_0 = 0$, and $\sum_{n=1}^{\infty}(1-a_n) = \infty$

$$M_n(z) = (m_n \circ \cdots \circ m_1)(z) = \frac{z + a_n}{1 + a_n z}.$$

Then

$$M_n(z) \to 1 \quad \text{as } n \to \infty,$$

for $z \in \overline{\mathbb{D}} \setminus \{-1\}$.



<u>Blaschke product example 1</u> $\exists\, \mu_n \searrow 1/3$ s.t. if

$$b_n(z) = \left(\frac{z+\mu_n}{1+\mu_n z}\right)^2$$

then

- $B_n = b_n \circ \cdots \circ b_1$ satisfies $\sum_{n=1}^{\infty} (1 - B_n(0)) = \infty$,
- $B_n(z) \to 1$ as $n \to \infty$, for $z \in \mathbb{D} \cup (\partial\mathbb{D} \cap D(1,\rho))$, $\rho > 0$.

<u>'Proof'</u> Each b_n has an attracting f.p. at 1, with arc A_n s.t. $b_n(A_n) \subset A_n$, and $\bigcap_n A_n = \{1\}$.
Want to choose μ_n s.t.

$$b_n(A_n) \subset A_{n+1} \quad n \geq 1,$$

$$\Rightarrow \quad B_n \to 1 \text{ as } n \to \infty \text{ on } A_0. \quad \blacksquare$$

$\mu = 1/3$ parabolic

Work in half-plane.

<u>Qn</u> Is $\{\zeta \in \partial\mathbb{D} : B_n(\zeta) \not\to 1\}$ of positive measure?

Blaschke product example 2 If $a_n \uparrow 1$, $a_0 = 0$, $\sum (1-a_n) = \infty$,

$$b_n(z) = M_n\left(\left(M_{n-1}^{-1}(z)\right)^2\right), \quad \text{where} \quad M_n = \frac{z+a_n}{1+a_n z},$$

then

- $B_n(z) = (b_n \circ \cdots \circ b_1)(z) = M_n(z^{2^n}) \longrightarrow 1$, for $z \in \mathbb{D}$,
- $(B_n(\zeta))$ is dense in $\partial\mathbb{D}$ for a.e. $\zeta \in \partial\mathbb{D}$.

used in the cardioid ex.

'Proof' Key fact: let $g(z) = z^2$ and $\varepsilon_n = 1 - a_n$

$$A_n = \{\zeta \in \partial\mathbb{D} : g^n(\zeta) \in D(-1, \varepsilon_n)\}, \quad \text{where } \sum_{n=1}^{\infty} \varepsilon_n = \infty.$$

Then

$$\{\zeta : \zeta \in A_n \text{ i.o.}\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n \text{ has full measure.}$$

Petrov version of Borel-Cantelli 2 $E_n \subset [0,1]$, $\sum_{n=1}^{\infty} |E_n| = \infty$,

$$|E_m \cap E_n| \leq C|E_m||E_n| \Rightarrow \left|\{x \in [0,1] : x \in E_n \text{ i.o}\}\right| \geq 1/C.$$

Theorem Let U be a SCWD of f, a TEF, with orbit (U_n).
If $\exists\, z_0 \in U$ s.t

$$\sum_{n=1}^{\infty} \operatorname{dist}(f^n(z_0), \partial U_n)^{\frac{1}{2}} < \infty,$$

then, for almost every $\zeta \in \partial U$, w.r.t. harmonic measure,

$$|f^n(\zeta) - f^n(z_0)| \to 0 \text{ as } n \to \infty.$$

Löwner's lemma: $f : \mathbb{D} \to \mathbb{D}$ holomorphic, $z_0 \in \mathbb{D}$, $E \subseteq \partial\mathbb{D}$:

$$\omega(z_0, f^{-1}(E), \mathbb{D}) \leqslant \omega(f(z_0), E, \mathbb{D}).$$

Translate to $f : U \to V$.

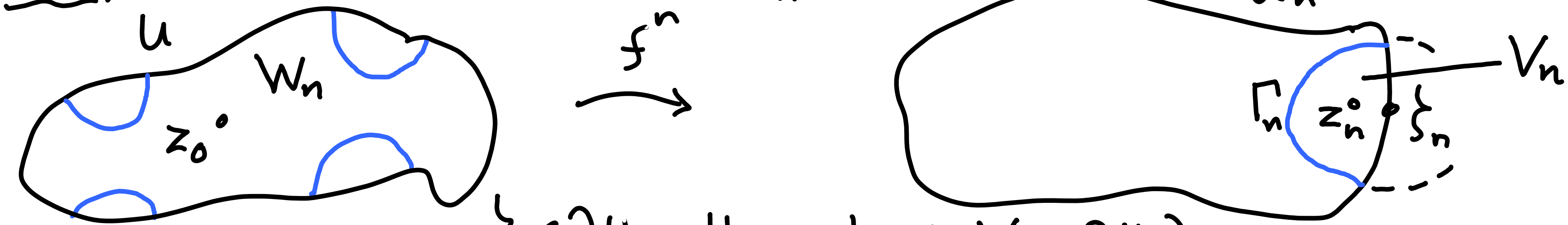
Borel–Cantelli 1 $E_n \subset [0,1]$, $\sum |E_n| < \infty$:

$$\{x \in [0,1] : x \in E_n \text{ i.o.}\} \text{ has measure } 0.$$

Milloux–Schmidt inequality

$$\omega(z, \Gamma, V) \leqslant C(r)|z - \zeta|^{1/2}, \text{ for } z \in V.$$

Proof Fix $r>0$ small. Put $z_n = f^n(z_0)$.



$\zeta_n \in \partial U_n$, $|\zeta_n - z_n| = \mathrm{dist}(z_n, \partial U_n)$

Let

V_n be cmpnt of $S_n = U_n \cap D(\zeta_n, r)$ s.t. $z_n \in V_n$, $\Gamma_n = \partial V_n \cap \overline{U}_n$

W_n be cmpnt of $f^{-n}(V_n)$ s.t. $z_0 \in W_n$.

By Löwner's lemma and Milloux-Schmidt,

$$\omega(z_0, f^{-n}(\Gamma_n), W_n) \leq \omega(z_n, \Gamma_n, V_n) \leq C(r)\, |z_n - \zeta_n|^{1/2},$$

by maximum principle,

$$\omega(z_0, f^{-n}(\overline{U}_n \setminus \overline{V}_n), U) \leq \omega(z_0, f^{-n}(\Gamma_n), W_n).$$

Hence

$\{\zeta \in \partial U : |f^n(\zeta) - \zeta_n| \geq r \text{ i.o.}\}$ has harm. measure 0,

so $\{\zeta \in \partial U : f^n(\zeta) - \zeta_n \to 0\}$ has full harm. measure. ∎

Thanks for your attention!