Uniformity in internal dynamics of wandering domains

And maybe a new example

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Outline

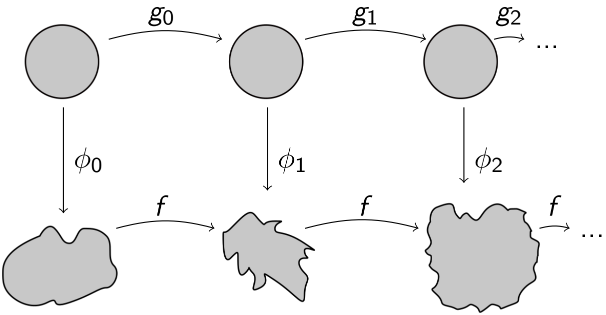
Periodic Fatou components

Fatou, $\sim$ 1920

Denjoy-Wolff theorem $\Rightarrow$
A periodic Fatou component fits neatly into one of five types

Simply connected wandering domains

Benini, Evdoridou, Fagella, Rippon, & Stallard, 2019



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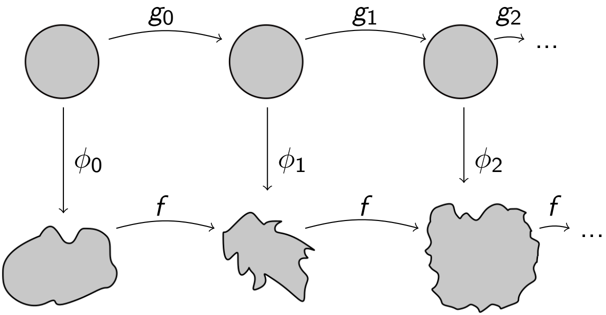

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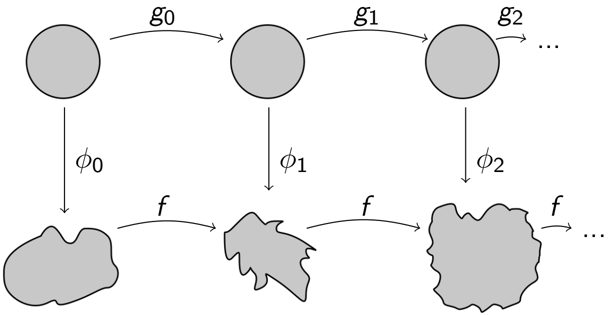

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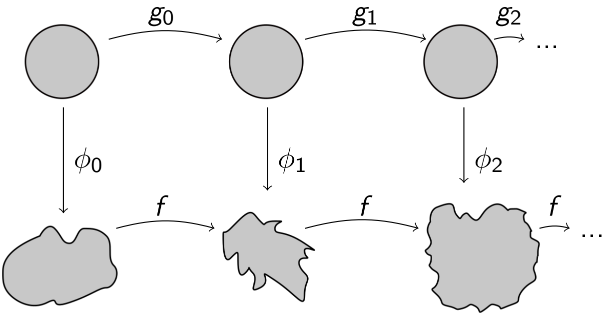

- The $g_n : \mathbb{D} \to \mathbb{D}$ form a non-autonomous sequence
- The Riemann maps ϕ_n preserve the hyperbolic metric
  - ! Look at the long-term behaviour of $d_{U_n}(f^n(z), f^n(w))$

First classification theorem

Benini et al., 2019

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  - $d_{U_n}(f^n(z), f^n(w)) \to c(z, w) > 0$ for all $z, w \in U$

(iii) Eventually isometric
  - $\exists N$ such that $d_{U_n}(f^n(z), f^n(w)) = c(z, w) > 0$ for all $n \geq N$ for all $z, w \in U$ (except for a countable set of pairs)

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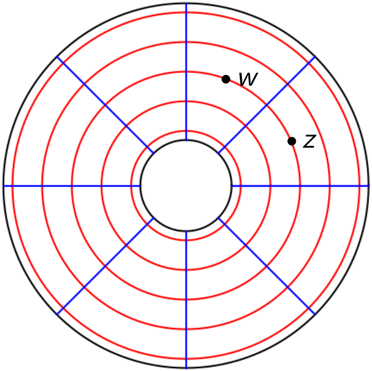
Trimodal domains

The model (F, 2021)

$f : \mathbb{C} \to \mathbb{C}$ entire, U a doubly-connected WD



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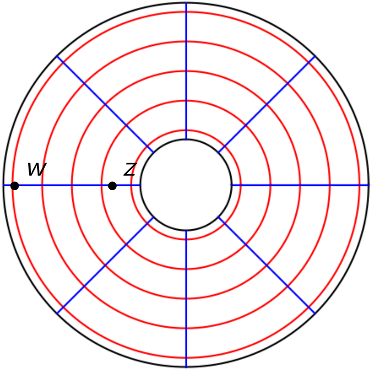

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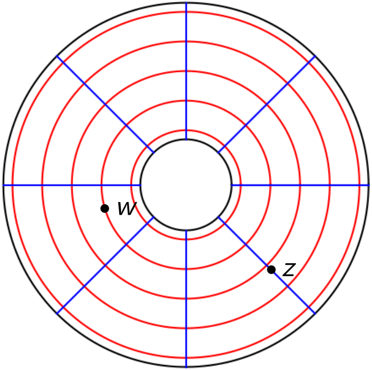

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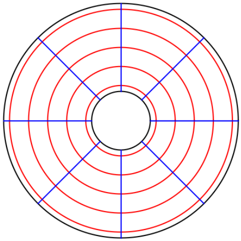
In other words... (F, 2021)



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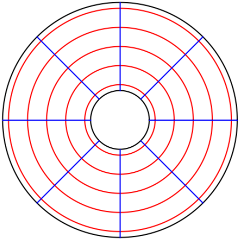

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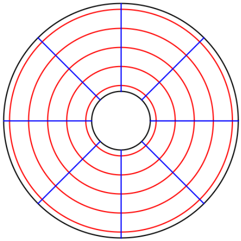

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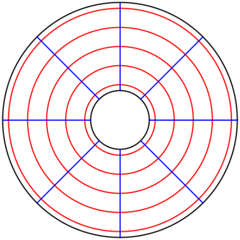

- The collections $\mathcal{C}$ and $\mathcal{L}$ form transversal foliations of $A(R)$
- Points in the same leaf of $\mathcal{C}$ get arbitrarily closer
- Points in the same leaf of $\mathcal{L}$ stay the same distance apart
- Any other pair gets closer, but not too close!

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- In multiply connected WDs: that would be nice...
  - ... but sometimes we'll need a base point $z_0 \in U$
  - ! We say that U is contracting (semi-contr., event. isom.) relative to z_0 if $d_{U_n}(f^n(z), f^n(z_0))$ behaves in the prescribed way for every $z \in U$

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- If V is eventually isometric, then so is U

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- If U is multiply connected...
  - Unimodal contracting and eventually isometric MCWDs exist
  - What about unimodal semi-contracting?

The example

There exists a meromorphic function f with an infinitely connected wandering domain U and a non-empty open subset $V \subset U$ such that U is semi-contracting relative to any $z_0 \in V$.

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4. Apply Shishikura's principles of surgery and get a qc-conjugate meromorphic map f

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2. f has an infinitely connected wandering domain $U \supset A$
3. A is semi-contracting
4. U is semi-contracting relative to $z_0 \in A$

Thank you for your attention!