Explosion points of Zorich maps

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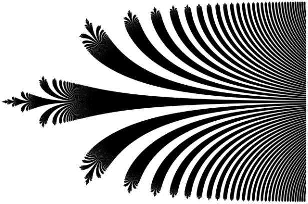
Theorem (Devaney and Krych, 1984)

For $0 < \lambda \leq 1/e$ the Julia set $\mathcal{J}(E_\lambda)$ consists of uncountably many, disjoint curves each of which has a finite endpoint and goes off to infinity.



Source: wikipedia

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- Totally separated $\Rightarrow$ Totally disconnected (all connected components are points).

Let A_λ denote the set of endpoints of $\mathcal{J}(E_\lambda)$, when $0 < \lambda \leq 1/e$.

Theorem (Mayer, 1990)

A_λ is totally separated but $A_\lambda \cup \{\infty\}$ is connected. So ∞ is an explosion point for $A_\lambda \cup \{\infty\}$.

Explosion points in dynamics have been studied in many different contexts by many people (Rempe, Alhabib, Evdoridou, Sixsmith,...).

Quasiregular maps

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- Holomorphic maps send infinitesimally small circles to infinitesimally small circles.
- Quasiregular maps send infinitesimally small spheres to infinitesimally small ellipsoids of bounded eccentricity.
- Holomorphic maps locally stretch the space evenly in all directions.
- Quasiregular maps locally stretch the space by different amounts in each direction but the ratio $\frac{\text{maximum stretch}}{\text{minimum stretch}}$ remains bounded.

Examples

- An easy one: $(x, y) \to (2x, y)$.
- Winding maps (i.e. in polar coordinates $(r, \theta) \to (r, k\theta)$)

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Indeed:

- Quasiregular maps are open, discrete and differentiable a.e. (Reshetnyak 1967-68).
- There is an analogue of Picard's Theorem (Rickman 1980).
- We can develop a function theory in $\mathbb{R}^d$, $d > 2$ for quasiregular maps.

Definition

A quasiregular map in $\mathbb{R}^d$ is said to be of *polynomial type* if $\lim_{x\to\infty} f(x) = \infty$. If this limit does not exist we say that f is of *transcendental type*.

Dynamics of quasiregular maps

Definition (Julia set for quasiregular maps, Bergweiler, Nikcs)

Let f be a quasiregular map on $\mathbb{R}^d$. We define the Julia set to be $\{x \in \mathbb{R}^d : \text{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} f^n(U)\right) = 0, U \text{ any open neighboorhood of x}\}$

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This new Julia set has many of the properties of the classical Julia set

Theorem (Bergweiler, Nicks 2013-14)

Let f be a quasiregular map on $\mathbb{R}^d$. Then assuming $\deg f > K$

- $\mathcal{J}(f)$ is non empty.
- $\mathcal{J}(f)$ is completely invariant.

Moreover if $\operatorname{cap} \mathcal{J}(f) > 0$ then:

- $\mathcal{J}(f)$ is perfect.
- $\mathcal{J}(f^p) = \mathcal{J}(f)$.
- $\mathcal{J}(f) = \overline{O_f^-(x)}$, where $x \in \mathcal{J}(f) \setminus E(f)$.

Another way of looking at the exponential map

- Define $h : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^2$ as $h(y) = (\cos y, \sin y)$.

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  $\{(x, y) : \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}\}$.

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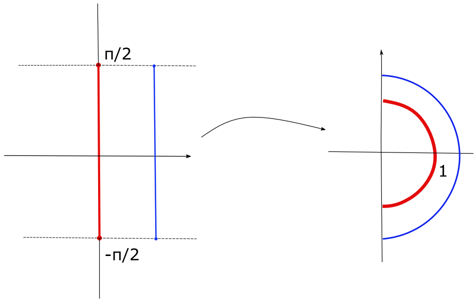

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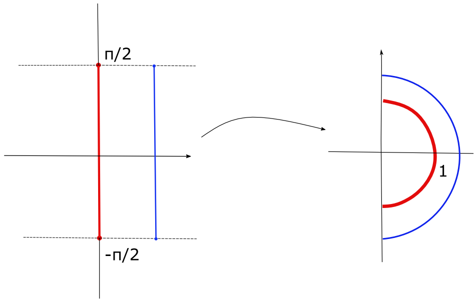

- Then extend E to the whole plane by reflecting across the boundaries of strips in the domain and across the imaginary axis in the range.

Zorich maps

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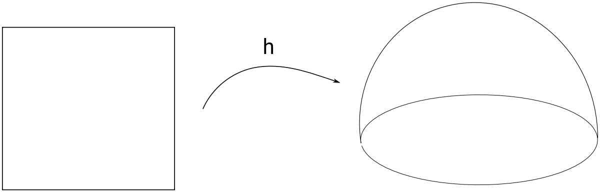
First consider an L bi-Lipschitz, sense-preserving map h that maps the square

$$Q := \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\right\}$$

to the upper hemisphere

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\}.$$



Then define $Z : Q \times \mathbb{R} \to \mathbb{R}^3$ as $Z(x_1, x_2, x_3) = e^{x_3} h(x_1, x_2)$. The map Z maps the square beam $Q \times \mathbb{R}$ to the upper half-space.

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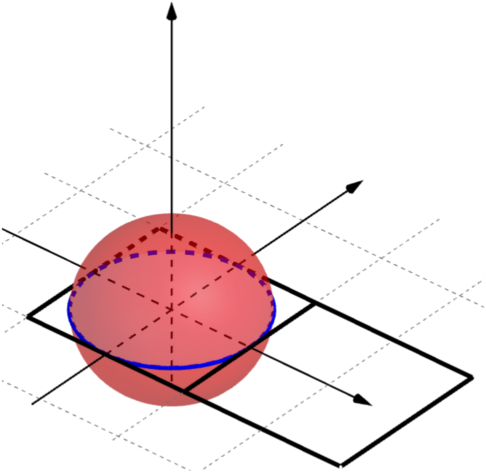
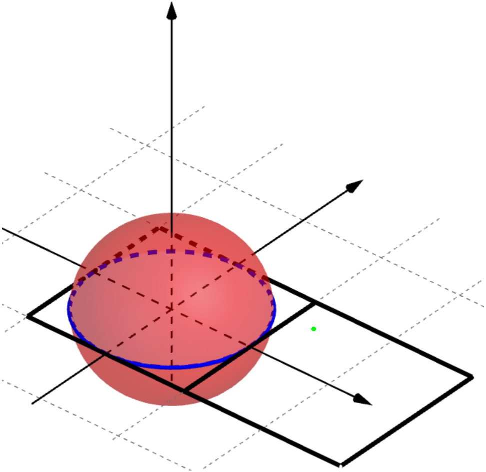
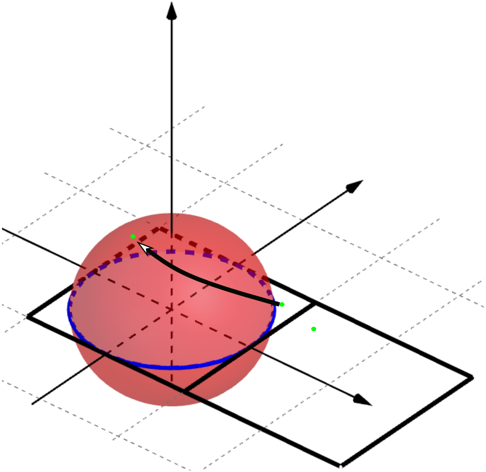
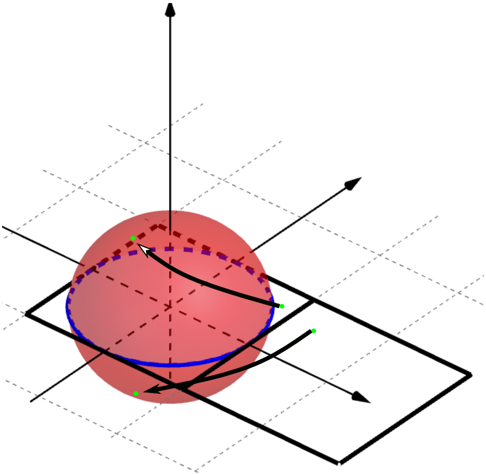
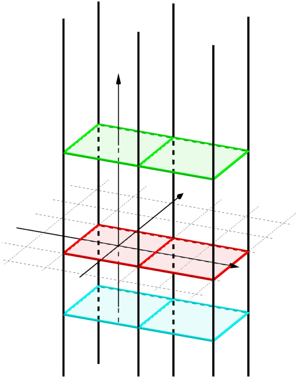
By repeatedly reflecting now, across the sides of the square beam in the domain and the $x_1 x_2$ plane in the range, we get a map $Z : \mathbb{R}^3 \to \mathbb{R}^3$.



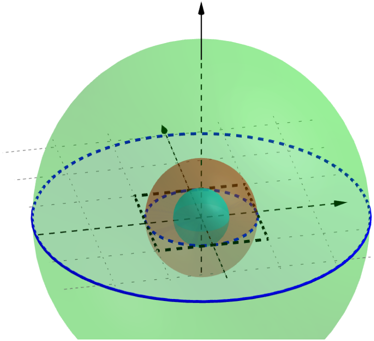








$$Z(x_1, x_2, x_3) = e^{x_3} h(x_1, x_2)$$



Properties:

- Z is doubly periodic, meaning that $Z(x_1 + 4, x_2, x_3) = Z(x_1, x_2 + 4, x_3) = Z(x_1, x_2, x_3)$.
- Unlike the exponential map, Z has a non empty branch set meaning that

$$B_Z := \{x \in \mathbb{R}^3 : Z \text{ is not locally homeomorphic at } x\} \neq \emptyset.$$

- Z is quasiregular, omits 0 and has an essential singularity at infinity, just like the exponential map on the plane.

Julia sets of Zorich maps

For the family of maps $Z_\nu = \nu Z$, $\nu > 0$ in analogy with the exponential family we have the following.

Theorem (Bergweiler, 2010 and Bergweiler, Nicks, 2014)

For all $0 < \nu < e^{(-\log L + L)}$ the Julia set of the Zorich map $\mathcal{J}(Z_\nu)$ consists of uncountably many, disjoint curves each of which has a finite endpoint and goes off to infinity.

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For all $0 < \nu < e^{(-\log L + L)}$ the Julia set of the Zorich map $\mathcal{J}(Z_\nu)$ consists of uncountably many, disjoint curves each of which has a finite endpoint and goes off to infinity.

Let $\mathcal{A}_\nu$ denote the set of endpoints of those curves.

Theorem (T.)

For all $0 < \nu < e^{(-\log L + L)}$, the point at infinity is an explosion point for the set $\mathcal{A}_\nu \cup \{\infty\}$.

Ideas for the proof

- First method: Try to prove directly that $\mathcal{A}_\nu \cup \{\infty\}$ is connected and that $\mathcal{A}_\nu$ is totally separated.

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  - This is what Mayer did in his proof.
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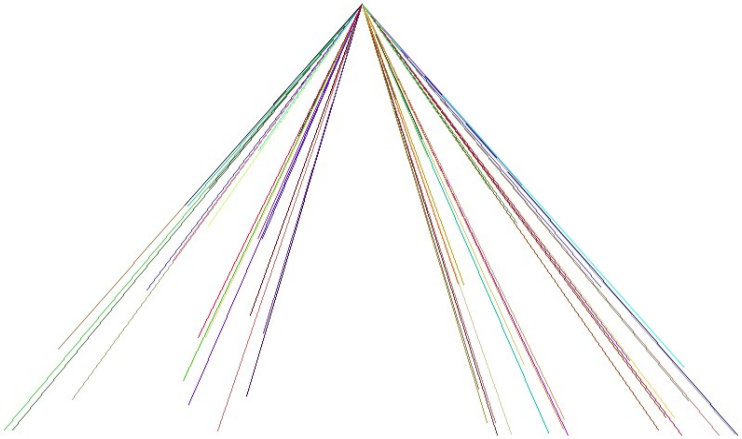
  - This is what Mayer did in his proof.
  - It heavily relies on the Riemann mapping theorem. Not available in higher dimensions.
- Second method: Show that the Julia set $\mathcal{J}(Z_\nu) \cup \{\infty\}$, $0 < \nu < e^{(-\log L + L)}$ is a so called Lelek fan with top at ∞.
  - For exponential maps that was first proven by Aarts and Oversteegen.
  - Used by Alhabib+Rempe to study explosion points for escaping endpoints.

A Lelek fan is a continuum (compact and connected metric space) that satisfies certain properties.



Source: J. Charatonik, P. Krupski, and P. Pyrih, Examples in Continuum Theory,

https://www2.karlin.mff.cuni.cz/ pyrih/e/e2001v1/c/ect/node87.html

Lelek fans

Definition

A continuum X is a Lelek fan with top at x_0 if :

1. X is hereditarily unicoherent. Meaning that for any subcontinua $X_1, X_2 \subset X$, $X_1 \cap X_2$ is connected.
2. X is uniquely arcwise connected.
3. x_0 is the only common endpoint of at least three arcs that are otherwise disjoint.
4. For any sequence of points $y_n \in X$ converging to $y \in X$ the unique arcs $[x_0, y_n]$ converge to $[x_0, y]$ in the Hausdorff metric.

   A point that is on the boundary of every arc that contains it is called an endpoint of X.
5. The endpoints in X are dense in X.

Lelek fans

Theorem (Lelek, 1961)

There is a Lelek fan with top x_0, for which x_0 is an explosion point for the set endpoints$\cup\{x_0\}$.

Theorem (Charatonik and independently Bula, Oversteegen, $\approx$1990)

Any two Lelek fans are homeomorphic to each other.

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Any two Lelek fans are homeomorphic to each other.

Since being an explosion point is a topological property, these two theorems imply that all Lelek fans have their top as an explosion point for the set endpoints$\cup\{x_0\}$.

Theorem (T.)

For $0 < \nu < e^{(-\log L + L)}$ the Julia set $\mathcal{J}(Z_\nu) \cup \{\infty\}$ is a Lelek fan with top at ∞.

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Proof

- Direct verification of properties (1)-(5) does not work.
- No information on how the hairs accumulate on each other $\Rightarrow$ We cannot really prove (4)
- For this reason we need a topological model of $\mathcal{J}(Z_\nu) \cup \{\infty\}$ which is geometrically easier to work with.
- Inspired by Aarts' and Oversteegen's straight brush model we define a higher dimensional analogue called 3-d straight brush.

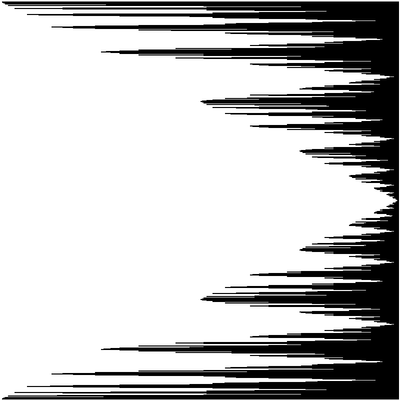


Image produced using code written by Alexandre DeZotti

3-d straight brush

Definition

A 3-d Straight Brush B is a subset of

$$\{(y, a_1, a_2) \in \mathbb{R}^3 : y \geq 0, (a_1, a_2) \in (\mathbb{R} \setminus \mathbb{Q})^2\}$$

with the following properties:

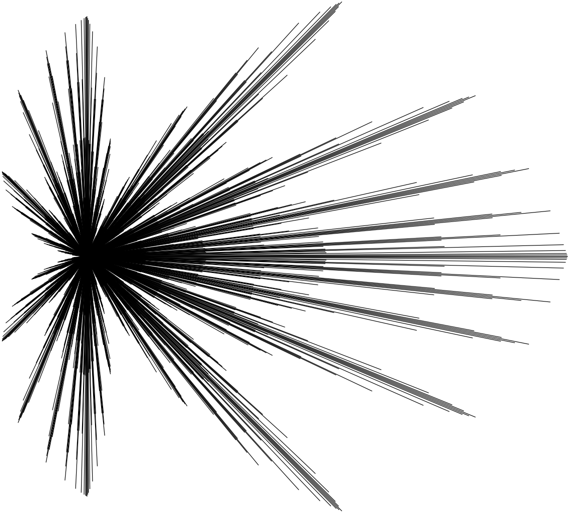
1. For every $(a_1, a_2) \in \mathbb{R}^2$, there is a $t_{(a_1,a_2)} \in [0, \infty]$ such that $(t, a_1, a_2) \in B$ if and only if $t \geq t_{(a_1,a_2)}$
2. The set of (a_1, a_2) with $t_{(a_1,a_2)} < \infty$ is dense in $(\mathbb{R} \setminus \mathbb{Q})^2$. Also, for any such (a_1, a_2) there exist sequences $(a_1, a_{n,2})$, $(a_1, b_{n,2})$, $(c_{n,1}, a_2)$, $(d_{n,1}, a_2)$, such that $a_{n,2} \uparrow a_2$, $b_{n,2} \downarrow a_2$, $c_{n,1} \uparrow a_1$, $d_{n,1} \downarrow a_1$. Moreover it is true that $t_{(a_1,a_{n,2})} \to t_{(a_1,a_2)}$ and similarly for the other sequences.
3. B is a closed subset of $\mathbb{R}^3$.

Conclusion

Due to the clear geometric structure of 3-d straight brushes it is much easier to show that they are Lelek fans.

Proposition

The one point compactification $B \cup \{\infty\}$ of any 3-d straight brush is a Lelek fan with top ∞.



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Theorem (Aarts and Oversteegen, 1993)

For $0 < \lambda \leq \frac{1}{e}$ there is a straight brush B and a homeomorphism from $B \cup \{\infty\}$ onto $\mathcal{J}(E_\lambda) \cup \{\infty\}$.

Theorem (T.)

For $0 < \nu < e^{(-\log L + L)}$ there exists a 3-d straight brush B and a homeomorphism from $B \cup \{\infty\}$ onto $\mathcal{J}(Z_\nu) \cup \{\infty\}$.

Thank you!