Elliptic (non-local) PDEs with singular data - Brezis' theory of reduced measures

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Abstract

Let E be a locally compact separable metric space and m be a Radon measure on E with full support. The presentation is devoted to the existence problem for the following equation

$$-Au + Vu = f(\cdot, u) + \mu, \tag{1}$$

where A is a self-adjoint linear operator on $L^2(E;m)$ generating a Markov semigroup, V is a locally (quasi-)integrable non-negative function on E, $f : E \times \mathbb{R} \to \mathbb{R}$ is a real Carathéodory function satisfying the sign condition, i.e.

$$f(x, y) \cdot y \leq 0, \quad x \in E,\, y \in \mathbb{R},$$

and μ is a Borel measure on E. The model example of a local operator which fits into our framework is uniformly elliptic divergence form diffusion operator

$$Au = \sum_{i,j=1}^{d} (a_{i,j} u_{x_i})_{x_j},$$

whereas a model example of a non-local operator fitting into the framework is the fractional Laplacian

$$Au = \Delta^\alpha u(x) = c_\alpha \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \frac{u(y) - u(x)}{|x - y|^{d+2\alpha}}\, dy$$

for $\alpha \in (0, 1)$. It is well known that the mechanism of existence and non-existence of a solution, hidden in the equation (1), is very subtle and sensitive to the change of data. In case $f \equiv 0$, we will give necessary and sufficient condition for the existence of a solution to (1). Whereas, in case $V \equiv 0$, we will provide, using Brezis' theory of reduced measures (see [1]–[3]), a characterization of the class of Borel measures for which (1) is solvable. We will end the presentation with a discussion on the Chern-Simons equation

$$-\Delta u = e^u(1 - e^u) + \mu$$

and its variants (see [4]).

[1] Brezis, H., Marcus, M., Ponce, A.C.: A new concept of reduced measure for nonlinear elliptic equations. *C. R. Math. Acad. Sci.* **339**, 169–174 (2004)
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[3] Klimsiak, T.: Reduced measures for semilinear elliptic equations involving Dirichlet operators. *Calc. Var.* 55:78 (2016)
[4] Ponce, A.C., Presoto, A.E.: Limit solutions of the Chern-Simons equation. *Nonlinear Anal.* **84** (2013) 91–102.