Function recovery in L_2

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We consider the problem of recovering a function $f\colon \Omega \to \mathbf{C}$ belonging to some class F based on a finite number of function values. The class F reflects our a priori knowledge about the function. Our theory works, for instance, if Ω is a compact domain or manifold and F is a compact subset of the space of bounded functions on Ω. The error is measured in a worst case scenario and with respect to the L_2-distance. We recently obtained the following result:

If the Kolmogorov widths of F in L_2 show a polynomial decay of order $\alpha > 1/2$,
then there is a sampling-based algorithm that achieves the same rate of convergence.

The algorithm is a (weighted) least squares estimator. Surprisingly, it achieves a better rate of convergence than the famous Smolyak algorithm for classical tensor product spaces. The proof is based on concentration results for random matrices and a sparsification theorem due to Marcus/Spielman/Srivastava which goes under the name Weaver's theorem or Kadison-Singer problem. For Hilbert classes F, we obtain that algorithms using function values are asymptotically as powerful as algorithms using arbitrary linear measurements (like Fourier coefficients or values of derivatives).

We discuss these result and also address the following questions: What does the algorithm look like? What can be said in the case of a low convergence rate $\alpha \leq 1/2$? What results do we obtain for the tractability of the approximation problem in high dimensional settings? We also present results for function recovery in the uniform norm and discuss related results from the literature.

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