Which problems can be solved by randomized algorithms?

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Let F and G be separable Banach spaces. A linear problem is given by a continuous linear operator $S : F \to G$. The aim is to approximate $S(f)$ for every $f \in F$ by means of an algorithm. Approximation arror of an algorithm A is given by

$$e^{det}(A, S) = \sup_{f \in F, \|f\|=1} \|S(f) - A(f)\|_G \quad \text{if } A \text{ is a deterministic algorithm,}$$

and

$$e^{ran}(A, S) = \sup_{f \in F, \|f\|=1} \mathbb{E}\left[\|S(f) - A(f)\|_G\right] \quad \text{if } A \text{ is a deterministic algorithm.}$$

Denote by $e^{det}(n, S)$ the minimal error of a deterministic algorithm using at most n information evaluation. We say that the problem is solvable in a deterministic setting when

$$\lim_{n\to\infty} e^{det}(n, S) = 0.$$

We define $e^{ran}(S, n)$ analogously and say that the problem S is solvable in the randomized setting if

$$\lim_{n\to\infty} e^{ran}(n, S) = 0.$$

In this talk we focus on the situation when we are allowed to use any continuous linear functional as an information.

The main question is: which problems are solvable? It is well-known that in the deterministic setting the problem given by S is solvable exactly when S is compact. In this talk we give a partial answer to the question in the randomized setting. The two main results are the following:

1. If the operator S is not finitely strictly singular then the problem given by S is not solvable in the randomized setting.
2. If G is additionally assumed to be a Hilbert space then there is a large class of Banach spaces F (characterized by the behaviour of Gaussian measures) for which $S : F \to G$ is solvable in the randomized setting if and only if S is compact.