How much randomness is needed for high-confidence Monte Carlo integration?

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We study Monte Carlo methods for integrating smooth functions based on n function evaluations.

The classical way of assessing the precision ε of a randomized integration method is based on the *root mean squared error* (RMSE) or the *mean error*. Optimal Monte Carlo error rates in terms of the mean error are well known for classical Sobolev spaces $W_p^r\left([0,1]^d\right)$ as well as for spaces $\mathbf{W}_p^{r,\mathrm{mix}}\left([0,1]^d\right)$ of dominating mixed smoothness. Some of the known methods seem to be very randomness efficient, in recent years a randomly shifted and dilated Frolov rule has been analysed in [1, 2] and shown to be optimal in many spaces while using only $2d$ random numbers. If, however, the error is measured in terms of small error ε with high probability $1-\delta$, the so-called *probabilistic* error criterion, see [3], some of the supposedly optimal methods turn out to be suboptimal with the error $\varepsilon = e(n,\delta)$ depending polynomially on δ^{-1} instead of the polynomial dependence on $\log \delta^{-1}$ we hope for.

Methods restoring optimality in the probabilistic error setting either rely on concentration of mass (e.g. Hoeffding's inequality) or probability amplification (e.g. median-of-means) and usually require $\mathcal{O}(n)$ fully independent samples, see [3]. However, it turns out that we may relax the independence, namely, that *k-wise independence* with $k \asymp \log(\frac{1}{\delta})$ is enough to achieve close to optimal confidence intervals. Furthermore, there are known constructions for families of k-wise independent *discrete* random variables. Counting the number of random bits is also the only way of conducting a thorough analysis concerning the questions of how much randomness is required for close to optimal performance of Monte Carlo algorithms. Hence, discretization is an additional hurdle we have to account for in this setting.

Restricted Monte Carlo methods that only use a small amount of random bits have been studied in [4] for the RMSE criterion. We here conduct a similar study for the probabilistic error criterion of restricted Monte Carlo methods, including lower and upper bounds.

Joint work with: Daniel Rudolf.

References

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