The unreasonable effectiveness of operator algebras

bridging between geometry and quantum theory

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Unreasonable effectiveness?

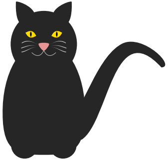
Reprinted from *Communications in Pure and Applied Mathematics*, Vol. 13, No. I (February 1960). New York: John Wiley & Sons, Inc. Copyright © 1960 by John Wiley & Sons, Inc.

THE UNREASONABLE EFFECTIVENSS OF MATHEMATICS IN THE NATURAL SCIENCES

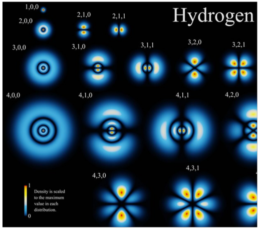
Eugene Wigner

- *Miracle* of the success of the axioms of quantum mechanics as formulated by Dirac.
- Language of linear operators on Hilbert spaces.

- One of the most experimentally successful physical theories;
- Valid at the atomic and subatomic scale;
- Arose from theories to explain observations that could not be reconciled with classical physics:
  - black-body radiation;
  - double slit experiment;
  - photo-electric effect (Einstein, Nobel prize)



- Physical quantities like e.g. energy, (angular) momentum, assume discrete values (for instance integer multiples of a fixed value).
- Particle-wave duality: objects have characteristics of both particles and waves;
- Uncertainty principle: there are limits to how accurately the value of physical quantities can be measured;
- Probabilistic interpretation: the theory cannot predict with certainty, only give probabilities.



- We cannot measure the position q and the momentum p of a particle with absolute precision.
- The more accurately we know one of these values, the less accurately we know the other:

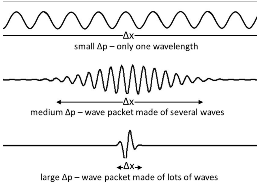
$$\Delta p \Delta q \geq \frac{h}{4\pi}.$$

- Not a shortcoming of our measuring devices, rather an intrinsic feature of quantum mechanics.
- Common to many *quantum* phenomena we cannot explain classically.

Remark

The same phenomenon appears in wave mechanics:

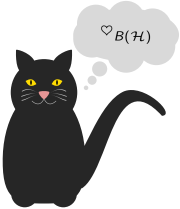
- small variation on momentum corresponds to few wavelengths.. the wave is delocalised (all over the place).
- small variation on position means we have a localised wave (wave packet), which is made by superimposing several waves. Hence, multiple wave lengths, big variation on momentum.

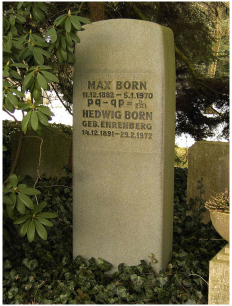


The Dirac(–von Neumann) axioms of quantum mechanics:

- A state ψ of the quantum system is a unit vector of $\mathcal{H}$ up to scalar multiples (i.e., a ray in $\mathcal{H}$);
- The observables of the quantum system are the (possibly unbounded) self-adjoint operators on $\mathcal{H}$;
- The expected value of an observable A for a system in a state ψ is given by the inner product $\langle \psi, A\psi \rangle$.

Von Neumann: reformulation with bounded operators.





- Matrix mechanics (Born, Heisenberg, Jordan);
- p and q are the self-adjoint operators representing position and momentum of a particle.
- Non-commutativity of operators:

$$[p, q] = i\hbar 1.$$

- Consequence: Heisenberg uncertainty principle (Kennard, 1927):

The proof only uses the following facts:

- A and B are two Hermitian operators;
- The two operators do not commute: $-i[A, B] = 1$;
- Using the Cauchy–Schwartz inequality we obtain:

$$\Delta(A)\Delta(B) \geq \frac{1}{4}.$$

In allen Fällen kann man also wohl ruhig behaupten, daß die Prezisionsmaße bei einer solcher gleichzeitiger Mäßung von zwei kanonisch konjugierten Variabeln stets der Relation [...] unterliegen werden.

(E. H. Kennard)

This is an operator theoretic result.

For X a compact Hausdorff space, consider

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous}\}.$$

The set $C(X)$ comes with

- vector space structure: for $f, g \in C(X)$ and $\lambda \in \mathbb{C}$

$$(\lambda f + g)(x) := \lambda f(x) + g(x), \qquad \forall x \in X;$$

- commutative product: for $f, g \in C(X)$:

$$(fg)(x) := f(x)g(x), \quad \forall x \in X;$$

- unit: the function identically equal to 1; and
- an involution $* : C(X) \to C(X)$ given by

$$f^*(x) := \overline{f(x)}.$$

There is a natural norm on the space $C(X)$, given by

$$\|f\| = \sup_{x \in X} |f(x)|. \tag{1}$$

with respect to which $C(X)$ is *a Banach *-algebra*.

The norm satisfies

$$\boxed{\|f^* f\| = \|f\|^2.}$$

$C(X)$ is a *commutative C^*-algebra*.

Example

Let X consist of n-points. $C(X) \simeq \mathbb{C}^n$ with the usual vector space structure, coordinate-wise multiplication and complex conjugation, and norm

$$\|(z_1, \ldots z_n)\|^2 = \max\{\overline{z_i} z_i \; | i = 1, \ldots, n\}$$

Any point $P \in X$ can be thought of as a functional

$$\sigma_P : C(X) \to \mathbb{C}, \quad \sigma_P(f) := f(P),$$

and it satisfies

$$\sigma_P(fg) = \sigma_P(f)\sigma_P(g), \quad \sigma_P(1) = 1,$$

i.e. σ_P is a *character* (also, a *pure state*).

All characters on $C(X)$ are of this form and the set of characters $\Sigma(C(X))$ is *homeomorphic* to X.

Theorem (Gelfand–Naimark)

*Let A be a commutative unital C^*-algebra. Then there is a *-isomorphism*

$$\boxed{A \simeq C(\Sigma(A))}$$

of commutative C^*-algebras.

Definition

A C*-algebra is a Banach *-algebra A with the property that

$$\boxed{\|a^* a\| = \|a\|^2},$$

for all $a \in A$.

Some examples:

- The algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices with conjugate transpose and the operator norm

$$\|A\| = \sup_{x \in \mathbb{C}^n, \|x\|=1} \|Ax\|;$$

- The algebra $B(H)$ of bounded operators on a Hilbert space, with operator adjoint, and operator norm

$$\|A\| = \sup_{x \in H, \|x\|=1} \|Ax\|;$$

- The algebra $\mathcal{K}(H) \subseteq B(H)$ of compact operators.

$B(H)$ is the prototypical example of C*-algebra.

Theorem (Gelfand–Naimark–Seagal)

Let A be a C^*-algebra. Then there exist a Hilbert space H and an injective $*$-homomorphism $\pi : A \to B(H)$.

Every C*-algebra can be embedded into the bounded operators on a Hilbert space.

- The bounded observables of the quantum mechanical system are defined to be the self-adjoint elements in a C*-algebra A.
- The states of the quantum mechanical system are to be the *states* of the C*-algebra, i.e. the normalized positive linear functionals ω on A.
- The value $\omega(A)$ of a state ω on an element $a \in A$ is the expectation value of the observable a for the quantum system is in the state ω.

If the $A = B(\mathcal{H})$, then the bounded observables are just the *bounded* self-adjoint operators on $\mathcal{H}$. If ψ is a unit vector in $\mathcal{H}$, then $\omega_\psi(a) := \langle \psi, a\psi \rangle$ is a state on A.

Start from the canonical commutation relation on $\mathbb{R}$

$$PQ - QP = -i\hbar. \tag{2}$$

The above equation has no solution in terms of bounded operators.

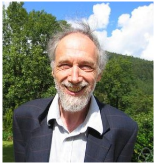
However, exponentiating to 1-parameter groups we get the Weyl form of the CCR's:

$$e^{itQ}e^{isP} = e^{-ist}e^{isP}e^{itQ}. \tag{3}$$

Conversely, any pair of 1-parameter groups U, V on $L^2(\mathbb{R})$ satisfying

$$\boxed{U(t)V(s) = e^{-ist}V(s)U(t)}$$

give rise to the CCR equation (2).



Noncommutative geometry (NCG) is a general mathematical framework introduced by A. Connes in the 80's. Inspired by the noncommutativity of *quantum theory*:

$$\boxed{AB \neq BA}$$

Idea

Motivated from Gelfand duality, look at noncommutative C*-algebras of operators as algebras of functions on some *noncommutative space*.

Study geometric problems using the language of functional analysis and operator algebra.

GEOMETRY **OPERATOR ALGEBRA**

The circle:

$$S^1 := \{z \in \mathbb{C} \mid \bar{z}z = 1\}.$$

The C*-algebra $C(S^1)$ is the closure of the *Laurent polynomials*

$$\frac{\mathbb{C}[\zeta, \bar{\zeta}]}{\langle \bar{\zeta}\zeta = 1 \rangle}.$$

We represent $C(S^1)$ via multiplication operators on the Hilbert space

$$H = L^2(S^1) \simeq \ell^2(\mathbb{Z}).$$

Under this isomorphism, multiplication by $e^{2\pi i\theta}$ is mapped to the bilateral shift

$$U(e_n) = (e_{n+1}), \quad U^*(e_n) = e_{n-1}.$$

$C(S^1)$ is the smallest C^*-subalgebra of $B(\ell^2(\mathbb{Z}))$ that contains the *unitary* U.

Now instead consider the Hilbert space $\ell^2(\mathbb{N})$ and the shift operator

$$T(e_n) = (e_{n+1}) \qquad T^*(e_n) = \begin{cases} e_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}.$$

The *Toeplitz algebra* $\mathcal{T}$ is the smallest C*-subalgebra of $B(\ell^2(\mathbb{N}))$ that contains T.

It is not commutative since $T^*T = \mathrm{Id}$ and $TT^* = 1 - P_{\ker(T^*)}$.
Elements of $\mathcal{T}$ commute up to compact operators:

$$\boxed{0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.}$$

Consequences: Noether–Goberg–Krein index theorem, Cuntz's proof of Bott periodicity, PV-exact sequence for crossed products by $\mathbb{Z}$, and more.
The algebra $C(S^1)$ is the "boundary" of a noncommutative disk.

Theorem (Serre–Swan)

For a compact Hausdorff space X and a locally trivial complex vector bundle $E \to X$, the module of sections $\Gamma(X, E)$ is a finitely generated projective $C(X)$-module. Conversely every finitely generated projective $C(X)$-module arises in this way.

$K_0(A) := \{[P] - [Q] : P, Q$ finitely generated projective modules$\}$.

For a compact manifold M the group

$$K_0(C(M)) \simeq K^0(M),$$

the Atiyah–Hirzebruch topological K-theory group (formal differences of equivalence classes of vector bundles $E \to M$.)

K-Homology as Dual theory, thanks to Atiyah's observation: Fredholm operators pair with vector bundles to give integers.

Definition

A *spectral triple* $(\mathcal{A}, H, D)$ for A consists of

- a Hilbert space $H = H_+ \oplus H_-$ such that A is represented on $H_\pm$,
- $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ for a closed operator $D_+ : H_+ \to H_-$, $D_- := D_+^*$ and such that $a(1+D^2)^{-1}$ is compact a operator,
- a dense subalgebra $\mathcal{A} \subset A$ such that for $a \in \mathcal{A}$ the commutators $[D_+, a]$ are bounded.

$K^0(A) := \{[(A, H, D)] : \text{homotopy classes of spectral triples}\}$.

For a manifold M, $(L^2(\wedge^* T^* M), d + d^*) \in K^*(C(M))$ is a spectral triple.

$[D] \in K^0(A)$ pairs with $[P] \in K_0(A)$ by choosing a connection

$$\nabla : P \to P \otimes_{\mathcal{A}} \Omega^1_D, \quad \Omega^1_D := \left\{ \sum a_i [D, b_i] : a_i, b_i \in \mathcal{A} \right\}$$

forming a connection operator

$$1 \otimes_{\nabla} D : P \otimes_A H \to P \otimes_A H$$

and taking its index:

$$K^0(A) \times K_0(A) \to \mathbb{Z}, \qquad (D, P) \mapsto \mathrm{Ind}(1 \otimes_{\nabla} D)$$

Kasparov '80s: For a pair of C^*-algebras, a graded abelian group $KK_*(A, B)$, together with an associative, bilinear product

$$KK_i(A, B) \times KK_j(B, C) \to KK_{i+j}(A, C).$$

The Kasparov KK-theory groups recover K-theory and K-homology:

$$KK_*(\mathbb{C}, A) \simeq K_*(A) \qquad KK_*(A, \mathbb{C}) \simeq K^*(A).$$

The Atiyah–Singer index pairing is a special case of the KK-product:

$$KK_0(\mathbb{C}, A) \times KK_0(A, \mathbb{C}) \to KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}.$$

Semi-split extensions of C*-algebras

$$0 \to I \to A \to Q \to 0$$

give rise to elements in $KK_1(Q, I)$.

The SSH model: a lattice model with chiral symmetry. On the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \ell^2(\mathbb{Z})$ we consider the one dimensional Hamiltonian

$$H := \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes \mathrm{Id}_n \otimes U + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes \mathrm{Id}_n \otimes U^* + m\sigma_2 \otimes \mathrm{Id}_n \otimes \mathrm{Id}, \tag{4}$$

The model possess a chiral symmetry, implemented by the unitary operator $J = \sigma_3 \otimes \mathrm{Id}_n \otimes \mathrm{Id}$, and a spectral gap at $m = 0$.

Theorem (The bulk-edge correspondence)

Consider the Hamiltonian (4) and its half-space restriction. If U_F is the Fermi unitary and $\mathrm{Ch}_1(U_F)$ its winding number, and $\mathrm{tr}(\widehat{J}\widehat{P}_\delta)$ the boundary invariant, we have

$$\mathrm{Ch}_1(U_F) = \mathrm{Tr}(\tilde{J}\tilde{P}(\delta)).$$

The above equality follows from the K-theory six-term exact sequence coming from the Toeplitz extension.

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.
We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

Eugene Wigner (1902–1995)