On skew Brownian motion and its approximations

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Baby Steps Beyond the Horizon 2022

Outline

- Brownian motion and its variants in $[0, \infty)$
- Semi-permeable membranes
- Skew Brownian motion
- A kinetic model

Brownian motion (in 1D)

Usually defined as a family $w(t), t \geq 0$ of random variables

such that

1. $w(0) = 0$,
2. increments are independent: $w(t_n) - w(t_{n-1}), \ldots, w(t_1) - w(t_0)$ independent for $t_n > t_{n-1} > \ldots > t_0 \geq 0$
3. $w(t) - w(s) \sim \mathcal{N}(0, t - s)$,

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A BM starting at $x \in \mathbb{R}$: $x + w(t), t \geq 0$.

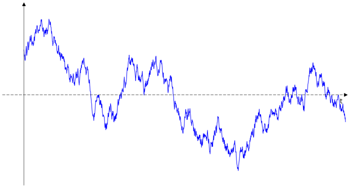
Described by a single operator: $f \mapsto \frac{1}{2} f''$

Reflected (reflecting) Brownian motion

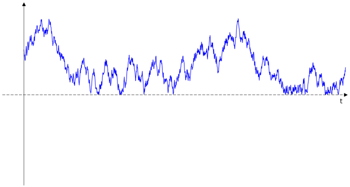
RBM

a process with values in $[0, \infty)$

$$|x + w(t)|.$$



Unrestricted BM



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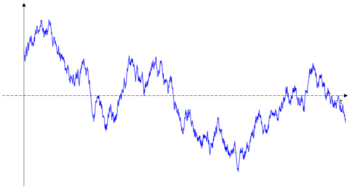

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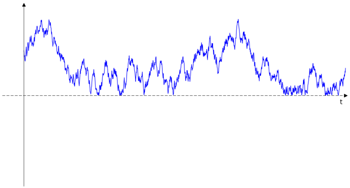

Reflecting BM

'Same' operator restricted to f such that $f'(0) = 0$.

Stopped Brownian motion

SBM

a process with values in $[0, \infty)$

$$= \begin{cases} x + w(t), & t < \tau_x, \\ 0, & t \geq \tau_x, \end{cases}$$

where $\tau_x = \inf\{t \geq 0; x + w(t) = 0\}$.

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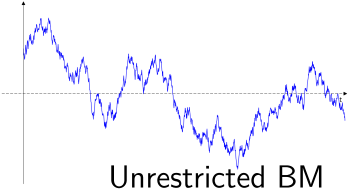

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Minimal Brownian motion

MBM

a process with values in $(0,\infty)$

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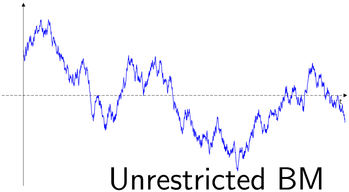

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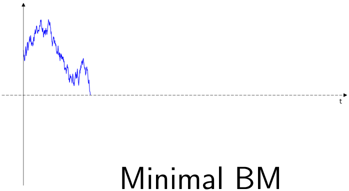

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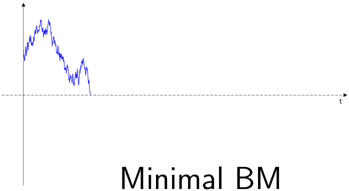

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Three extreme cases of boundary conditions

type of BM	minimal	reflecting	stopped
boundary condition	$f(0) = 0$	$f'(0) = 0$	$f''(0) = 0$
particle	removed	reflected	captured

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For processes in $[0, \infty)$, combination of these three:

$$\frac{1}{2} p f''(0) - (1 - p) f'(0) + c f(0) = 0,$$

$p \in [0, 1], c \geq 0.$

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Below,

- $p = 1$ — elementary exit
- $p = 0$ — elastic.

Elementary exit BM: $\frac{1}{2}f''(0) + cf(0) = 0$

EEBM

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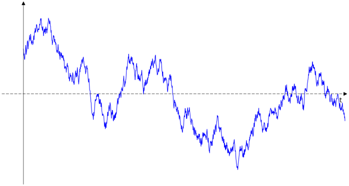
$\tau_x = \inf\{t \ge 0; x + w(t) = 0\}$, T – ind., $\Pr(T > s) = \mathrm{e}^{-cs}, s \ge 0$.



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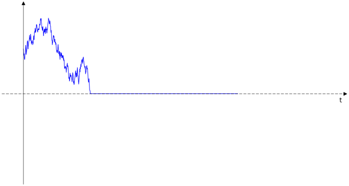

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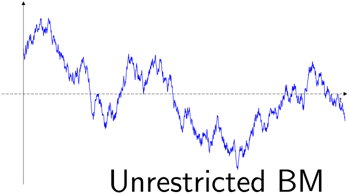

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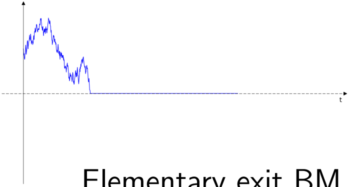
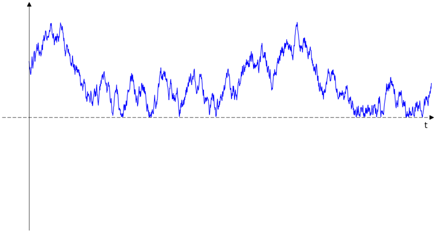


Elementary exit BM

c — rate of probability mass escape

Elastic Brownian motion (1): $f'(0) = cf(0)$

'How many zeros' of RBM?

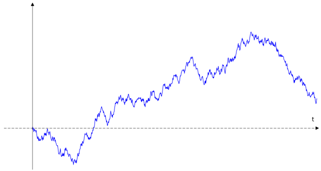


Amazingly,

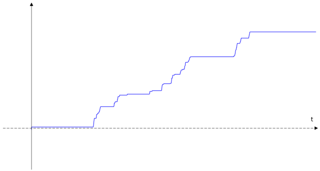
$$\widetilde{w}(t) := m(t) - w(t), \text{ where } m(t) := \max_{s\in[0,t]} w(t), \qquad t \geq 0$$

is also a reflecting Brownian motion (starting at $x = 0$).

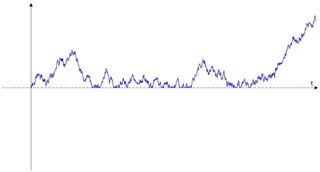
Elastic Brownian motion (2): $f'(0) = cf(0)$



(a) standard BM



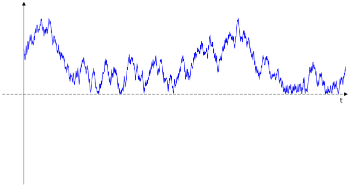
(b) maximum of BM

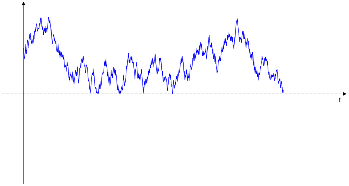


(c) difference

Elastic Brownian motion (3): $f'(0) = cf(0)$

Set of zeros is Cantor-like.
σ (local time) measures time spent at the boundary.





$\sigma \geq T \longrightarrow$ EBM undefined

Summary (and something new)

- In the boundary condition

$$\tfrac{1}{2}pf''(0) - (1-p)f'(0) + cf(0) = 0,$$

    - the larger the $p \in [0,1]$ the more sticky the boundary,
    - the smaller the $p \in [0,1]$ the more rigid the boundary,
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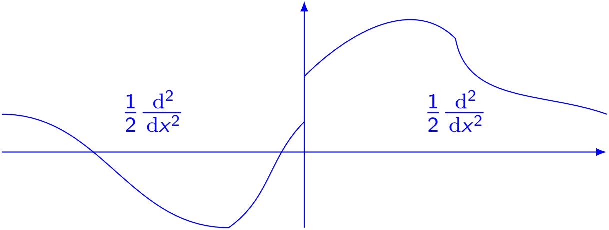
- In

$$\tfrac{1}{2}pf''(0) - (1-p)f'(0) + cf(0) = \boxed{d \int_{(0,\infty)} f \,\mathrm{dP}},$$

  where P – probability measure,
  - $d \leq c$ specifies the probability of starting anew after escaping through $x = 0$,
  - P is the distribution of the starting anew position.

Semi-permeable membranes (snapping out BM)

Test functions:



Transmission conditions ($\mu, \nu \geq 0$ — permeability coefficients):

$$f'(0+) = \mu(f(0+) - f(0-)), \qquad f'(0-) = \nu(f(0+) - f(0-)).$$

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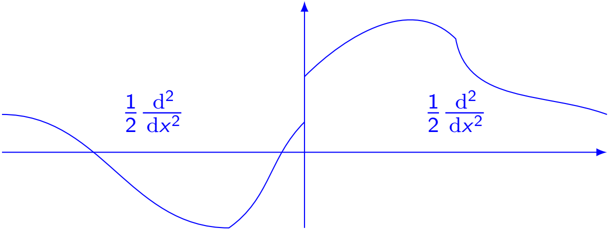

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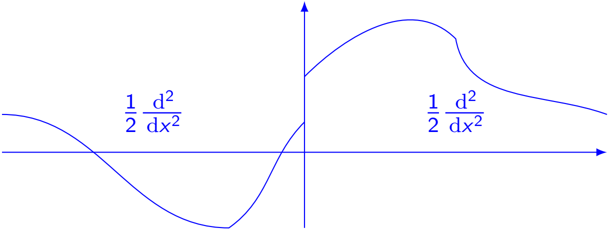

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As $n \to \infty$, the membrane 'disappears',and yet asymmetry remains

$$\mu f'(0-) = \nu f'(0+), \qquad f(0+) = f(0-).$$

Technicalities

- Convergence of semigroups
- In L^1 and C
- Weak convergence of processes
- Convergence of cosine families — uniform with respect to time!
- Generalizations: graphs.

Is it permeable or not?

$$\mu f'(0-) = \nu f'(0+), \qquad f(0+) = f(0-).$$

Orwell 'Animal Farm':

All animals are equal but some animals are more equal than others.

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- but rather complicated
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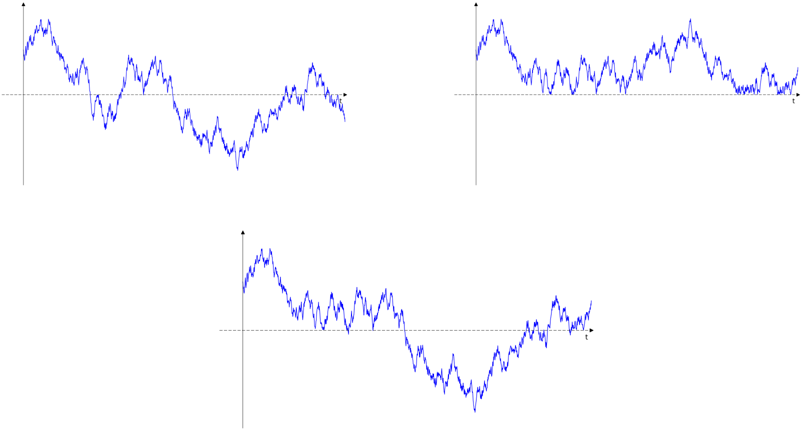
Key parameter

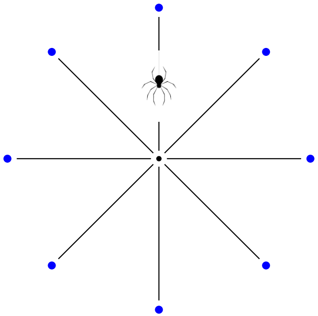
$$\alpha := \frac{\nu}{\nu + \mu} \in [0, 1].$$

Construction of skew BM



$$\alpha := \frac{\nu}{\nu + \mu} \text{— probability of not reflecting excursion downwards}$$

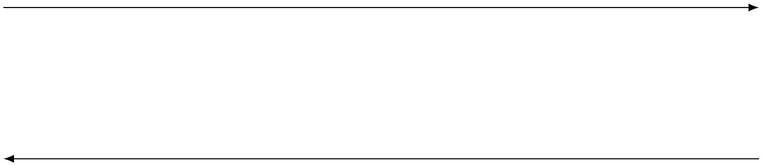
A number of generalizations



$$\sum_{i=1}^{n} \alpha_i f_i'(0) = 0$$

Walsh's spider, see also Portenko, Lejay.

Kinetic model with interface



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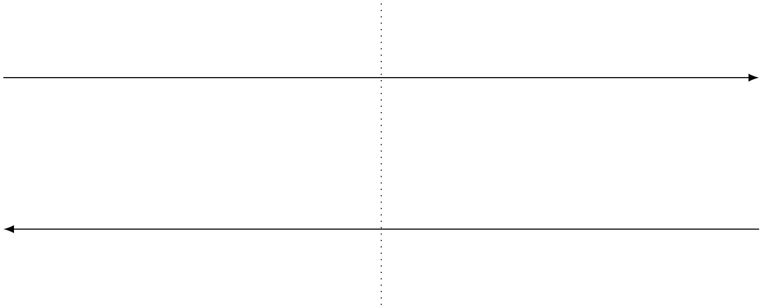

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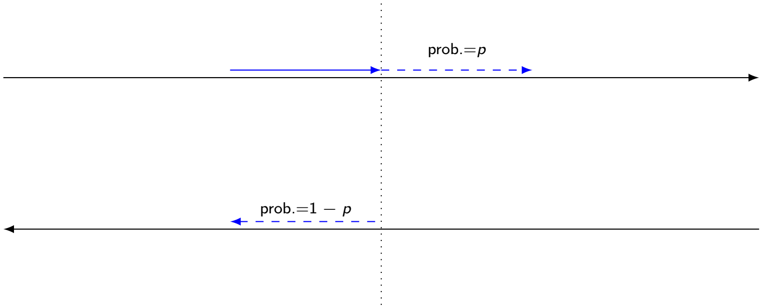

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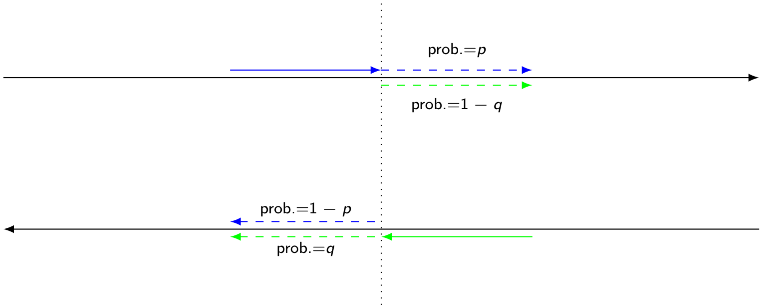

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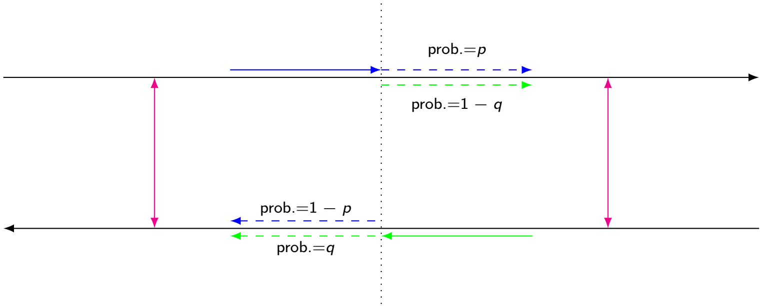

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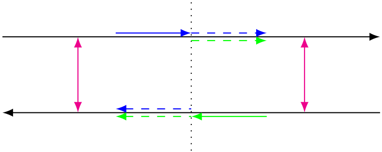

Diffusion approximation

- intensity of jumps — $\frac{1}{\epsilon^2}$
- particle's velocity — $\frac{1}{\epsilon}$

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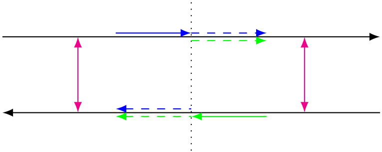

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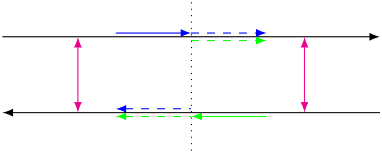

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With interface
$\longrightarrow$ skew BM $pf'(0+) = qf'(0-)$ $\longrightarrow$ $\alpha = \frac{p}{p+q}$.

Model kinetyczny — podsumowanie

Interface modifies the limit BM:

by turning it into skew BM with probability of not reflecting excursions

$$\alpha = \frac{p}{p+q}.$$

Thank you