Random walks in random environment

Dariusz Buraczewski

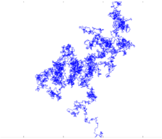
Baby Steps Beyond the Horizon

Bedlewo, August 31th, 2022

Why random walks?

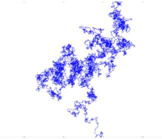
Why random walks?

How behaves a gas molecule in the air?



Why random walks?

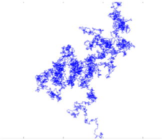
How behaves a gas molecule in the air?



In 1827 Robert Brown described behaviour of pollen of some plants immersed in water.

Why random walks?

How behaves a gas molecule in the air?

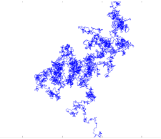


In 1827 Robert Brown described behaviour of pollen of some plants immersed in water.

In 1904/5 Marian Smoluchowski and Albert Einstein modeled the motion of individual particles.

Why random walks?

How behaves a gas molecule in the air?



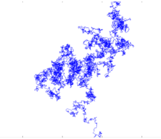
In 1827 Robert Brown described behaviour of pollen of some plants immersed in water.


In 1904/5 Marian Smoluchowski and Albert Einstein modeled the motion of individual particles.

Stock market fluctuations: S&P 500 Index

Why random walks?

How behaves a gas molecule in the air?



In 1827 Robert Brown described behaviour of pollen of some plants immersed in water.

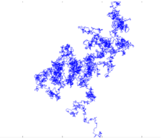
In 1904/5 Marian Smoluchowski and Albert Einstein modeled the motion of individual particles.

Stock market fluctuations: S&P 500 Index

All these examples led to a mathematical model of Brownian motion, which is a random process B_t with continuous trajectories, independent and stationary increments. Brownian motion has various surprising properties...

Why random walks?

How behaves a gas molecule in the air?



In 1827 Robert Brown described behaviour of pollen of some plants immersed in water.

In 1904/5 Marian Smoluchowski and Albert Einstein modeled the motion of individual particles.

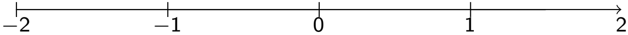
Stock market fluctuations: S&P 500 Index

All these examples led to a mathematical model of Brownian motion, which is a random process B_t with continuous trajectories, independent and stationary increments. Brownian motion has various surprising properties...

One way to better understand all these phenomena
is to introduce a simpler model: random walks

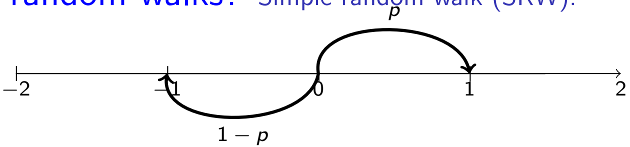
What are random walks? Simple random walk (SRW):

What are random walks? Simple random walk (SRW):



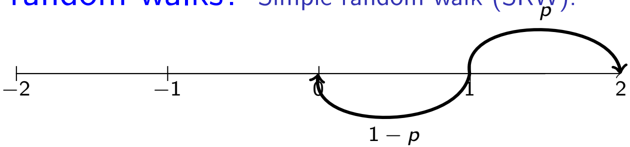
What are random walks?

Simple random walk (SRW):

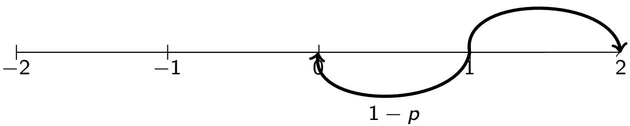


What are random walks?

Simple random walk (SRW):



What are random walks? Simple random walk (SRW):

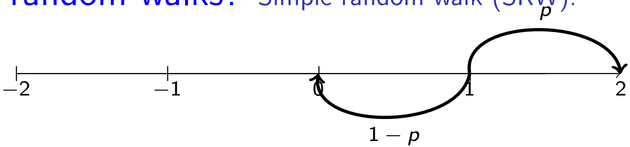


We need to define the process in a mathematical language:

$$X_0 = 0,\ X_{n+1} = X_n \pm 1,\ X_n = Y_1 + \ldots + Y_n,$$

where $\{Y_k\}_{k\in\mathbb{N}}$ are independent and $\mathbb{P}[Y = 1] = p = 1 - \mathbb{P}[Y = -1]$.

What are random walks? Simple random walk (SRW):



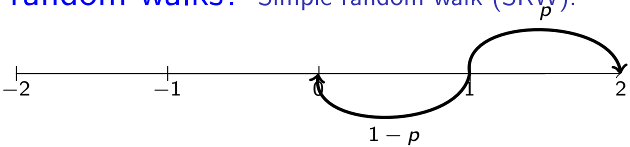
We need to define the process in a mathematical language:

$$X_0 = 0,\ X_{n+1} = X_n \pm 1,\ X_n = Y_1 + \ldots + Y_n,$$

where $\{Y_k\}_{k\in\mathbb{N}}$ are independent and $\mathbb{P}[Y = 1] = p = 1 - \mathbb{P}[Y = -1]$.
Fundamental questions:

- Does the process return to 0? (recurrence/transience),
- What is the rate of convergence to $+\infty$ if $p > 1/2$? (law of large numbers)
- What it the typical distance of the process from its mean (from 0 if p $= 1/2$)? (central limit theorem)

What are random walks? Simple random walk (SRW):



We need to define the process in a mathematical language:

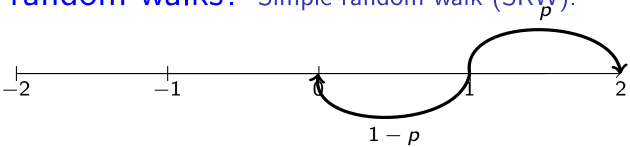
$$X_0 = 0,\ X_{n+1} = X_n \pm 1,\ X_n = Y_1 + \ldots + Y_n,$$

where $\{Y_k\}_{k \in \mathbb{N}}$ are independent and $\mathbb{P}[Y = 1] = p = 1 - \mathbb{P}[Y = -1]$.
Fundamental questions:

- Does the process return to 0? (recurrence/transience),
- What is the rate of convergence to $+\infty$ if $p > 1/2$? (law of large numbers)
- What it the typical distance of the process from its mean (from 0 if $\mathrm{p} = 1/2$)? (central limit theorem)

Random walk can be defined on more general structures (on graphs), but today we will discuss only random walks on $\mathbb{Z}$.

What are random walks? Simple random walk (SRW):



We need to define the process in a mathematical language:

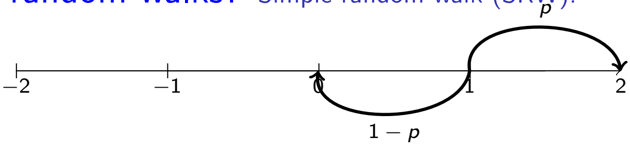
$$X_0 = 0,\ X_{n+1} = X_n \pm 1,\ X_n = Y_1 + \ldots + Y_n,$$

where $\{Y_k\}_{k\in\mathbb{N}}$ are independent and $\mathbb{P}[Y = 1] = p = 1 - \mathbb{P}[Y = -1]$.
Fundamental questions:

- Does the process return to 0? (recurrence/transience),
- What is the rate of convergence to $+\infty$ if $p > 1/2$? (law of large numbers)
- What it the typical distance of the process from its mean (from 0 if p $= 1/2$)? (central limit theorem)

Random walk can be defined on more general structures (on graphs), but today we will discuss only random walks on $\mathbb{Z}$.
Is this model the right one? Does it describe the real word correctly?

What are random walks? Simple random walk (SRW):



We need to define the process in a mathematical language:

$$X_0 = 0,\ X_{n+1} = X_n \pm 1,\ X_n = Y_1 + \ldots + Y_n,$$

where $\{Y_k\}_{k\in\mathbb{N}}$ are independent and $\mathbb{P}[Y = 1] = p = 1 - \mathbb{P}[Y = -1]$.
Fundamental questions:

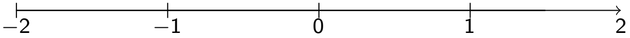
- Does the process return to 0? (recurrence/transience),
- What is the rate of convergence to $+\infty$ if $p > 1/2$? (law of large numbers)
- What it the typical distance of the process from its mean (from 0 if $\mathrm{p} = 1/2$)? (central limit theorem)

Random walk can be defined on more general structures (on graphs), but today we will discuss only random walks on $\mathbb{Z}$.
Is this model the right one? Does it describe the real word correctly?

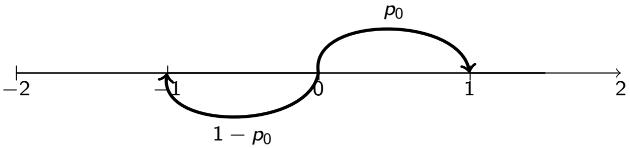
In many practical cases the environment in which the particle moves is highly irregular, due to factors such as defects, impurities, fluctuations, porosity etc.

How to model mathematically these defects?

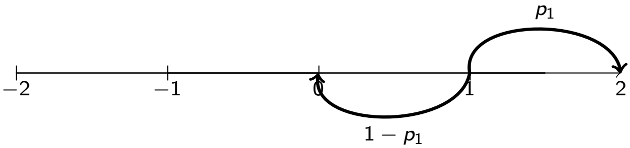
Random walk in random environment (RWRE) (Solomon; Kesten, Kozlov, Spitzer, 1970's)



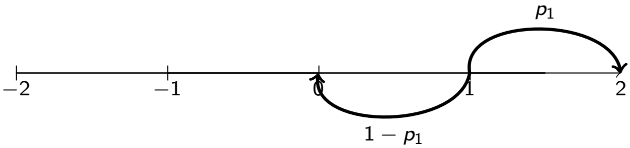
Random walk in random environment (RWRE) (Solomon; Kesten, Kozlov, Spitzer, 1970's)



Random walk in random environment (RWRE) (Solomon; Kesten, Kozlov, Spitzer, 1970's)



Random walk in random environment (RWRE) (Solomon; Kesten, Kozlov, Spitzer, 1970's)

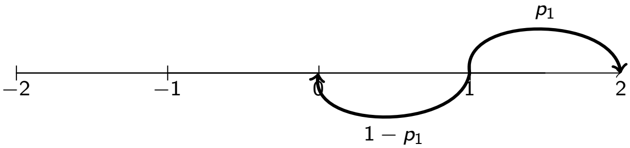


$\omega = \{p_k\}_{k \in \mathbb{Z}}$ is a given environment, $p_k \in (0,1)$ are iid.

$X = \{X_n\}_{n \in \mathbb{N}}$ is a random walk in random environment (RWRE)

$$P_\omega[X_{n+1} = k+1 \,|X_n = k] = p_k$$
$$P_\omega[X_{n+1} = k-1 \,|X_n = k] = 1 - p_k.$$

Random walk in random environment (RWRE) (Solomon; Kesten, Kozlov, Spitzer, 1970's)



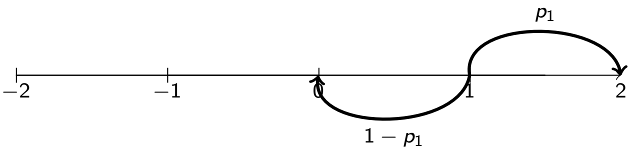
$\omega = \{p_k\}_{k\in\mathbb{Z}}$ is a given environment, $p_k \in (0,1)$ are iid.

$X = \{X_n\}_{n\in\mathbb{N}}$ is a random walk in random environment (RWRE)

$$P_\omega[X_{n+1} = k+1 \,|X_n = k] = p_k$$
$$P_\omega[X_{n+1} = k-1 \,|X_n = k] = 1 - p_k.$$

P_ω – quenched probability.

Random walk in random environment (RWRE) (Solomon; Kesten, Kozlov, Spitzer, 1970's)



$\omega = \{p_k\}_{k\in\mathbb{Z}}$ is a given environment, $p_k \in (0,1)$ are iid.

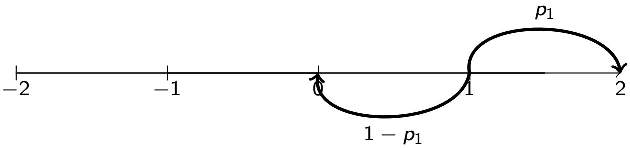
$X = \{X_n\}_{n\in\mathbb{N}}$ is a random walk in random environment (RWRE)

$$P_\omega[X_{n+1} = k+1 \,|X_n = k] = p_k$$
$$P_\omega[X_{n+1} = k-1 \,|X_n = k] = 1 - p_k.$$

P_ω – quenched probability. Define the annealed probability $\mathbb{P}$ viz.

$$\mathbb{P}[X \in A,\, \omega \in B] = \int_B P_\omega[X \in A]\, P(\mathrm{d}\omega).$$

Random walk in random environment (RWRE) (Solomon; Kesten, Kozlov, Spitzer, 1970's)



$\omega = \{p_k\}_{k \in \mathbb{Z}}$ is a given environment, $p_k \in (0,1)$ are iid.

$X = \{X_n\}_{n \in \mathbb{N}}$ is a random walk in random environment (RWRE)

$$P_\omega[X_{n+1} = k+1 \,|X_n = k] = p_k$$
$$P_\omega[X_{n+1} = k-1 \,|X_n = k] = 1 - p_k.$$

P_ω – quenched probability. Define the annealed probability $\mathbb{P}$ viz.

$$\mathbb{P}[X \in A,\, \omega \in B] = \int_B P_\omega[X \in A]\, P(\mathrm{d}\omega).$$

There are two ways to observe the process:

- quenched, under P_ω, then $\{X_n\}$ is a Markov chain;
- annealed, under $\mathbb{P}$, then $\{X_n\}$ **is not** a Markov chain.

Theorem [Solomon '75, Recurrence and transience]. Let $\rho = \frac{1-p}{p}$

- If $\mathbb{E} \log \rho = 0$, then $\liminf X_n = -\infty$, $\limsup X_n = \infty$, $\mathbb{P}$ a.s.
- If $\mathbb{E} \log \rho < 0$, then $X_n \to \infty$, $\mathbb{P}$ a.s.

Theorem [Solomon '75, Recurrence and transience]. Let $\rho = \frac{1-p}{p}$

- If $\mathbb{E} \log \rho = 0$, then $\liminf X_n = -\infty$, $\limsup X_n = \infty$, $\mathbb{P}$ a.s.
- If $\mathbb{E} \log \rho < 0$, then $X_n \to \infty$, $\mathbb{P}$ a.s.

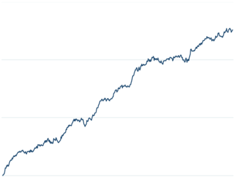
Let us simulate SRW and RWRE

Theorem [Solomon '75, Recurrence and transience]. Let $\rho = \frac{1-p}{p}$

- If $\mathbb{E} \log \rho = 0$, then $\liminf X_n = -\infty$, $\limsup X_n = \infty$, $\mathbb{P}$ a.s.
- If $\mathbb{E} \log \rho < 0$, then $X_n \to \infty$, $\mathbb{P}$ a.s.

Let us simulate SRW and RWRE

SRW with $p > 1/2$

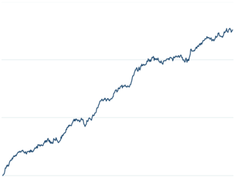


Theorem [Solomon '75, Recurrence and transience]. Let $\rho = \frac{1-p}{p}$

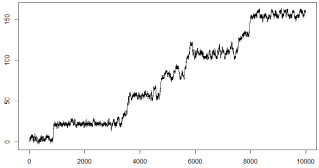
- If $\mathbb{E} \log \rho = 0$, then $\liminf X_n = -\infty$, $\limsup X_n = \infty$, $\mathbb{P}$ a.s.
- If $\mathbb{E} \log \rho < 0$, then $X_n \to \infty$, $\mathbb{P}$ a.s.

Let us simulate SRW and RWRE

SRW with $p > 1/2$



RWRE with $\mathbb{E} \log \rho < 0$ and $\mathbb{P}(\rho > 1) > 0$

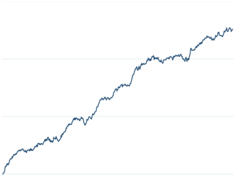


Theorem [Solomon '75, Recurrence and transience]. Let $\rho = \frac{1-p}{p}$

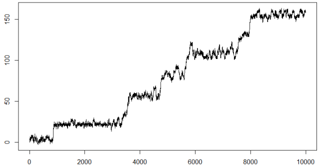
- If $\mathbb{E} \log \rho = 0$, then $\liminf X_n = -\infty$, $\limsup X_n = \infty$, $\mathbb{P}$ a.s.
- If $\mathbb{E} \log \rho < 0$, then $X_n \to \infty$, $\mathbb{P}$ a.s.

Let us simulate SRW and RWRE

SRW with $p > 1/2$



RWRE with $\mathbb{E} \log \rho < 0$ and $\mathbb{P}(\rho > 1) > 0$



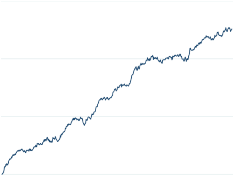
How to measure the force that pushes the walker to $+\infty$? Let us consider $n \mapsto \sum_{k=1}^{n} \log \rho_k$

Theorem [Solomon '75, Recurrence and transience]. Let $\rho = \frac{1-p}{p}$

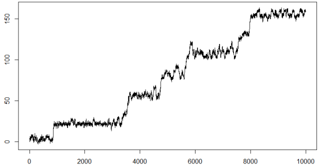
- If $\mathbb{E}\log\rho = 0$, then $\liminf X_n = -\infty$, $\limsup X_n = \infty$, $\mathbb{P}$ a.s.
- If $\mathbb{E}\log\rho < 0$, then $X_n \to \infty$, $\mathbb{P}$ a.s.

Let us simulate SRW and RWRE

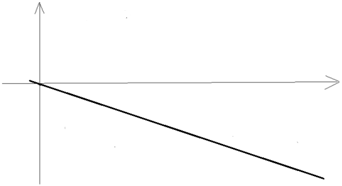
SRW with $p > 1/2$



RWRE with $\mathbb{E}\log\rho < 0$ and $\mathbb{P}(\rho > 1) > 0$



How to measure the force that pushes the walker to $+\infty$? Let us consider $n \mapsto \sum_{k=1}^{n} \log\rho_k$



SRW	RWRE
$p_k = p > 1/2$ is constant	$\{p_k\}$ are i.i.d., $\mathbb{E} \log \rho < 0$ and $\mathbb{P}[\rho > 1] > 0$

Law of Large Numbers

$\frac{X_n}{n} \to v > 0$ a.s.	$\frac{X_n}{n} \to v \geq 0$ a.s.

Central Limit Theorem

$\frac{X_n - nv}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$	$\frac{X_n - nv}{a_{\alpha,n}} \xrightarrow{d} L_\alpha$ [depending on α, $a_{\alpha,n} = \sqrt{n}, n^\alpha, n^{1/\alpha}$]

Large deviations

$\mathbb{P}(\lvert X_n/n - v\rvert \geq \varepsilon)$ is exp. small	$\mathbb{P}(X_n < (v-\varepsilon)n)$ is polyn. small

Many walkers

3 walkers meet i.o.	arbitrary many walkers meet i.o.

Limit theorems for sums of independent and identically distributed random variables:

- CLT: if $\mathbb{E}Y^2 < \infty$, then $\frac{Y_1+\cdots+Y_n-n\mathbb{E}X}{\mathrm{Var Y}\cdot\sqrt{n}} \xrightarrow{d} N(0,1)$

Limit theorems for sums of independent and identically distributed random variables:

- CLT: if $\mathbb{E}Y^2 < \infty$, then $\frac{Y_1+\cdots+Y_n-n\mathbb{E}X}{\mathrm{VarY}\cdot\sqrt{n}} \xrightarrow{d} N(0,1)$
- stable laws (particular case): if $Y > 0$, $\mathbb{P}[Y > t]t^\alpha \to C$ for some $\alpha < 2$, then $\frac{Y_1+\cdots+Y_n-nv}{n^{1/\alpha}} \xrightarrow{d} \mathcal{L}_\alpha$.

Limit theorems for sums of independent and identically distributed random variables:

- CLT: if $\mathbb{E}Y^2 < \infty$, then $\frac{Y_1+\cdots+Y_n-n\mathbb{E}X}{\mathrm{VarY}\cdot\sqrt{n}} \xrightarrow{d} N(0,1)$
- stable laws (particular case): if $Y > 0$, $\mathbb{P}[Y > t]t^\alpha \to C$ for some $\alpha < 2$, then $\frac{Y_1+\cdots+Y_n-nv}{n^{1/\alpha}} \xrightarrow{d} \mathcal{L}_\alpha$.

Recall $\rho = \frac{1-p}{p}$

Theorem [Kesten, Kozlov, Spitzer '75, Central Limit Theorem for RWRE] Assume RWRE is transient ($\mathbb{E}[\log\rho] < 0$) and $\mathbb{E}\rho^\alpha = 1$ for some α, then

$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha.$$

If $\alpha > 2$, then $a_n = \sqrt{n}$ and $L_\alpha = N(1,\sigma^2)$. Otherwise the limit and normalization are related to the parameter α and the corresponding stable law $\mathcal{L}_\alpha$.

Let $\rho = \frac{1-p}{p}$.

Let $\rho = \frac{1-p}{p}$.

Theorem (Kesten, Kozlov, Spitzer '75)

Central limit theorem

$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha$$

Let $\rho = \frac{1-p}{p}$.

Theorem (Kesten, Kozlov, Spitzer '75)

Central limit theorem

$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha$$

Let $T_n = \inf\{k : X_k = n\}$. One needs to prove

$$\frac{T_n - (1/v)n}{n^{1/\alpha}} \Rightarrow \mathcal{L}_\alpha$$

Let $\rho = \frac{1-p}{p}$.

Theorem (Kesten, Kozlov, Spitzer '75)

Central limit theorem

$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha$$

Let $T_n = \inf\{k : X_k = n\}$. One needs to prove

$$\frac{T_n - (1/v)n}{n^{1/\alpha}} \Rightarrow \mathcal{L}_\alpha$$

$$T_n =$$

Let $\rho = \frac{1-p}{p}$.

Theorem (Kesten, Kozlov, Spitzer '75)

Central limit theorem

$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha$$

Let $T_n = \inf\{k : X_k = n\}$. One needs to prove

$$\frac{T_n - (1/v)n}{n^{1/\alpha}} \Rightarrow \mathcal{L}_\alpha$$

$$\begin{aligned} T_n &= \# \text{ of steps during } [0, T_n) \\ &= \end{aligned}$$

Let $\rho = \frac{1-p}{p}$.

Theorem (Kesten, Kozlov, Spitzer '75)

Central limit theorem

$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha$$

Let $T_n = \inf\{k : X_k = n\}$. One needs to prove

$$\frac{T_n - (1/v)n}{n^{1/\alpha}} \Rightarrow \mathcal{L}_\alpha$$

$$\begin{aligned} T_n =& \# \text{ of steps during } [0, T_n) \\ =& \# \text{ of steps to the right during } [0, T_n) \\ &+ \# \text{ of steps to the left during } [0, T_n) \\ =& \end{aligned}$$

Let $\rho = \frac{1-p}{p}$.

Theorem (Kesten, Kozlov, Spitzer '75)

Central limit theorem

$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha$$

Let $T_n = \inf\{k : X_k = n\}$. One needs to prove

$$\frac{T_n - (1/v)n}{n^{1/\alpha}} \Rightarrow \mathcal{L}_\alpha$$

$$\begin{aligned} T_n &= \# \text{ of steps during } [0, T_n) \\ &= \# \text{ of steps to the right during } [0, T_n) \\ &\quad + \# \text{ of steps to the left during } [0, T_n) \\ &= n + 2 \cdot \# \text{ of steps to the left during } [0, T_n) \\ &= \end{aligned}$$

Let $\rho = \frac{1-p}{p}$.

Theorem (Kesten, Kozlov, Spitzer '75)

Central limit theorem

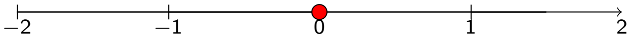
$$\frac{X_n - vn}{a_n} \Rightarrow L_\alpha$$

Let $T_n = \inf\{k : X_k = n\}$. One needs to prove

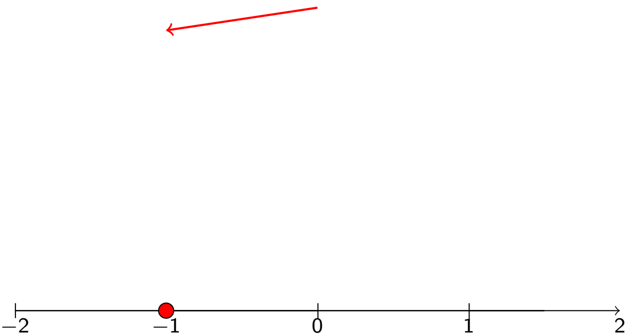
$$\frac{T_n - (1/v)n}{n^{1/\alpha}} \Rightarrow \mathcal{L}_\alpha$$

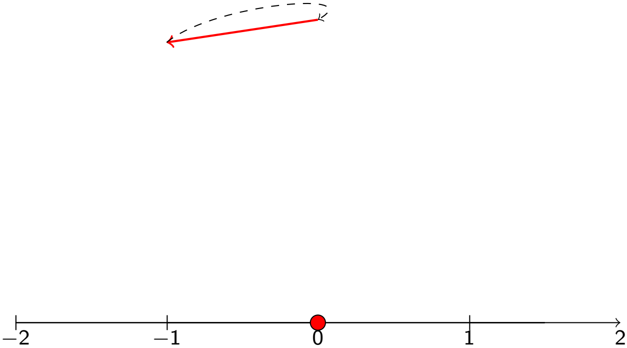
$$\begin{aligned} T_n &= \#\text{ of steps during } [0, T_n) \\ &= \#\text{ of steps to the right during } [0, T_n) \\ &\quad + \#\text{ of steps to the left during } [0, T_n) \\ &= n + 2 \cdot \#\text{ of steps to the left during } [0, T_n) \\ &= n + 2 \cdot U_n \end{aligned}$$

Goal: prove limit theorems for U_n (the number of steps to the left during $[0, T_n)$)

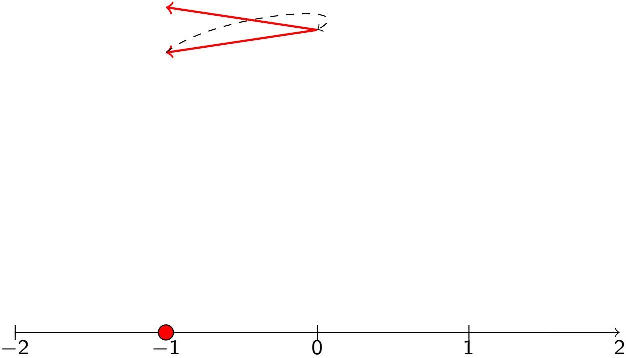


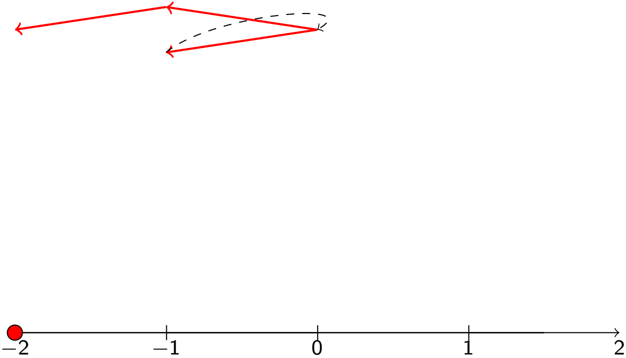
−2
−1
0
1
2


−2
−1
0
1
2

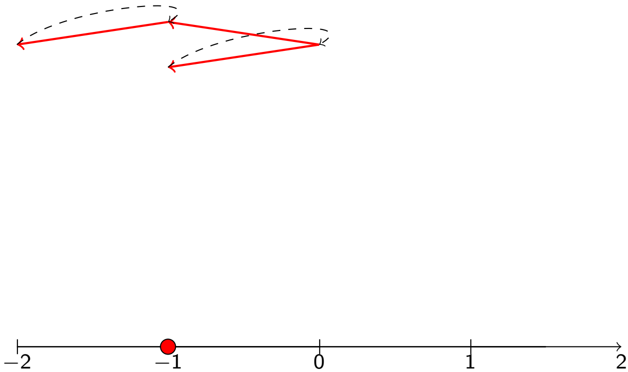


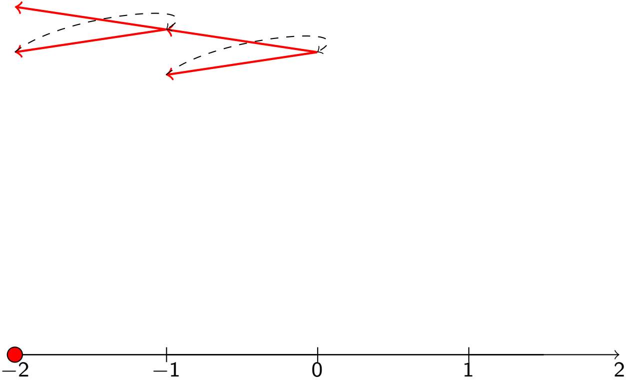
−2
−1
0
1
2

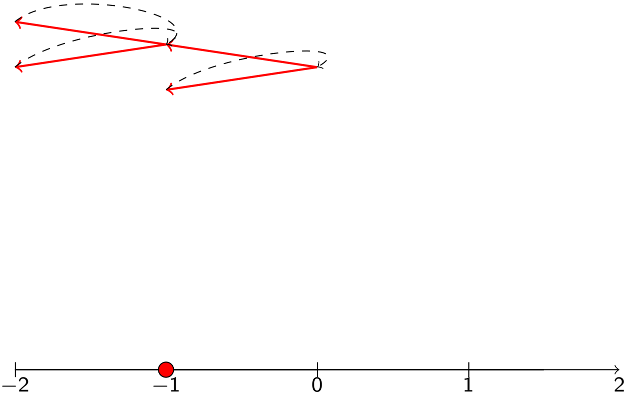

−2
−1
0
1
2

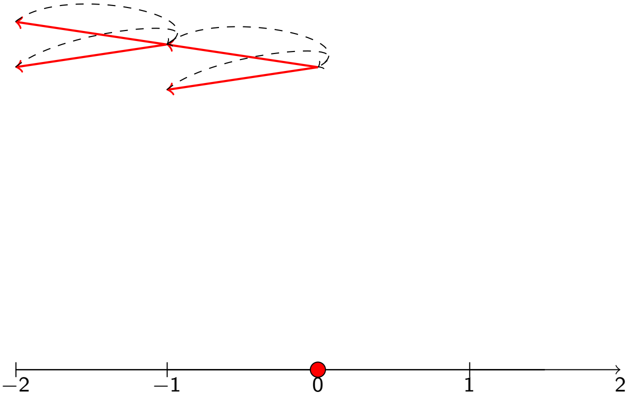


−2
−1
0
1
2

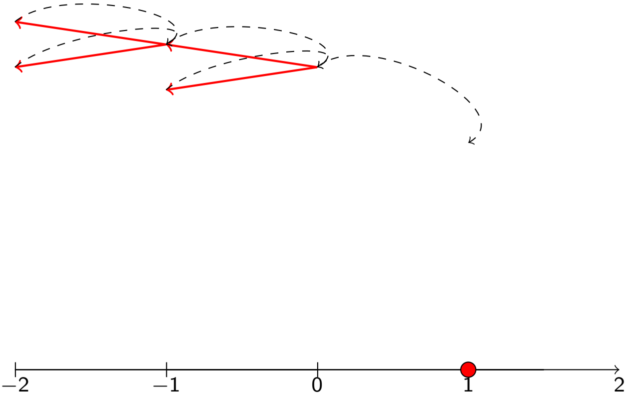

−2
−1
0
1
2


−2
−1
0
1
2

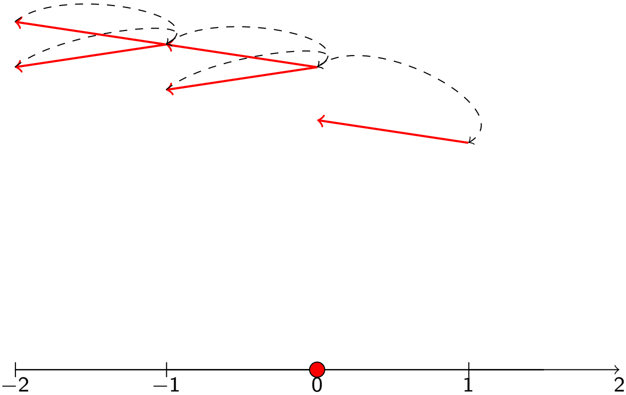

−2
−1
0
1
2



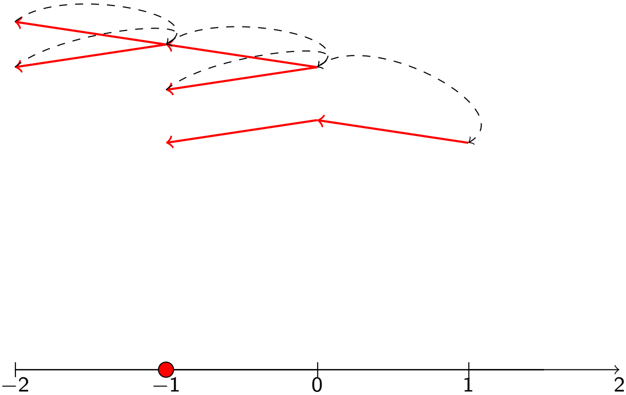
−2
−1
0
1
2



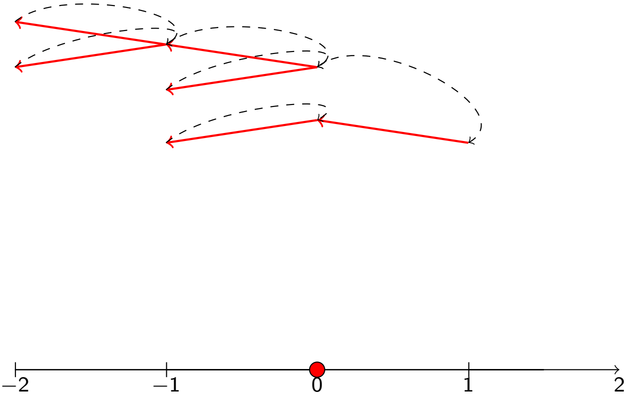
−2
−1
0
1
2



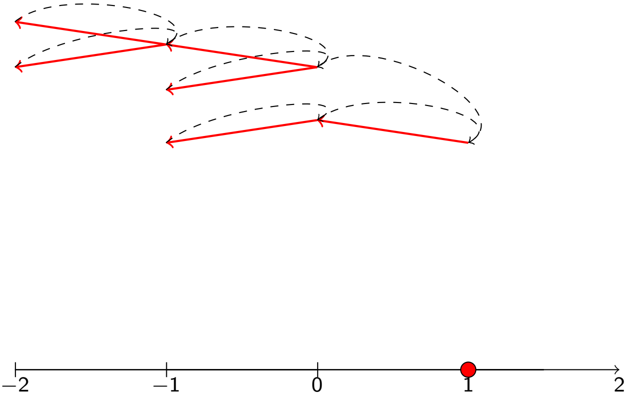
−2
−1
0
1
2



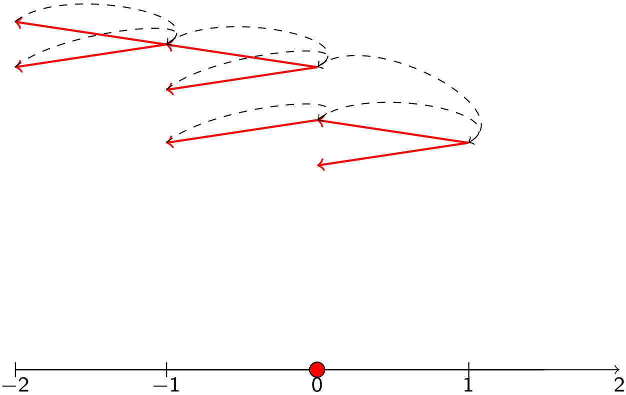
−2
−1
0
1
2



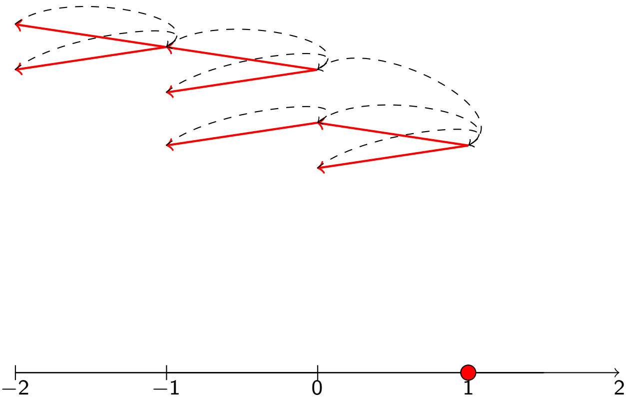
−2
−1
0
1
2



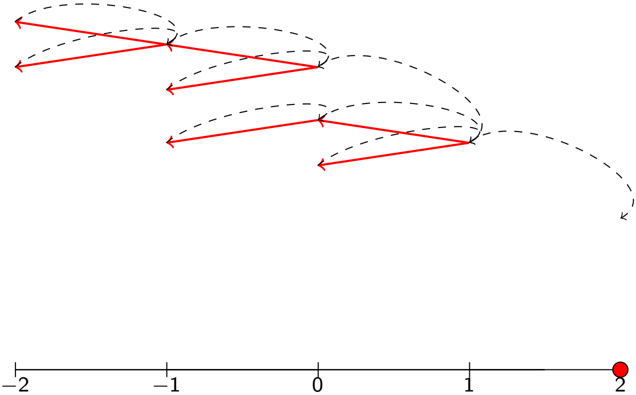
−2
−1
0
1
2

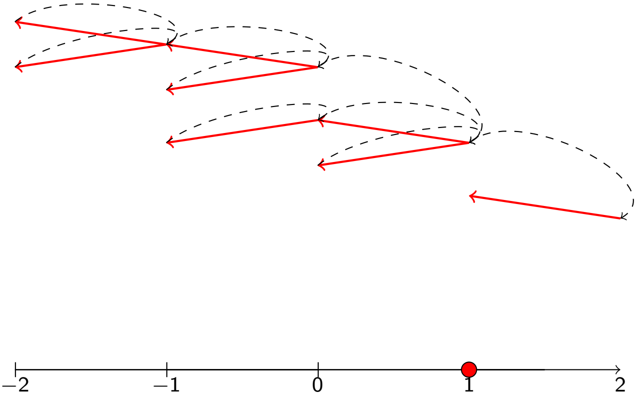


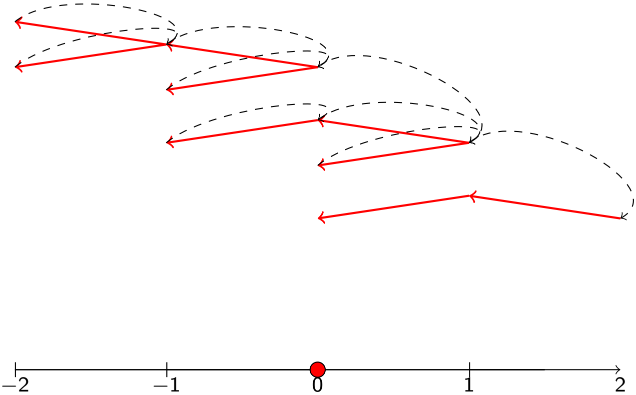
−2
−1
0
1
2

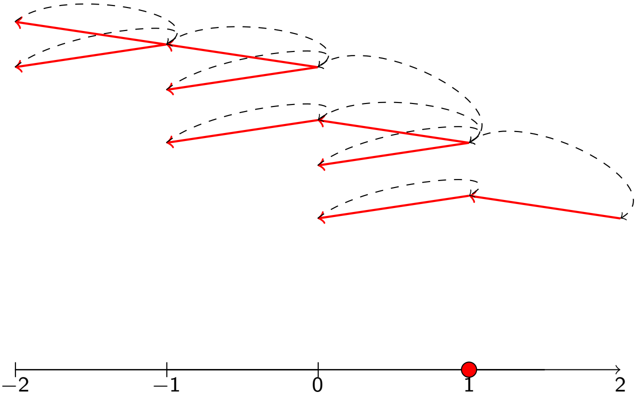


−2
−1
0
1
2

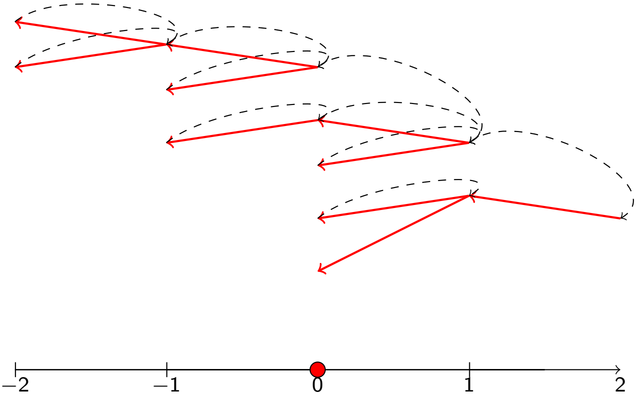

−2
−1
0
1
2

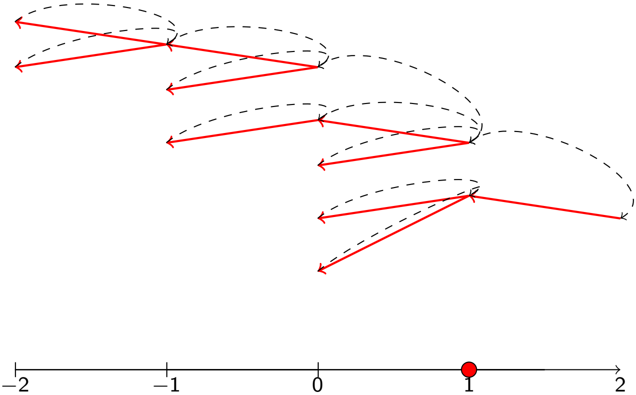

−2
−1
0
1
2

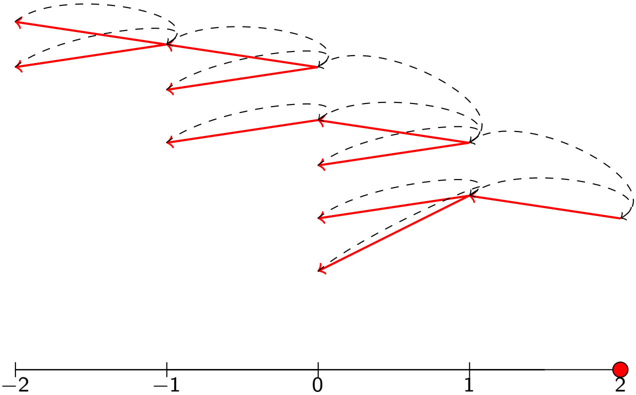

−2
−1
0
1
2

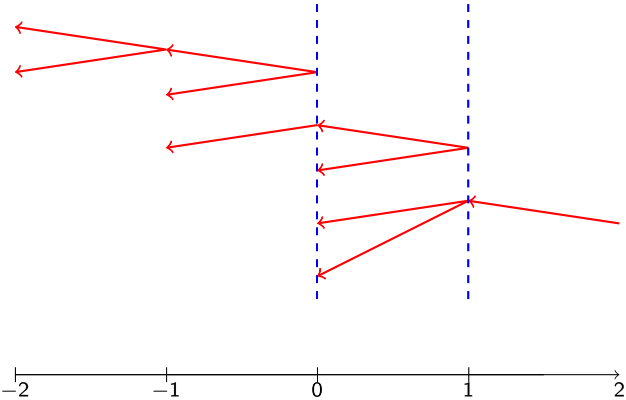


−2
−1
0
1
2

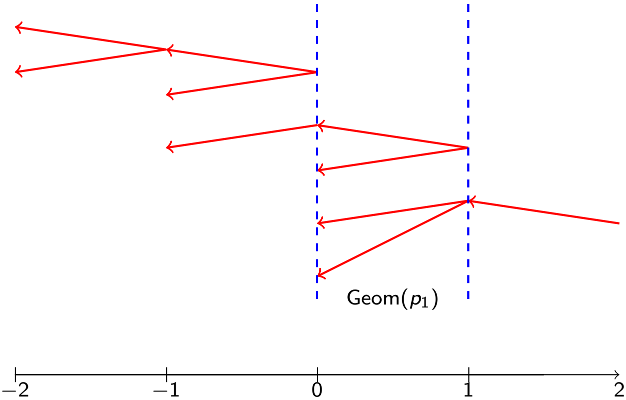

−2
−1
0
1
2

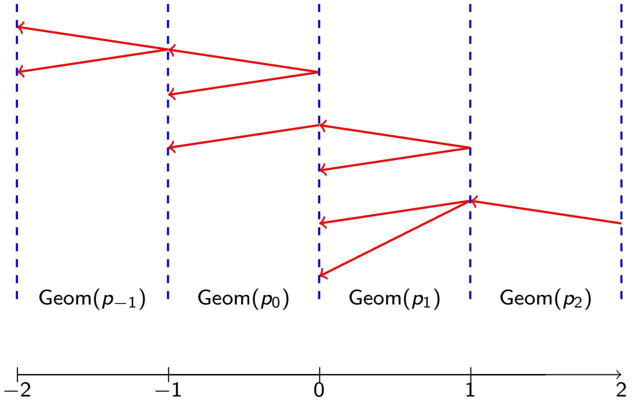

−2
−1
0
1
2


−2
−1
0
1
2



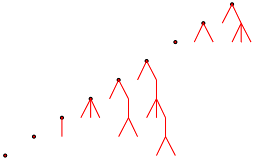
−2
−1
0
1
2


Geom(p_1)
−2
−1
0
1
2



Branching process in random environment with one immigrant

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

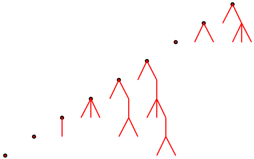


$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$

where $V_j^k \sim \mathrm{Geom}(p_k)$.

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$
where $V_j^k \sim \mathrm{Geom}(p_k)$.



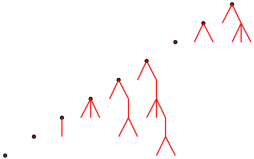
Goal: find limit theorems for $Z_1 + \ldots + Z_n$

- $\tau_1 = \inf\{n \geq 1 :\ Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$

where $V_j^k \sim \mathrm{Geom}(p_k)$.



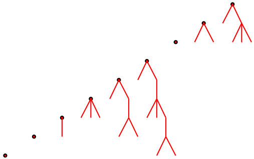
Goal: find limit theorems for $Z_1 + \ldots + Z_n$

- $\tau_1 = \inf\{n \geq 1 :\ Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;
- τ_k - the kth extinction time; $W_k = \sum_{j=\tau_{k-1}+1}^{\tau_k} Z_j$.

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$

where $V_j^k \sim \mathrm{Geom}(p_k)$.



Goal: find limit theorems for $Z_1 + \ldots + Z_n$

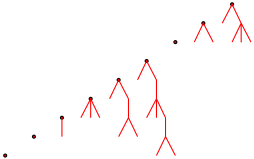
- $\tau_1 = \inf\{n \geq 1 :\ Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;
- τ_k - the kth extinction time; $W_k = \sum_{j=\tau_{k-1}+1}^{\tau_k} Z_j$.

Then $Z_1 + \ldots + Z_n \approx W_1 + \ldots W_{n/\mathbb{E}\tau}$ and $\{W_k\}$ are i.i.d.

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$

where $V_j^k \sim \mathrm{Geom}(p_k)$.



Goal: find limit theorems for $Z_1 + \ldots + Z_n$

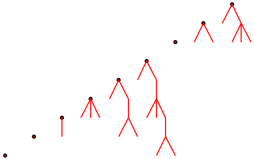
- $\tau_1 = \inf\{n \geq 1 : \ Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;
- τ_k - the kth extinction time; $W_k = \sum_{j=\tau_{k-1}+1}^{\tau_k} Z_j$.

Then $Z_1 + \ldots + Z_n \approx W_1 + \ldots W_{n/\mathbb{E}\tau}$ and $\{W_k\}$ are i.i.d.

How to estimate the size of W_k? Let $R_n = \mathbb{E}_\omega Z_n$, recall $\rho = \frac{1-p}{p}$, then $\mathbb{E}_\omega V_j^k = \rho_k$.

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0, \; Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$
$$\text{where } V_j^k \sim \mathrm{Geom}(p_k).$$



Goal: find limit theorems for $Z_1 + \ldots + Z_n$

- $\tau_1 = \inf\{n \geq 1 : \; Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;
- τ_k - the kth extinction time; $W_k = \sum_{j=\tau_{k-1}+1}^{\tau_k} Z_j$.

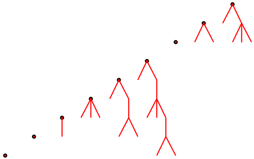
Then $Z_1 + \ldots + Z_n \approx W_1 + \ldots W_{n/\mathbb{E}\tau}$ and $\{W_k\}$ are i.i.d.

How to estimate the size of W_k? Let $R_n = \mathbb{E}_\omega Z_n$, recall $\rho = \frac{1-p}{p}$, then $\mathbb{E}_\omega V_j^k = \rho_k$.

$R_n = \mathbb{E}_\omega Z_n = \rho_n(\mathbb{E}_\omega Z_{n-1} + 1) = \rho_n(R_{n-1} + 1)$ random difference equation (RDE).

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$
where $V_j^k \sim \mathrm{Geom}(p_k)$.



Goal: find limit theorems for $Z_1 + \ldots + Z_n$

- $\tau_1 = \inf\{n \geq 1 :\ Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;
- τ_k - the kth extinction time; $W_k = \sum_{j=\tau_{k-1}+1}^{\tau_k} Z_j$.

Then $Z_1 + \ldots + Z_n \approx W_1 + \ldots W_{n/\mathbb{E}\tau}$ and $\{W_k\}$ are i.i.d.

How to estimate the size of W_k? Let $R_n = \mathbb{E}_\omega Z_n$, recall $\rho = \frac{1-p}{p}$, then $\mathbb{E}_\omega V_j^k = \rho_k$.

$R_n = \mathbb{E}_\omega Z_n = \rho_n(\mathbb{E}_\omega Z_{n-1} + 1) = \rho_n(R_{n-1} + 1)$ random difference equation (RDE).

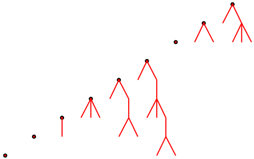
Let ν be the stationary measure of R_n, that is if $R \sim \nu$, then $R \stackrel{d}{=} \rho(R+1)$.

It turns out that $\mathbb{P}[W > t] \sim \mathbb{P}[R > t] \sim Ct^{-\alpha}$, where α is the parameter such that $\mathbb{E}\rho^\alpha = 1$.

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$

where $V_j^k \sim \mathrm{Geom}(p_k)$.



Goal: find limit theorems for $Z_1 + \ldots + Z_n$

- $\tau_1 = \inf\{n \ge 1 : \ Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;
- τ_k - the kth extinction time; $W_k = \sum_{j=\tau_{k-1}+1}^{\tau_k} Z_j$.

Then $Z_1 + \ldots + Z_n \approx W_1 + \ldots W_{n/\mathbb{E}\tau}$ and $\{W_k\}$ are i.i.d.

How to estimate the size of W_k? Let $R_n = \mathbb{E}_\omega Z_n$, recall $\rho = \frac{1-p}{p}$, then $\mathbb{E}_\omega V_j^k = \rho_k$.

$R_n = \mathbb{E}_\omega Z_n = \rho_n(\mathbb{E}_\omega Z_{n-1} + 1) = \rho_n(R_{n-1} + 1)$ random difference equation (RDE).

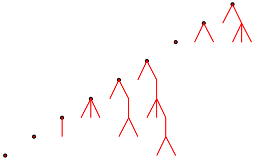
Let ν be the stationary measure of R_n, that is if $R \sim \nu$, then $R \stackrel{d}{=} \rho(R+1)$.

It turns out that $\mathbb{P}[W > t] \sim \mathbb{P}[R > t] \sim Ct^{-\alpha}$, where α is the parameter such that $\mathbb{E}\rho^\alpha = 1$.

Therefore if $\alpha < 2$, then $\frac{Z_1+\ldots+Z_n}{n^{1/\alpha}} \sim \frac{W_1+\ldots W_{n/\mathbb{E}\tau}}{n^{1/\alpha}} \xrightarrow{d} \mathcal{L}_\alpha$.

$\{Z_n\}$ is branching process in random environment with one immigrant, that is Z_n is the population at time n.

$$Z_0 = 0,\ Z_k = \sum_{j=1}^{Z_{k-1}+1} V_j^k,$$
where $V_j^k \sim \mathrm{Geom}(p_k)$.



Goal: find limit theorems for $Z_1 + \ldots + Z_n$

- $\tau_1 = \inf\{n \geq 1 : \ Z_n = 0\}$ - the first extinction time;
- $W_1 = \sum_{j=1}^{\tau_1} Z_j$ - total population before the first extinction time;
- τ_k - the kth extinction time; $W_k = \sum_{j=\tau_{k-1}+1}^{\tau_k} Z_j$.

Then $Z_1 + \ldots + Z_n \approx W_1 + \ldots W_{n/\mathbb{E}\tau}$ and $\{W_k\}$ are i.i.d.

How to estimate the size of W_k? Let $R_n = \mathbb{E}_\omega Z_n$, recall $\rho = \frac{1-p}{p}$, then $\mathbb{E}_\omega V_j^k = \rho_k$.

$$R_n = \mathbb{E}_\omega Z_n = \rho_n(\mathbb{E}_\omega Z_{n-1} + 1) = \rho_n(R_{n-1}+1)$$ random difference equation (RDE).

Let ν be the stationary measure of R_n, that is if $R \sim \nu$, then $R \stackrel{d}{=} \rho(R+1)$.

It turns out that $\mathbb{P}[W > t] \sim \mathbb{P}[R > t] \sim Ct^{-\alpha}$, where α is the parameter such that $\mathbb{E}\rho^\alpha = 1$.

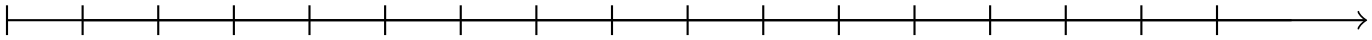
Therefore if $\alpha < 2$, then $\frac{Z_1+\ldots+Z_n}{n^{1/\alpha}} \sim \frac{W_1+\ldots W_{n/\mathbb{E}\tau}}{n^{1/\alpha}} \xrightarrow{d} \mathcal{L}_\alpha$.

Summarizing: RWRE $\leftrightarrow$ BPREI $\leftrightarrow$ RDE

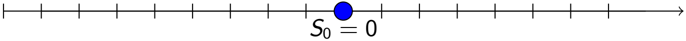
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE)

In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,

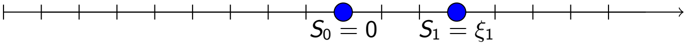
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



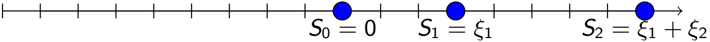
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



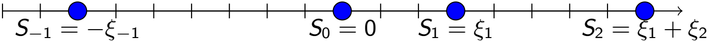
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



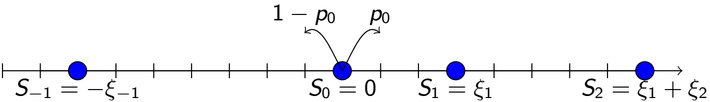
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



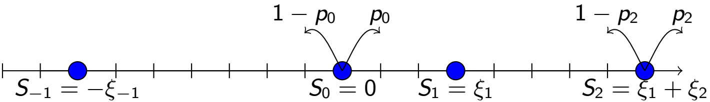
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



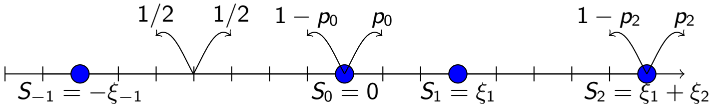
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



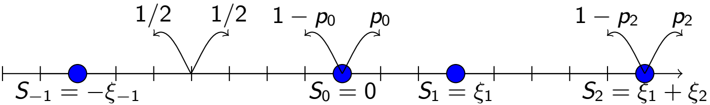
In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,

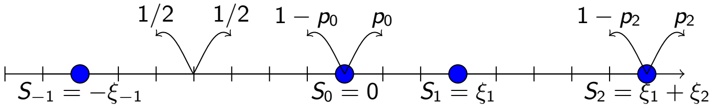


In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



Lemma: If $\mathbb{E}\log\rho < 0$ (for $\rho = \frac{1-p}{p}$) and $\mathbb{E}\log\xi < \infty$, then $X_n \to +\infty$.

In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



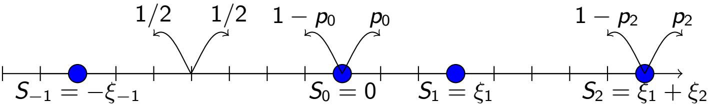
Lemma: If $\mathbb{E}\log\rho < 0$ (for $\rho = \frac{1-p}{p}$) and $\mathbb{E}\log\xi < \infty$, then $X_n \to +\infty$.

Theorem [B, Dyszewski, Iksanov, Marynych, Roitershtein]:

If $\mathbb{P}(\xi > x) \sim x^{-\beta}$, $\beta \in (0,1)$ and $\mathbb{E}\rho^\alpha < 1$ for $\beta/2 < \alpha$, then

$$\frac{X_n}{\sqrt{n}} \Rightarrow \chi$$

In 2017 Matzavinos, Roitershtein and Seol introduced random walk in sparse random environment (RWSRE) Parameters: $\{p_k\}_{k\in\mathbb{Z}}$ iid, $p_k \in (0,1)$, $\{\xi_k\}_{k\in\mathbb{Z}}$ iid, $\xi_k \in \mathbb{N}$,



Lemma: If $\mathbb{E}\log\rho < 0$ (for $\rho = \frac{1-p}{p}$) and $\mathbb{E}\log\xi < \infty$, then $X_n \to +\infty$.

Theorem [B, Dyszewski, Iksanov, Marynych, Roitershtein]:

If $\mathbb{P}(\xi > x) \sim x^{-\beta}$, $\beta \in (0,1)$ and $\mathbb{E}\rho^\alpha < 1$ for $\beta/2 < \alpha$, then

$$\frac{X_n}{\sqrt{n}} \Rightarrow \chi$$

Work in progress: understand the role of randomness ... (joint with P. Dyszewski and A. Kołodziejska):

1. Quenched limit theorem for RWSRE. We fix the environment. Then for $\alpha < 2$ ($\mathbb{E}\rho^\alpha = 1$) the limit in distribution of T_n does not exists, but one can consider the limit in a weaker sense ...
2. Large deviations for RWSRE.