Infinite structures: between mathematics and computer sciences

Mirna Džamonja

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“From finite to infinite” EU project FINTOINF H2020- No.1010232

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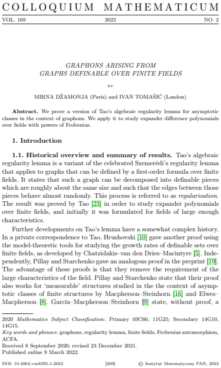
Some results


COLLOQUIUM MATHEMATICUM

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GRAPHONS ARISING FROM GRAPHS DEFINABLE OVER FINITE FIELDS

BY

MIRNA DŽAMONJA (Paris) and IVAN TOMAŠIĆ (London)

Abstract. We prove a version of Tao's algebraic regularity lemma for asymptotic classes in the context of graphons. We apply it to study expander difference polynomials over fields with powers of Frobenius.

1. Introduction

1.1. Historical overview and summary of results. Tao's algebraic regularity lemma is a variant of the celebrated Szemerédi's regularity lemma that applies to graphs that can be defined by a first-order formula over finite fields. It states that such a graph can be decomposed into definable pieces which are roughly about the same size and such that the edges between those pieces behave almost randomly. This process is referred to as *regularisation*. The result was proved by Tao [23] in order to study expander polynomials over finite fields, and initially it was formulated for fields of large enough characteristics.

Further developments on Tao's lemma have a somewhat complex history. In a private correspondence to Tao, Hrushovski [10] gave another proof using the model-theoretic tools for studying the growth rates of definable sets over finite fields, as developed by Chatzidakis–van den Dries–Macintyre [5]. Independently, Pillay and Starchenko gave an analogous proof in the preprint [19]. The advantage of these proofs is that they remove the requirement of the large characteristics of the field. Pillay and Starchenko state that their proof also works for 'measurable' structures studied in the the context of asymptotic classes of finite structures by Macpherson–Steinhorn [16] and Elwes–Macpherson [8]. García–Macpherson–Steinhorn [9] state, without proof, a

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Key words and phrases: graphons, regularity lemma, finite fields, Frobenius automorphism, ACFA.

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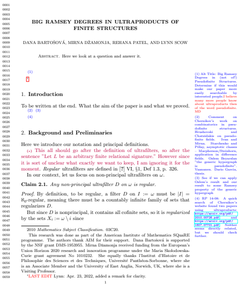
Figure: M. Dž.-I. Tomašić, *Graphons Arising From Graphs Definable over Finite Fields* Colloquium Mathematicum 169-2 (2022) pg. 269-306


BIG RAMSEY DEGREES IN ULTRAPRODUCTS OF FINITE STRUCTURES

DANA BARTOŠOVÁ, MIRNA DŽAMONJA, REHANA PATEL, AND LYNN SCOW

ABSTRACT. Here we look at a question and answer it.

(1)

1

1. Introduction

To be written at the end. What the aim of the paper is and what we proved.

(2) (3)

(4)

2. Background and Preliminaries

Here we introduce our notation and principal definitions.

(∗) This all should go after the definition of ultrafilters, so after the sentence "Let L be an arbitrary finite relational signature." However since it is sort of unclear what exactly we want to keep, I am ignoring it for the moment. *Regular* ultrafilters are defined in [?] VI, §1, Def 1.3, p. 326.

In our context, let us focus on non-principal ultrafilters on ω.

Claim 2.1. *Any non-principal ultrafilter D on ω is regular.*

Proof. By definition, to be regular, a filter D on $I := \omega$ must be $|I| = \aleph_0$-regular, meaning there must be a countably infinite family of sets that regularizes D.

But since D is nonprincipal, it contains all cofinite sets, so it is *regularized* by the sets $X_i := \omega \setminus i$ since

2010 *Mathematics Subject Classification.* 03C20.

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[1] LAST EDIT Lynn: Apr. 23, 2022, added a remark for clarity.

1

(1) Alt Title: Big Ramsey Degrees in (not of!) Pseudofinite Structures. Determine if this would make our paper more easily searchable by interested people. I believe many more people know about ultraproducts then of the word pseudofinite. MD

(2) Comment on Chernikov's work on combinatorics in pseudofinite structures. Hrushovski and Chatzidakis on pseudo-finite fields. Ivan and Mirna. Starchenko and Pillay, asymptotic classes – Macpherson/Steinhorn, application in difference fields. Ozlem Beyarslan "the generic hypergraph is pseudofinite". Vaananen. Dario Garcia, thesis.

(3) See if we can apply Ozlem's result and our result to some Ramsey property of the generic hypergraph

(4) RP 14-08: A quick search of Chernikov's website found two papers using ultraproducts: https://arxiv.org/pdf/2010.00726.pdf and https://arxiv.org/pdf/1607.07701.pdf Neither seems directly related, but we should check further.

Figure: In preparation D. Bartošova, M. Dž, R. Patel and L. Scow *Big Ramsey Degrees in Ultraproducts of Finite Structures*

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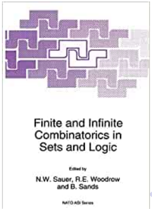
Figure: Finite and Infinite Combinatorics 1991



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CH is like the P=NP problem for set theory.

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A *(neat) simplified* $(\omega, 1)$*-morass* is a system $\mathcal{M} = \langle \theta_\alpha : \alpha \leq \omega \rangle, \langle \mathfrak{F}_{\alpha,\beta} : \alpha < \beta \leq \omega \rangle$ such that

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2. for $\alpha < \beta < \omega$, $\mathfrak{F}_{\alpha,\beta}$ is a finite set of order preserving functions from θ_α to θ_β,
3. $\mathfrak{F}_{\alpha,\omega}$ is a set of order preserving functions from θ_α to ω_1 such that $\bigcup_{f \in \mathfrak{F}_{\alpha,\omega}} f``\theta_\alpha = \omega_1$,
4. for all $\alpha < \beta < \gamma \leq \omega$ we have that $\mathfrak{F}_{\alpha,\gamma} = \{f \circ g : g \in \mathfrak{F}_{\alpha,\beta} \text{ and } f \in \mathfrak{F}_{\beta,\gamma}\}$,
5. $\mathfrak{F}_{\alpha,\alpha+1}$ always contains the identity function id_α on θ_α and either this is all, or $\mathfrak{F}_{\alpha,\alpha+1} = \{\mathrm{id}_\alpha, h_\alpha\}$ for some h_α such that there is a *splitting point* β with $h_\alpha \restriction \beta = \mathrm{id}_\alpha \restriction \beta$ and $h_\alpha(\beta) > \theta_\alpha$,
6. for every $\beta_0, \beta_1 < \omega$ and $f_l \in \mathfrak{F}_{\beta_l,\omega}$ for $l < 2$ there is $\gamma < \omega$ with $\beta_0, \beta_1 < \gamma$, a function $g \in \mathfrak{F}_{\gamma,\omega}$ and $f'_l \in \mathfrak{F}_{\beta_l,\gamma}$ such that $f_l = g \circ f'_l$ for $l < 2$.

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3. the structure on C^* is defined so that for each $\alpha < \omega$ and $f \in \mathfrak{F}_{\alpha,\omega}$, the function f is a $\mathfrak{C}$-embedding from $\mathrm{dom}(f)$ to $\mathrm{ran}(f) \restriction \mathcal{L}(\mathfrak{C})$.

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Then $\mathcal{M}'\mathcal{K} = \mathcal{M}\mathcal{K}$ (up to isomorphism).

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An application : constructions of homogeneous graphs of size $\aleph_1$. A homogeneous anti-metric space of size $\aleph_1$ (solved an open problem). A Ramsey conclusion...

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Examples of structures constructed by a morass often live in one Cohen real extension example a Souslin tree (Velleman). Other reals ?