The entropy of the 1×2 LEGO brick

Søren Eilers
eilers@math.ku.dk

Department of Mathematical Sciences
University of Copenhagen

August 30, 2022
Baby Steps Beyond the Horizon

Content

LEGOland 2002

LEGO Company profile 2004

LEGO facts and figures

- It would take 40,000,000,000 LEGO bricks stacked on top of each other to reach from the Earth to the Moon.

- A LEGO set is sold across the counter somewhere in the world every 7 seconds.

- The eight robots in the LEGO Warehouse in Billund can move 660 crates of LEGO bricks an hour.

- Children all over the world spend 5 billion hours a year playing with LEGO bricks.

- There are 102,981,500 different ways of combining six eight-stud bricks of the same colour.

- On average each person on earth owns 52 LEGO bricks.

Formalizing the question

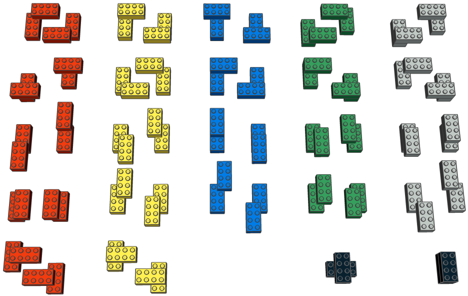
Definition

Let b_n denote the number of contiguous buildings that can be constructed with n 2×4 LEGO bricks, all sides parallel to the axes, and identified up to rotation in the XY-plane and translation in all of $\mathbb{R}^3$.

Let t_n denote the number of those buildings that are "towers", i.e. of height n.

Contiguous means that if you lift one brick, the whole building follows suit.

The number 102981500



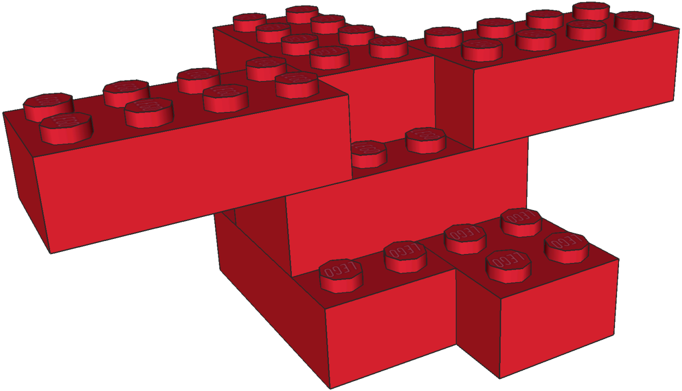
Counting towers

$$t_n = \frac{1}{2}(46^{n-1} - 2^{n-1}) + 2^{n-1}$$

Observation

$$\frac{1}{2}(46^5 - 2^5) + 2^5 = 102981504$$

Forgotten buildings!



2004 computations

$b_1 = 1$	
$b_2 = 24$	Kirk Christiansen
$b_3 = 1560$	Anonymous
$b_4 = 119580$	E
$b_5 = 10116403$	E
$b_6 = 915103765$	E

LEGO Company profile 2006

Selected LEGO statistics

- More than 400,000,000 children and adults will play with LEGO bricks this year.
- LEGO products are on sale in more than 130 countries.
- If you built a column of about 40,000,000,000 LEGO bricks, it would reach the moon.
- Approx. four LEGO sets are sold each second.
- There are 915,103,765 different ways of combining six eight-stud bricks of the same colour.
- On average every person on earth has 52 LEGO bricks.
- With a production of about 306 million tyres a year, the LEGO Group is the world's largest tyre manufacturer.
- If all the LEGO sets sold over the past 10 years were placed end to end, they would reach from London, England, to Perth, Australia.

LEGO House


LEGO
Home of the Brick


LEGO
6 pcs
Combo #329,439,255
Søren
Visit 8/1/2018
app-code C-569-026-918

A112389

$b_1 = 1$	
$b_2 = 24$	
$b_3 = 1560$	
$b_4 = 119580$	E 2004
$b_5 = 10116403$	E 2004
$b_6 = 915103765$	E 2004
$b_7 = 85747377755$	Abrahamsen-E 2005
$b_8 = 8274075616387$	Abrahamsen-E 2005
$b_9 = 816630819554486$	Nilsson 2014
$b_{10} = 82052796578652749$	Simon 2018

Content

How does $(b_n)_{n\in\mathbb{N}}$ grow?

- Does
$$\lim_{n\to\infty} \sqrt[n]{b_n}$$
exist in $\mathbb{R}$? If so, what is the limit?
- Does
$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n}$$
exist in $\mathbb{R}$? If so, what is the limit?
- Can one choose α, β, γ so that
$$\lim_{n\to\infty} \frac{b_n}{\alpha n^\beta \gamma^n} = 1?$$

Fixed buildings

Definition

Let b_n denote the number of buildings that can be constructed with n 2×4 LEGO bricks, identified up to rotation in the XY-plane and translation in all of $\mathbb{R}^3$.

Definition

Let f_n denote the number of buildings that can be constructed containing a base brick $[0, 2] \times [0, 4] \times [0, 1]$ along with n other 2×4 LEGO bricks, in such a way that only the base brick intersects $\mathbb{R}^2 \times (-\infty, 1)$

A112389 vs A123830

$b_1 = 1$	
$b_2 = 24$	$f_1 = 46$
$b_3 = 1560$	$f_2 = 2596$
$b_4 = 119580$	$f_3 = 194834$
$b_5 = 10116403$	$f_4 = 15834801$
$b_6 = 915103765$	$f_5 = 1395436949$
$b_7 = 85747377755$	$f_6 = 128352319891$
$b_8 = 8274075616387$	$f_7 = 12224079725173$
$b_9 = 816630819554486$	$f_8 = 1193967045643245$
$b_{10} = 82052796578652749$	$f_9 = 118973723976420310$

$$b_n \leq f_{n+1} \leq 4b_{n+1}$$

$$f_n f_m \leq f_{n+m}$$

$$b_n \leq f_{n+1} \leq 4 b_{n+1}$$
$$f_n f_m \leq f_{n+m}$$

Proposition [Durhuus-E]

$$\lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{f_n} \in [0, \infty]$$

insert 0 into 5 from Above perpenDicularly	05AD
insert 3 into 0 from Above parallEllly	30AE
move Forward to next brick	F
move Forward to next brick	F
insert 2 into 4 from Above parallEllly	24AE
move Forward to next brick	F
insert 0 into 6 from Above parallEllly	06AE
move Forward to next brick	F
insert 3 into 0 from Above parallEllly	30AE
move Forward to next brick	F

We see that any building counted by f_n can be described in $5n$ hexadecimal digits.

Proposition

$$\lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{f_n} \leq \lim_{n\to\infty} \sqrt[n]{16^{5n}} = 16^5 = 1048576$$

In fact, compressing the specification using

0	1	2	3	4	5	6	7
Stud			A/B	Hole			D/E

we see that $5n/2$ hexadecimal digits suffice.

Proposition

$$\lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{f_n} \leq \lim_{n\to\infty} \sqrt[n]{16^{5n/2}} = 16^{5/2} = 1024$$

Theorem [Durhuus-E]

$$\lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{f_n} = \gamma \in [78, 177]$$

- Upper bounds by Klarner-Rivest branches and twigs.
- Lower bounds by computer counts of “fat” buildings.

Content

Smallest nontrivial brick in arbitrary dimension

Let $d \geq 2$. A $2 \times 1^{d-1}$ brick in $\mathbb{R}^d$ is a cuboid

$$[\mathbf{x}, \mathbf{y}] = [x_1, y_1] \times [x_2, y_2] \times [x_d, y_d] = [\mathbf{x}', \mathbf{y}'] \times [x_d, y_d]$$

where for some $i_0 \in \{1, \ldots, d-1\}$, we have

$$y_{i_0} - x_{i_0} = 2$$

and for all $i \in \{1, \ldots, d\} \backslash \{i_0\}$

$$y_i - x_i = 1$$

Arbitrary dimension

A **building** with n such bricks is a collection

$$[\mathbf{x}^1, \mathbf{y}^1], \ldots, [\mathbf{x}^n, \mathbf{y}^n]$$

satisfying

- $\mathbf{x}^i \in \mathbb{Z}^d$
- $(\mathbf{x}^i, \mathbf{y}^i) \cap (\mathbf{x}^j, \mathbf{y}^j) = \emptyset$ when $i \neq j$
- $\displaystyle\bigcup_{i=1}^{n} (\mathbf{x}^i, \mathbf{y}^i) \times [x_d^i, y_d^i]$ is connected.

It is **fixed** when it contains $[0, 2] \times [0, 1]^{d-1}$ and that is the only brick intersecting $\mathbb{R}^{d-1} \times (-\infty, 1)$.

Question

Definition

Let $b_n^{(d)}$ denote the number of buildings of n $2 \times 1^{d-1}$ bricks, identified up to isometries on $\mathbb{R}^{d-1}$ and translations in all of $\mathbb{R}^d$.

Definition

Let $f_n^{(d)}$ denote the number of fixed buildings of $n+1$ $2 \times 1^{d-1}$ bricks.

Observation

$$\gamma_d = \lim_{n\to\infty} \sqrt[n]{f_n^{(d)}} = \lim_{n\to\infty} \sqrt[n]{b_n^{(d)}}$$

exists.

Questions

- What is γ_d?

Crude bounds are $2d - 1 \le \gamma_d \le 16d - 14$.

- Does γ_d grow like $cd + O(1/d)$? If so, what is c?
- Is

$$\lim_{n\to\infty} \frac{f_{n+1}^{(d)}}{f_n^{(d)}} = \gamma_d?$$

- Can one choose α_d, β_d so that

$$\lim_{n\to\infty} \frac{f_n^{(d)}}{\alpha_d n^{\beta_d} \gamma_d^n} = 1?$$

Least squares, but how?

If $f_n = \alpha n^\beta \gamma^n$, we have

$$\frac{f_{n+1}}{f_n} = \frac{\alpha(n+1)^\beta \gamma^{(n+1)}}{\alpha n^\beta \gamma^n} = \gamma \left(1 + \frac{1}{n}\right)^\beta$$

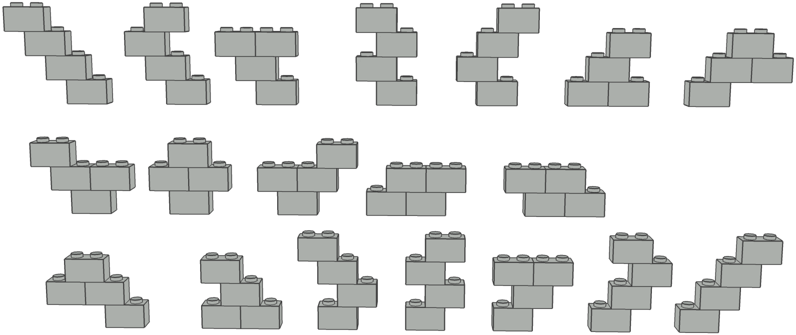
Content

A building counted by $b_n^{(1)}$ is **strict** when no pair of bricks are connected though two studs and holes.
A building counted by $f_n^{(1)}$ is a **pyramid** when all bricks rest, directly or indirectly, on the base brick.

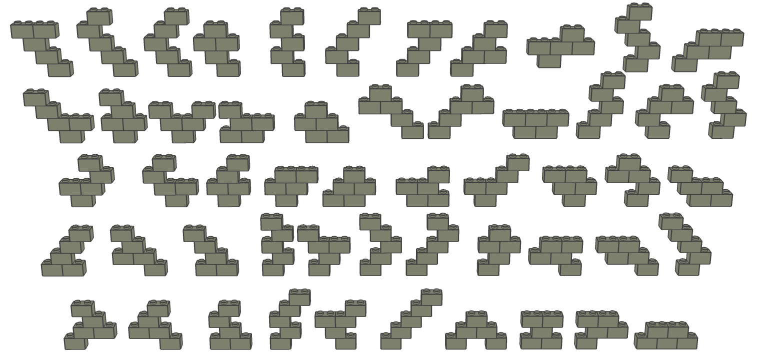
Definitions

- $\dot{b}_n^{(1)}$ counts the strict buildings up to isometries in $\mathbb{R}$ and translation in $\mathbb{R}^2$.
- $f_n^{(1),\triangle}$ counts the (fixed) pyramids.
- $\dot{f}_n^{(1),\triangle}$ counts the (fixed) strict pyramids.

$\dot{b}_4^{(1)} = 10$, $\dot{f}_3^{(1),\triangle} = 13$



$\dot{b}_5^{(1)} = 33$, $\dot{f}_4^{(1),\triangle} = 35$



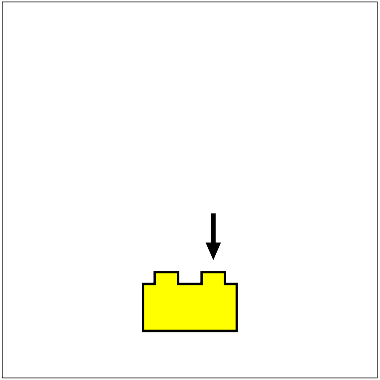
Theorem [Bousquet-Melou, Rechnitzer]

$$f_n^{(1),\triangle} = \binom{2n+1}{n+1}$$

- A1700
- $\lim_{n\to\infty} \sqrt[n]{f_n^{(1),\triangle}} = \lim_{n\to\infty} f_{n+1}^{(1),\triangle} / f_n^{(1),\triangle} = 4$
- $\sum_{n=0}^{\infty} f_n^{(1),\triangle} x^n = \dfrac{2}{1 - 4x + \sqrt{1-4x}}$
- $\lim_{n\to\infty} \dfrac{f_n^{(1),\triangle}}{\frac{2}{\sqrt{\pi}} n^{-1/2} 4^n} = 1$

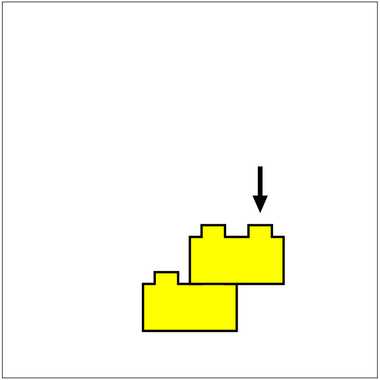
Purely positive string

1 1 0 1 0 0



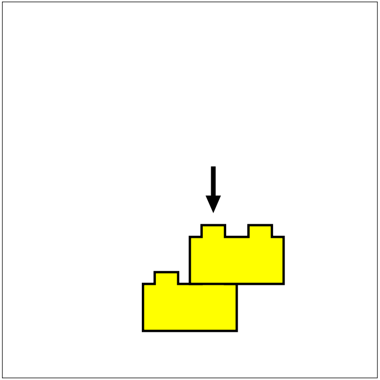
Purely positive string

1	1	0	1	0	0



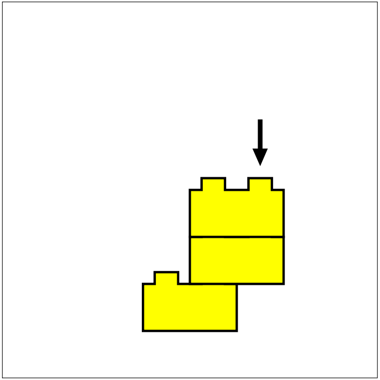
Purely positive string

1	1	0	1	0	0



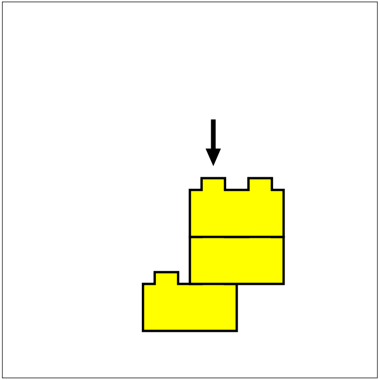
Purely positive string

1	1	0	1	0	0



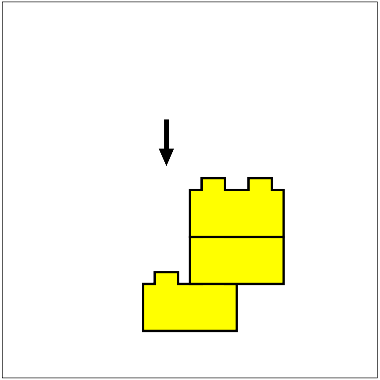
Purely positive string

1	1	0	1	0	0



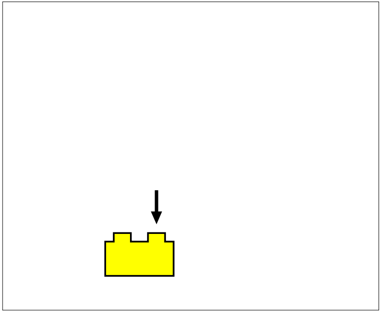
Purely positive string

1 1 0 1 0 0



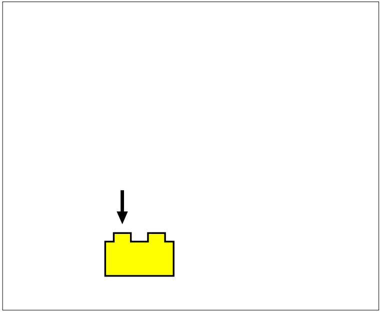
Durhuus-E, mixed case

1 0 0 0 1 1



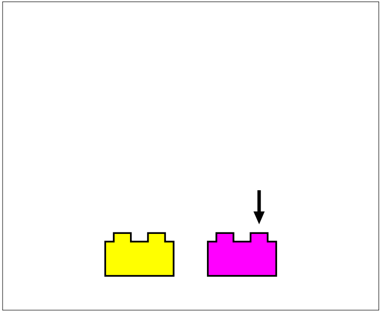
Durhuus-E, mixed case

1	0	0	0	1	1



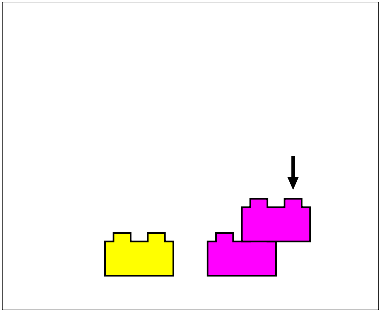
Durhuus-E, mixed case

1	0	0	0	1	1



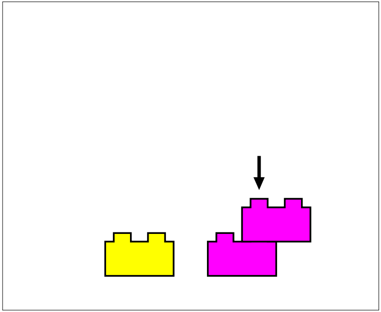
Durhuus-E, mixed case

1	0	0	0	1	1



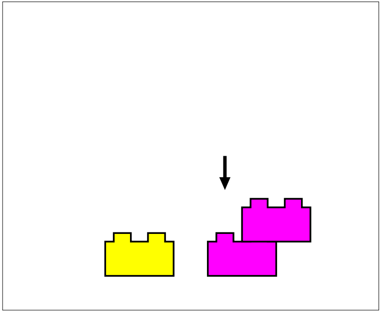
Durhuus-E, mixed case

1	0	0	0	1	1



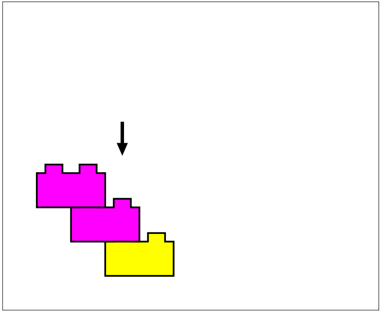
Durhuus-E, mixed case

1 0 0 0 1 1

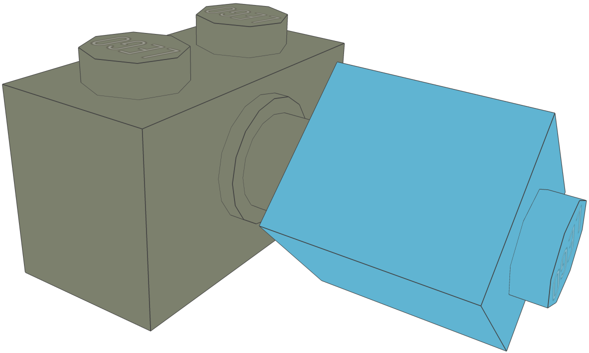


Durhuus-E, mixed case

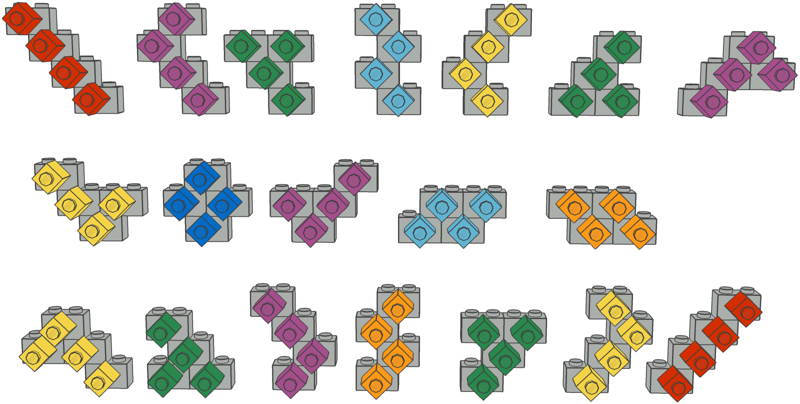
1	0	0	0	1	1



Translation in the strict case



$\dot{b}_4^{(1)} = 10$, $\dot{f}_3^{(1),\triangle} = 13$



Tetris


NEXT
TETRIS
12
LINES
LEFT
AHEAD
HIGH SCORE
10000
ROUND 7
CREDITS 0
SCORE 41900
LINES 0
NEXT

Theorem [Dhar et al]

$$\dot{f}_n^{(1),\triangle} = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{2k+1}{k+1}$$

- A5773
- $\lim_{n\to\infty} \sqrt[n]{\dot{f}_n^{(1),\triangle}} = \lim_{n\to\infty} \dot{f}_{n+1}^{(1),\triangle} / \dot{f}_n^{(1),\triangle} = 3$
- $\sum_{n=0}^{\infty} \dot{f}_n^{(1),\triangle} x^n = \frac{2x}{3x - 1 + \sqrt{1 - 2x - 3x^2}}$
- $\lim_{n\to\infty} \frac{\dot{f}_n^{(1),\triangle}}{\frac{1}{\sqrt{3\pi}} n^{-1/2} 3^n} = 1$

Observation

With p_n the number of free polyominoes, we have

$$p_n \le \dot{b}_n^{(1)} \le 8p_n$$

- A105
- $\lim_{n\to\infty} \sqrt[n]{\dot{b}_n^{(1)}} = \lim_{n\to\infty} \dot{b}_{n+1}^{(1)}/\dot{b}_n^{(1)} = \lambda$ [Klarner, Madras].
- $4.00253 \le \lambda \le 4.5252$
- Widely accepted estimate: $\lambda \approx 4.0626$
- Widely expected asymptotics: $0.3169\lambda^n/n$.

Wild speculation

$\gamma^{\square}$	$\triangle$	
$\cdot$	3	λ
	4	$\gamma^{(1)}$

Durhuus-E

Might $\gamma^{(1)} = 5$? Might $\gamma^{(1)} = \lambda + 1$?

Wild speculation

Probably not. Mølck Nilsson computed $f_n^{(1)}$ up to $n = 25$ using transfer matrix methods, and used Guttmann's differential approximant method to estimate

$$\gamma^{(1)} \approx 5.20295$$