Algebraically closed structures

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Plan of the talk

1. Introduction to general concepts of model theory.
2. The notion of a model companion.
3. Examples and applications.

What is model theory

- Model theory is a branch of logic. It was initiated by Tarski in 1930s.
- Model theory reached its current form mostly thanks to groundbreaking ideas and results of Shelah (mainly in 1970s) and Hrushovski (from 1980s till present).
- Currently model theory has connections with and applications to: diophantine geometry, algebraic geometry, algebraic dynamics, differential equations, combinatorics, ...

What is model theory about

- Analyzing definable properties of structures, where the terms "definable" and "structure" have a precise meaning coming from the first-order logic.
- The "first-order" assumption above may be relaxed sometimes but we will not get into that.
- In general, we have some fixed language L and then: L-formulas, L-sentences, L-theories, L-structures, and models of L-theories.
- I will just give some examples (next slide).

Model theory of fields

- Language: $L_r = \{+, \cdot, -, 0, 1\}$ (the language of rings).
- L_r-formulas, for example:
  - $\exists y \; x + x = y \cdot y$
  - $\forall x \; \exists y \; x = y \cdot y$
- L_r-sentences are L_r-formulas where all variables are quantified. For example: $\exists x \; x \cdot x = -1$
- L_r-theories: sets consisting of L_r-sentences. For example: the theory of commutative rings with 1, the theory of fields or the theory of algebraically closed fields.
- L_r-structures: sets M together with two specified functions $+^M, \cdot^M : M \times M \to M$, one specified function $-^M : M \to M$, and two specified elements $0^M, 1^M$,
- Models of L_r-theories. For example: the models of the theory of fields are exactly those L_r-structures which are fields.

Inductive theories

Let us fix a language L. If $(M_i)_i$ is a chain of L-structures, then the increasing union $M := \bigcup_i M_i$ is an L-structure as well.

Definition

A theory T is inductive, if for each chain of models of T, its union is also a model of T.

- From our mathematical experience, we know that for example the theory of groups is inductive (similarly for rings or fields).
- There is a general reason for that, which is the result below.

Theorem

A theory is inductive if and only if it is a $\forall\exists$-theory, that is: it can be axiomatized by $\forall\exists$-sentences.

Existentially closed models

Definition

Let M be a model of T. We say that M is an existentially closed model of T, if for any quantifier free L_M-formula $\chi(x)$ and any extension $M \subseteq N$ of models of T, we have that:

$$N \models \exists x\ \chi(x) \qquad \text{implies} \qquad M \models \exists x\ \chi(x).$$

Intuitively, all solvable in an extension of M "systems of (in)equations" (parameters from M) can be already solved in M.

Example

The class of existentially closed fields (that is: existentially closed models of the theory of fields) coincides with the class of algebraically closed fields.

Inductive theories and model companion

The next results says that inductive theories have many existentially closed models.

Theorem

Assume that T is inductive and M is a model of T. Then, there is an extension $M \subseteq N$ of models of T such that N is an existentially closed model of T.

The proof is similar to the construction of an algebraic closure of a field (add the solutions "one by one" and take the unions of chains on the limit steps).

Definition

For an inductive L-theory T, we call an L-theory T^* a model companion of T if the class of models of T^* coincides with the class of existentially closed models of T.

Model companions and non-companionable theories

1. The theory of sets has a model companion, which is the theory of infinite sets.
2. The theory of linear orders has a model companion, which is the theory of dense linear orders without endpoints.
3. The theory of fields has a model companion, which is the theory of algebraically closed fields.
4. The theory of fields with an automorphism has a model companion, which is called ACFA.
5. The theory of fields with a derivation has a model companion, which is called DCF.
6. The theory of commutative groups has a model companion: the theory of commutative divisible groups having infinitely many elements of order p for every prime p.
7. The theory of groups has no model companion.
8. The theory of commutative rings has no model companion.

Existence

- If we want to study model-theoretic properties of some class of algebraic objects (as fields with derivations or commutative groups), it is natural to start from a model companion of the corresponding theory (if it exists).
- Then, we get a nice theory of "large" objects in the class we are interested in.
- Analyzing the model-theoretic properties of such theories often leads to interesting applications. Some of them are discussed on next slides.

Applications I

- DCF_0 is the model companion of the theory of differential fields of characteristic 0. It was used by Hrushovski for applications in diophantine geometry (relative Mordell-Lang) and recently by Casale-Freitag-Nagloo to"give a complete proof of an assertion of Painlevé from 1895".
- $\mathrm{SCF}_{p,e}$ is the model companion of fields of characteristic $p > 0$ and inseparability degree $e > 0$ ($[K : K^p] = p^e$). It may be considered as a positive characteristic version of DCF_0 and it was also used by Hrushovski for applications in diophantine geometry.

Applications II

- ACFA is the model companion of the theory of fields with a fixed automorphism. It was used by Chatzidakis-Hrushovski (and others) for application in diophantine geometry and algebraic dynamics.
- ACVF is the model companion of the theory of valued fields. It was used by Hrushovski-Kazhdan for applications to the theory of motivic integration and by Hrushovski-Loeser for applications to the theory of Berkovich spaces.
- RCF is the model companion of the theory of ordered fields. It was used by Pila-Wilkie (and others) to show transcendence results and recently for the first proof of the (full) André-Oort conjecture.

Group actions on fields

- By a result of Hrushovski, the theory of fields with two commuting automorphisms does *not* have a model companion.
- Fields with two commuting automorphisms are the same as fields with the action of the group $(\mathbb{Z}, +) \times (\mathbb{Z}, +)$.
- Together with Hoffmann and (separately) Beyarslan, I studied model theory of actions of arbitrary groups on fields.
- For example, with Beyarslan we fully characterized those commutative torsion groups A for which the model companion of actions of A on fields exists.

Theory of commutative rings I

We will see that the theory of commutative rings does not have a model companion.

Exercise

Let R be a commutative ring and $r \in R$. TFAE.

1. The element r is not nilpotent.
2. There is a commutative ring extension $R \subseteq S$ and there is $s \in S \setminus \{0\}$ such that $s^2 = s$ and $r|s$ (in S).

Lemma

If R is existentially closed, then the second condition above is equivalent to: there is $s \in R \setminus \{0\}$ such that $s^2 = s$ and $r|s$ (in R).

Idea of argument This condition can be written as follows:

$$\exists x \, \exists y \ \ (x \cdot x = x) \wedge (x \neq 0) \wedge (r \cdot y = x).$$

Theory of commutative rings II

- The non-definable condition (of being nilpotent) is definable in existentially closed rings! This is the "source of problems".
- Let $L := L_r \cup \{c\}$ (a new constant symbol).
- For each $n > 0$, let φ_n be the following L-sentence:

$$c^n \neq 0 \ \wedge \ \text{“}c \text{ does not divide any idempotent element”}.$$

- Assume that T is the model companion of the theory of commutative rings and let $T' := T \cup \{\varphi_n \mid n > 0\}$.
- Using Lemma, T' is finitely satisfiable (any finite subset of T' has a model), since for any existentially closed ring R and for any $n \geqslant 0$, there is $r \in R$ such that $r^{n+1} = 0 \neq r^n$.
- By Compactness Theorem, T' has a model (R, c^R) but then the element $c^R \in R$ is not nilpotent, c^R does not divide any idempotent element from R, and R is existentially closed, contradicting Lemma.