When Fourier analysis meets ergodic theory and additive combinatorics

Mariusz Mirek

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Erdös–Turán conjecture (1936)

In 1936 Erdös and Turán realized that it ought to be possible to find arithmetic progressions of length k in any sufficiently dense set of integers.

- A set $E \subseteq \mathbb{N}$ is said to have positive *upper Banach density* if

$$d(E) = \limsup_{N\to\infty} \frac{\#(E \cap [1,N])}{N} > 0.$$

Conjecture (Erdös–Turán conjecture (1936))

Suppose that $E \subseteq \mathbb{N}$ has a positive upper Banach density. Then for any integer $k \geq 2$, there exist infinitely many arithmetic progressions:

$$\{x, x+n, x+2n, \ldots, x+kn\} \subset E.$$

Examples:

- $d(q\mathbb{N}+r) = 1/q$, for some $q \in \mathbb{N}$ and $r \in \{0, \ldots, q-1\}$.
- $d(\mathbb{P}) = 0$, if $\mathbb{P}$ is the set of primes, since $\#(\mathbb{P} \cap [1,N]) \sim \frac{N}{\log N}$.
- $d(E) = \frac{6}{\pi^2} > 0$, if E is the set of all square-free integers, that is integers which are divisible by no perfect square other than 1.
  - $10 = 2 \cdot 5$ is square-free,
  - $12 = 3 \cdot 4$ is not square-free, since $4 = 2^2$.

Szemerédi theorem

Theorem (Szemerédi theorem (1974))

Suppose that $E \subseteq \mathbb{N}$ has a positive upper Banach density. Then for any integer $k \geq 2$, there exist infinitely many arithmetic progressions:

$$\{x, x+n, x+2n, \ldots, x+kn\} \subset E.$$

- In 1953 Roth proved Erdös–Turán conjecture for $k = 3$ using classical Fourier methods.
- In 1974 Szemerédi proved Erdös–Turán conjecture for arbitrary integer $k \in \mathbb{N}$ using intricate arguments from combinatorics and graph theory.
- In 1977 Furstenberg used ergodic methods to give a conceptually new proof of Szemerédi's theorem using the multiple recurrence theorem.
- In 2001 Gowers gave a new quantitative proof of Szemerédi's theorem. Gowers developed the so-called higher order Fourier analysis.

Quantitative formulation of Szemerédi's theorem

Definition

Let $r_k(N)$ denote the size of the largest subset of $\{1, \ldots, N\}$ containing no configuration of the form $\{x, x+n, x+2n, \ldots, x+(k-1)n\}$ with $n \neq 0$.

Theorem (Roth (1953), classical Fourier methods)

One has that

$$r_3(N) \lesssim \frac{N}{\log \log N}.$$

Theorem (Szemerédi (1974) and Furstenberg (1977))

Szemerédi's theorem as well as Furstenberg's theorem give only

$$r_k(N) = o(N), \qquad k \in \mathbb{N}.$$

Theorem (Gowers (2001), higher order Fourier analysis)

For every $k \in \mathbb{N}$ there is $\gamma_k > 0$ such that

$$r_k(N) \lesssim \frac{N}{(\log \log N)^{\gamma_k}}.$$

Finitary version of Roth theorem

Theorem (Roth's theorem (1953))

For every $\delta \in (0,1]$ there is $N \in \mathbb{N}$ such that every $A \subseteq \mathbb{Z}_N$ satisfying $\#A \geq \delta N$ contains (AP3) an arithmetic progression of length three.

Proof:

Let $\widehat{f}(\xi) = N^{-1} \sum_{m \in \mathbb{Z}_N} e^{-2\pi i m \xi} f(m)$ denote the finite Fourier transform of a function $f : \mathbb{Z}_N \to \mathbb{C}$ on $\mathbb{Z}_N$. Setting $\alpha = \widehat{\mathbb{1}_A}(0) \geq \delta$, one has

$$
\begin{aligned}
N^{-2}\#\{(a,d) \in \mathbb{Z}_N^2 : a, a+d, a+2d \in A\} & \\
&= N^{-2} \sum_{x+y=2z} \mathbb{1}_A(x)\mathbb{1}_A(y)\mathbb{1}_A(z) \\
&= \alpha^3 + \sum_{\xi \in \mathbb{Z}_N \setminus \{0\}} \widehat{\mathbb{1}_A}(\xi)^2 \widehat{\mathbb{1}_A}(-2\xi).
\end{aligned}
$$

- The left-hand side is the probability that x, y, z all belong to A if you choose them randomly to satisfy the equation $x + y = 2z$.
- Without the constraint that $x + y = 2z$ this probability would be α^3, since each of x, y and z would have a probability α of belonging to A.
- So the term α^3 on the right-hand side can be thought of as "what one would expect", whereas the remainder is a measure of the effect of the dependence of x, y and z on each other. □

The current state of the art

Author / Authors	$r_3(N) \lesssim$
Roth (1953)	$\frac{N}{\log\log N}$
Heath–Brown (1987) and Szemerédi (1990)	$\frac{N}{(\log N)^c}$ for some $c > 0$
Bourgain (1999)	$\frac{N}{(\log N)^{1/2-o(1)}}$
Bourgain (2008)	$\frac{N}{(\log N)^{2/3-o(1)}}$
Sanders (2010)	$\frac{N}{(\log N)^{3/4-o(1)}}$
Sanders (2010)	$\frac{(\log\log N)^6 N}{\log N}$
Bloom (2014)	$\frac{(\log\log N)^4 N}{\log N}$
Schoen (2020)	$\frac{(\log\log N)^{3+o(1)} N}{\log N}$
Bloom and Sisask (2020)	$\frac{N}{(\log N)^{1+c}}$ for some $c > 0$

By Behrend (1946) we know that $r_3(N) \gtrsim N\exp(-c(\log N)^{1/2})$ for some absolute $c > 0$, so these results still leave much to be desired.

Measure-preserving systems

A measure-preserving system $(X, \mathcal{B}(X), \mu, T)$ is a σ-finite measure space $(X, \mathcal{B}(X), \mu)$ endowed with a measurable mapping $T \colon X \to X$, which preserves the measure μ, i.e. $\mu(T^{-1}[E]) = \mu(E)$ for all $E \in \mathcal{B}(X)$.

1. *The integer shift system* $(\mathbb{Z}, \mathbf{P}(\mathbb{Z}), |\cdot|, S)$ with $S : \mathbb{Z} \to \mathbb{Z}$ given by
$$S(x) := x + 1.$$
2. *The circle rotation system* $(\mathbb{T}, \mathcal{L}(\mathbb{T}), \mathrm{d}x, R_\alpha)$ with the rotation map $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $R_\alpha(x) := x + \alpha \pmod 1$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
3. *The circle-doubling system* $(\mathbb{T}, \mathcal{L}(\mathbb{T}), \mathrm{d}x, D_2)$ with the doubling map $D_2 : \mathbb{T} \to \mathbb{T}$ given by $D_2(x) := 2x \pmod 1$.
4. *The continued fraction system* $([0,1), \mathcal{L}([0,1)), \mu, T)$ with the Gauss measure
$$\mu(A) := \frac{1}{\log 2} \int_A \frac{\mathrm{d}x}{1+x},$$
and continued fraction map $T : [0,1) \to [0,1)$ given by $T(0) := 0$ and
$$T(x) := \frac{1}{x} \pmod 1, \qquad \text{when } x \neq 0.$$

Furstenberg's ideas and recurrence theorems

Theorem (Poincaré recurrence theorem (1890))

Given $(X, \mathcal{B}(X), \mu, T)$ *if* $\mu(X) < \infty$ *and* $E \in \mathcal{B}(X)$ *with* $\mu(E) > 0$ *then*

$$\mu(E \cap T^{-n}[E]) > 0 \quad \textit{for infinitely many} \quad n \in \mathbb{N}.$$

- If a cloud of gas initially confined in the left compartment of a vessel is released into the right empty compartment, then after a sufficiently long time, the gas particles will return to the left compartment.
- Furstenberg's multiple recurrence theorem asserts that for every $k \in \mathbb{N}$

  $$\mu(E \cap T^{-n}[E] \cap T^{-2n}[E] \cap \ldots \cap T^{-kn}[E]) > 0 \quad \text{for some} \quad n \in \mathbb{N}.$$
- Suppose that $E \subseteq \mathbb{N}$ has a positive upper Banach density. Then for any integer $k \geq 2$, we are looking for the configurations

  $$\{x, x+n, x+2n, \ldots, x+kn\} = \{x, S^n(x), S^{2n}(x), \ldots, S^{kn}(x)\} \subset E.$$

  where $S : \mathbb{Z} \to \mathbb{Z}$ is defined by $S(x) = x + 1$ for all $x \in \mathbb{Z}$.
- It is easy to see that it suffices to show that

  $$E \cap S^{-n}[E] \cap S^{-2n}[E] \cap \ldots \cap S^{-kn}[E] \neq \emptyset.$$

Furtsenberg's proof of Szemerédi's theorem

Theorem (Furtsenberg theorem (1977))

Let $(X, \mathcal{B}, \mu, T)$ *be a probability measure-preserving system* $\mu(X) = 1$ *and* $E \in \mathcal{B}(X)$ *with* $\mu(E) > 0$ *then for every* $k \in \mathbb{N}$ *we have*

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(E \cap T^{-n}[E] \cap T^{-2n}[E] \cap \ldots \cap T^{-kn}[E]) > 0.$$

In particular, for every $k \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that*

$$\mu(E \cap T^{-n}[E] \cap T^{-2n}[E] \cap \ldots \cap T^{-kn}[E]) > 0.$$

Theorem (Furtsenberg correspondence principle)

Given $A \subseteq \mathbb{N}$ *with* $d(A) > 0$ *there exists a probability measure-preserving system* $(X, \mathcal{B}, \mu, T)$ *and a set* $E \in \mathcal{B}(X)$ *such that* $\mu(E) = d(A)$ *and*

$$\begin{aligned} 0 &< \mu(E \cap T^{-n}[E] \cap T^{-2n}[E] \cap \ldots \cap T^{-kn}[E]) \\ &\leq d(A \cap S^{-n}[A] \cap S^{-2n}[A] \cap \ldots \cap S^{-kn}[A]), \qquad k \in \mathbb{N}, \end{aligned}$$

where $S : \mathbb{Z} \to \mathbb{Z}$ *is the shift operator defined by* $S(x) = x + 1$ *for all* $x \in \mathbb{Z}$.

Bergelson–Leibman theorem

Furstenberg's proof of Szemerédi's theorem was a major breakthrough in modern ergodic theory, which had also transformed the area of additive number theory and combinatorics as well as ergodic theory itself:

- partly because of the difficulty of Szemerédi's original proof;
- and partly because Furstenberg's proof has many natural extensions, which do not seem to follow from Szemerédi's approach. These include a polynomial Szemerédi theorem of Bergelson and Leibman:

Theorem (Bergelson and Leibman theorem (1996))

Given polynomials $P_1, \ldots, P_k \in \mathbb{Z}[\mathrm{n}]$ *each with zero constant term suppose that* $\mu(X) = 1$ *and* $E \in \mathcal{B}(X)$ *with* $\mu(E) > 0$*, then one has*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(E \cap T^{-P_1(n)}[E] \cap T^{-P_2(n)}[E] \cap \ldots \cap T^{-P_k(n)}[E]) > 0.$$

In particular, the subsets of integers with nonvanishing Banach density contain polynomial patterns of the form

$$x,\ x + P_1(n),\ x + P_2(n),\ \ldots,\ x + P_k(n).$$

Green–Tao theorem

Furstenberg's ergodic-theoretic proof of Szemerédi theorem was also the departure point for the modern additive combinatorics, where quantitative bounds for Szemerédi-type theorems play a central role.

- This line of investigations had been initiated by Gowers who introduced new ideas of the so-called higher order Fourier analysis.
- The latter concepts, in contrast to the ergodic qualitative approach, turned out to be very effective in obtaining quantitative bounds for long arithmetic progressions and resulted in many deep theorems:

Theorem (Green and Tao theorem (2004))

Suppose that $E \subseteq \mathbb{P}$ has a positive upper Banach density in the primes $\mathbb{P}$, i.e.

$$\limsup_{N\to\infty} \frac{\#(E \cap [1,N])}{\#(\mathbb{P} \cap [1,N])} > 0.$$

Then for any integer $k \geq 2$, there exist infinitely many arithmetic progressions:

$$\{x, x+n, x+2n, \ldots, x+kn\} \subset E.$$

The longest known arithmetic progression in the primes

- The longest and largest known AP-k is an AP-27, it was found on September 23, 2019 by Rob Gahan with an AMD R9 290 GPU.

$$a_n = 224584605939537911 + 81292139 \cdot 223092870 \cdot n$$

where $n = 0, 1, \ldots, 26$.

- The first known AP-26 was found on April 12, 2010 by Benoãt Perichon on a PlayStation 3 with software by Jarosław Wróblewski and Geoff Reynolds:

$$a_n = 43142746595714191 + 23681770 \cdot 223092870 \cdot n$$

where $n = 0, 1, \ldots, 25$.

- However, on January 18, 2007 Jarosław Wróblewski:

$$a_n = 468395662504823 + 205619 \cdot 223092870 \cdot n$$

where $n = 0, 1, \ldots, 23$ found the first AP-24. For this Wróblewski used a total of 75 computers: 15: 64-bit Athlons; 15: dual core 64-bit Pentium D 805; 30: 32-bit Athlons 2500; and 15: Durons 900.

(AP3) in the Piatetski–Shapiro primes

- In 1953 Piatetski–Shapiro considered the following subset of the primes
$$\mathbb{P}_\gamma = \mathbb{P} \cap \{\lfloor n^{1/\gamma} \rfloor : n \in \mathbb{N}\},$$
and established the following asymptotic formula
$$\#(\mathbb{P}_\gamma \cap [1, N]) \sim \frac{N^\gamma}{\log N}, \quad \text{as} \quad N \to \infty$$
for $\gamma \in \left(\frac{11}{12}, 1\right)$, where $\left(\frac{11}{12} \approx 0,916...\right)$.

Theorem (Roth's theorem for $\mathbb{P}_\gamma$, (M.M. 2015))

Assume that $\gamma \in (71/72, 1)$, $(71/72 \approx 0,9861...)$. Then every $A \subseteq \mathbb{P}_\gamma$ with positive relative upper density, i.e.
$$\limsup_{n \to \infty} \frac{|A \cap [1, n]|}{|\mathbb{P}_\gamma \cap [1, n]|} > 0,$$
contains a non–trivial three–term arithmetic progression.

- Green–Tao theorem does not settle whether $\mathbb{P}_\gamma$ contains non–trivial arithmetic progressions of length at least three, since
$$\limsup_{N \to \infty} \frac{\#(\mathbb{P}_\gamma \cap [1, N])}{\#(\mathbb{P} \cap [1, N])} = \limsup_{N \to \infty} \frac{N^\gamma}{\log N} \cdot \frac{\log N}{N} = \limsup_{N \to \infty} N^{\gamma - 1} = 0.$$

Ergodic averages as a tool to detect recurrent points

For a measurable function $f \in L^0(X)$ define the ergodic average by

$$A_N f(x) := \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x), \quad \text{for} \quad x \in X.$$

- If we set $f(x) = \mathbb{1}_E(x)$, then

$$A_N \mathbb{1}_E(x) = \frac{1}{N}\#\{0 \le n < N : T^n x \in E\}.$$

- Norm or pointwise convergence of $A_N f$ can be used to reprove the Poincaré recurrence theorem: if $\mu(X) = 1$, and $\mu(E) > 0$, then

$$\mu(E \cap T^{-n}[E]) > 0 \quad \text{for some} \quad n \in \mathbb{N}.$$

- In the early 1930's von Neumann and Birkhoff proved that for every $1 \le p < \infty$ and every $f \in L^p(X)$ there exists $f^* \in L^p(X)$ such that

$$\lim_{N\to\infty} A_N f(x) = f^*(x)$$

  for almost every $x \in X$ and in $L^p(X)$ norm.

Birkhoff's ergodic theorem

To establish that for every $1 \leq p < \infty$ and every $f \in L^p(X)$ there exists $f^* \in L^p(X)$ such that

$$\lim_{N\to\infty} A_N f(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x) = f^*(x) \tag{1}$$

one can proceed in two steps:

- **Step 1.** Quantitative version of ergodic theorem

$$\|\sup_{N\in\mathbb{N}} |A_N f|\|_{L^p(X)} \lesssim \|f\|_{L^p(X)} \quad \text{for} \quad p \in (1, \infty]. \tag{2}$$

  The bounds in (2) follow from the Hardy–Littlewood maximal inequality

$$\Big\|\sup_{N\in\mathbb{N}} \Big|\frac{1}{N}\sum_{n=1}^{N} f(x-n)\Big|\Big\|_{\ell^p(\mathbb{Z})} \lesssim \|f\|_{\ell^p(X)}, \quad \text{for} \quad p \in (1, \infty],$$

  which is $A_N f$ with the shift operator $T(x) = x - 1$ in (1).

- **Step 2.** Pointwise convergence on a dense class of functions in $L^p(X)$.

Convergence on a dense class

$$A_N f(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^n x)$$

- $\mathbf{I}_T = \{f \in L^2(X) : f \circ T = f\}$. If $f \in \mathbf{I}_T$, then

  $$A_N f = f,$$

  μ-almost everywhere.

- $\mathbf{J}_T = \{g \circ T - g : g \in L^2(X) \cap L^\infty(X)\}$. If $f \in \mathbf{J}_T$, then by telescoping

  $$|A_N f(x)| = \frac{1}{N}\left|\sum_{n=1}^{N} g(T^{n+1}x) - g(T^n x)\right| = \frac{1}{N}|g(T^{N+1}x) - g(Tx)| \xrightarrow[N\to\infty]{} 0.$$

- $\mathbf{I}_T \oplus \mathbf{J}_T$ is dense in $L^2(X)$.

Bourgain's pointwise ergodic theorem

In the early 1980's Bellow and Furstenberg asked independently about the poinwise convergence for polynomial ergodic averages

$$A_N^P f(x) := \frac{1}{N}\sum_{n=1}^{N} f(T^{P(n)}x) \quad \text{for} \quad x \in X,$$

where $P \in \mathbb{Z}[\mathrm{n}]$ is a polynomial of degree > 1.

- Furstenberg was motivated by the result of Sárközy: $S \subseteq \mathbb{Z}$ has positive upper Banach density, then there are $x, n \in \mathbb{N}$ such that $x, x + n^2 \in S$.
- Furstenberg proved norm convergence for $A_N^P f$ and deduced the polynomial Poincaré recurrence theorem: if $\mu(X) < \infty$ and $E \in \mathcal{B}(X)$ with $\mu(E) > 0$, then $\mu(E \cap T^{-P(n)}[E]) > 0$ for some $n \in \mathbb{N}$.

Bellow and Furstenberg question was very hard. Even for $P(n) = n^2$, since $(n+1)^2 - n^2 = 2n + 1$. For overcoming this problem, Bourgain used the ideas from the circle method of Hardy and Littlewood to show:

- $L^p(X)$ boundedness of the maximal function for any $1 < p \leq \infty$.
- Given an increasing sequence $(N_j : j \in \mathbb{N})$, for each $J \in \mathbb{N}$ one has

$$\Big(\sum_{j=0}^{J} \big\| \sup_{N_j \leq N < N_{j+1}} \big|A_N^P f - A_{N_j}^P f\big| \big\|_{L^2(X)}^2\Big)^{1/2} \leq o(J^{1/2})\|f\|_{L^2(X)}.$$

The current state of the art

Let $(X, \mathcal{B}(X), \mu, T)$ be a probability measure-preserving system $\mu(X) = 1$. Let $P_1, \ldots, P_k \in \mathbb{Z}[\mathrm{n}]$, and $f_1, \ldots, f_k \in L^\infty(X)$. Recall that

$$A_N^{P_1,\ldots,P_k}(f_1, \ldots, f_k)(x) = \frac{1}{N}\sum_{n=1}^{N} f_1(T^{P_1(n)}x) \ldots f_k(T^{P_k(n)}x). \tag{3}$$

Norm convergence of (3) on $L^2(X)$:

- Furstenberg (1977): $k = 2$ with $P_1(n) = an$, $P_2(n) = bn$, $a, b \in \mathbb{Z}$.
- Furstenberg–Weiss (1996): $k = 2$ with $P_1(n) = n$ and $P_2(n) = n^2$.
- Host and Kra (2002) and independently Ziegler (2004): any $k \in \mathbb{N}$ and arbitrary linear polynomials $P_i(n) = a_i n$ with $a_1, \ldots, a_k \in \mathbb{Z}$.
- Leibman (2005): for any $k \in \mathbb{N}$ and arbitrary $P_1, \ldots, P_k \in \mathbb{Z}[\mathrm{n}]$.

Pointwise convergence of (3) on $L^p(X)$:

- Bourgain (1990): for $k = 2$ with $P_1(n) = an$, $P_2(n) = bn$, $a, b \in \mathbb{Z}$.

Furstenberg–Bergelson–Leibman conjecture

One of the central open problems in pointwise ergodic theory (from the mid 1980's) is a conjecture of Furstenberg–Bergelson–Leibman:

Theorem (Furstenberg–Bergelson–Leibman conjecture)

Let $\mathbb{G}$ be a nilpotent group of measure preserving transformations of a probability space $(X, \mathcal{B}(X), \mu)$. Let $P_{j,i} \in \mathbb{Z}[\mathrm{n}]$ be polynomials and $T_1, \ldots, T_d \in \mathbb{G}$ and $f_1, \ldots, f_m \in L^\infty(X)$. Does the limit of the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T_1^{P_{1,1}(n)} \cdots T_d^{P_{1,d}(n)} x) \cdot \ldots \cdot f_m(T_1^{P_{m,1}(n)} \cdots T_d^{P_{m,d}(n)} x) \tag{4}$$

exist μ-almost everywhere on X as $N \to \infty$?

- The norm convergence in $L^2(X)$ for the averages (4) was established in the nilpotent setting by M. Walsh in 2012 .
- Bergelson and Leibman showed that $L^2(X)$ norm convergence for (4) may fail if $\mathbb{G}$ is a solvable group.
- The nilpotent setting is probably the most general setting where the conjecture of Furstenberg–Bergelson–Leibman might be true.

Recent contribution to the nilpotent setting

Linear and nilpotent variant of the Furstenberg–Bergelson–Leibman problem can be summarize as follows:

Theorem (M., Ionescu, Magyar, and Szarek (2021))

Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite space and let $T_1, \ldots, T_d : X \to X$ be a family of invertible and measure preserving transformations satisfying

$$[[T_i, T_j], T_k] = \mathrm{Id} \qquad \textit{for all} \quad 1 \leq i \leq j \leq k \leq d.$$

Then for every polynomials $P_1, \ldots, P_d \in \mathbb{Z}[\mathrm{n}]$ and every $f \in L^p(X)$ with $1 < p < \infty$ the averages

$$\frac{1}{N} \sum_{n=1}^{N} f\big(T_1^{P_1(n)} \cdots T_d^{P_d(n)} x\big)$$

converge for μ-almost every $x \in X$ and in $L^p(X)$ norm as $N \to \infty$.

- One can think that $T_1, \ldots, T_d$ belong to a nilpotent group of step two of measure preserving mappings of a σ-finite space $(X, \mathcal{B}(X), \mu)$.

Recent contribution to the bilinear setting

Thirty years after Bourgain's pointwise bilinear ergodic theorem for the averages with linear orbits

$$A_N^{\mathrm{an},b\mathrm{n}}(f,g)(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^{an}x)g(T^{bn}x) \qquad a,b\in\mathbb{Z}$$

jointly with Ben Kruse and Terry Tao we established the following theorem.

Theorem (M., Krause, and Tao, (2020))

Let $(X,\mathcal{B}(X),\mu,T)$ *be an invertibe* σ*-finite measure-preserving system, let* $P\in\mathbb{Z}[\mathrm{n}]$ *with* $\deg(P)\geq 2$*, and let* $f\in L^{p_1}(X)$ *and* $g\in L^{p_2}(X)$ *for some* $p_1,p_2\in(1,\infty)$ *with*

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}\leq 1.$$

Then the Furstenberg–Weiss averages

$$A_N^{\mathrm{n},P(\mathrm{n})}(f,g)(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^{n}x)g(T^{P(n)}x)$$

converge for μ*-almost every* $x\in X$ *and in* $L^p(X)$ *norm as* $N\to\infty$*.*

Key ideas

The proof of pointwise convergence for

$$A_N^{\mathrm{n},P(\mathrm{n})}(f,g)(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^n x)g(T^{P(n)}x)$$

is quite intricate, and relies on several deep results in the literature:

- the Ionescu–Wainger multiplier theorem (discrete Littlewood–Paley theory and paraproduct theory) — (HA)&(NT);
- The circle method of Hardy and Littlewood — (NT);
- the inverse theory of Peluse and Prendeville — (CO)&(NT);
- Hahn–Banach separation theorem — (FA);
- L^p-improving estimates of Han–Kovač–Lacey–Madrid–Yang (derived from the Vinogradov mean value theorem) — (HA)&(NT);
- Rademacher–Menshov argument combined with Khinchine's inequality — (HA)&(FA)&(PR);
- $L^p(\mathbb{R})$ bounds for a shifted square function — (HA);
- bounded metric entropy argument from Banach space theory— (CO)&(FA)&(PR);
- van der Corput type estimates in the p-adic fields — (HA)&(NT).

Inverse theorem of Peluse and Prendiville

An important ingredient in the proof is the inverse theorem of Peluse, which can be thought of as a counterpart of Weyl's inequality:

Theorem (Peluse, (2019/2020))

Let $m \geq 2$ *and* $P_1, \ldots, P_m \in \mathbb{Z}[\mathrm{n}]$ *each having zero constant term such that* $\deg P_1 < \ldots < \deg P_m$. *Let* $N \in \mathbb{N}$ *and* $\delta \in (0,1)$ *and assume that functions* $f_0, f_1, \ldots, f_m : \mathbb{Z} \to \mathbb{C}$ *are supported on* $[-N_0, N_0]$ *for some* $N_0 \simeq N^{\deg P_m}$, *and* $\|f_0\|_{L^\infty(\mathbb{Z})}, \|f_1\|_{L^\infty(\mathbb{Z})}, \ldots, \|f_m\|_{L^\infty(\mathbb{Z})} \leq 1$, *and suppose that*

$$\Big\| \frac{1}{N} \sum_{n=1}^{N} f_0(x) f_1(x - P_1(n)) \cdots f_m(x - P_m(n)) \Big\|_{L^1_x(\mathbb{Z})} \geq \delta N^{\deg P_m}.$$

Then there are $1 \leq q \lesssim \delta^{-O(1)}$ *and* $\delta^{O(1)} N^{\deg P_1} \lesssim M \leq N^{\deg P_1}$ *such that*

$$\Big\| \frac{1}{M} \sum_{y=1}^{M} f_1(x + qy) \Big\|_{L^1_x(\mathbb{Z})} \gtrsim \delta^{O(1)} N^{\deg P_m}$$

provided that $N \gtrsim \delta^{-O(1)}$.

Quantitative polynomial Szemerédi's

Let $r_{P_1,\dots,P_m}(N)$ denote the size of the largest subset of $\{1,\dots,N\}$ containing no configuration of the form $x, x+P_1(n),\dots,x+P_m(n)$ with $n \neq 0$.

- Berglson and Leibman showed proving polynomial multiple recurrence theorem that
$$r_{P_1,\dots,P_m}(N) = o_{P_1,\dots,P_m}(N),$$
whenever $P_1,\dots,P_m \in \mathbb{Z}[\mathrm{n}]$ and each having zero constant term.

Theorem (Gowers (2001), higher order Fourier analysis)

If $P_1(n) = n,\dots,P_m(n) = (m-1)n$ for every $m \in \mathbb{N}$ then there is $\gamma_m > 0$ such that
$$r_{P_1,\dots,P_m}(N) \lesssim \frac{N}{(\log\log N)^{\gamma_m}}.$$

- No bounds were known in general for the polynomial Szemerédi's theorem until a series of recent papers of Peluse and Prendiville.
- Peluse showed that there is a constant $\gamma_{P_1,\dots,P_m} > 0$ such that
$$r_{P_1,\dots,P_m}(N) \lesssim_{P_1,\dots,P_m} \frac{N}{(\log\log N)^{\gamma_{P_1,\dots,P_m}}}$$
answering a question posed by Gowers.

Commutative Furstenberg–Bergelson–Leibman conjecture

Ongoing project (Krause, M., Peluse, and Wright (2021))

Let $(X, \mathcal{B}(X), \mu)$ be a probability space equipped with commuting invertible measure-preserving maps $T_1, \dots, T_k : X \to X$. Consider $P_1, \dots, P_k \in \mathbb{Z}[\mathrm{n}]$ with distinct degrees and $f_1, \dots, f_k \in L^\infty(X)$. It is expected that the averages

$$A_N^{P_1,\dots,P_k}(f_1,\dots,f_k)(x) = \frac{1}{N}\sum_{n=1}^{N} f_1(T_1^{P_1(n)}x)\dots f_k(T_k^{P_k(n)}x)$$

converge for μ-almost every $x \in X$.

- There is some hope in the case when $T_1 = \dots = T_k = T$.
- We also have some promising thoughts for the following averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T_1^n x) g(T_2^{n^2} x)$$

that correspond to the configurations: $(x, y), (x + n, y), (x, y + n^2) \in \mathbb{Z}^2$.

Thank you!