Newton-like components in the Cebyshev-Halley family of degree n polynomials

Dan Paraschiv, Universitat de Barcelona

We prove the existence of infinitely connected immediate basins of attraction in the Chebyshev-Halley family of maps. Furthermore, we prove that, in this case, the unbounded component of the Julia set is homeomorphic to the Julia set of a Newton map.

Introduction

Root-finding algorithms are iterative methods which asymptotically converge to the zeros (or some of the zeros) of the non linear equation, say $g(z) = 0$. The universal and most studied root-finding algorithm is known as *Newton's method*. Two other of the best known root-finding algorithms are *Chebyshev's method* and *Halley's method*. They are included in the *Chebyshev-Halley family* of root-finding algorithms, which was introduced in [1] (see also [2], and [3]), and is defined as follows. Let g be a holomorphic map. Then

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_g(z_n)}{1 - \alpha L_g(z_n)}\right)\frac{g(z_n)}{g'(z_n)},$$

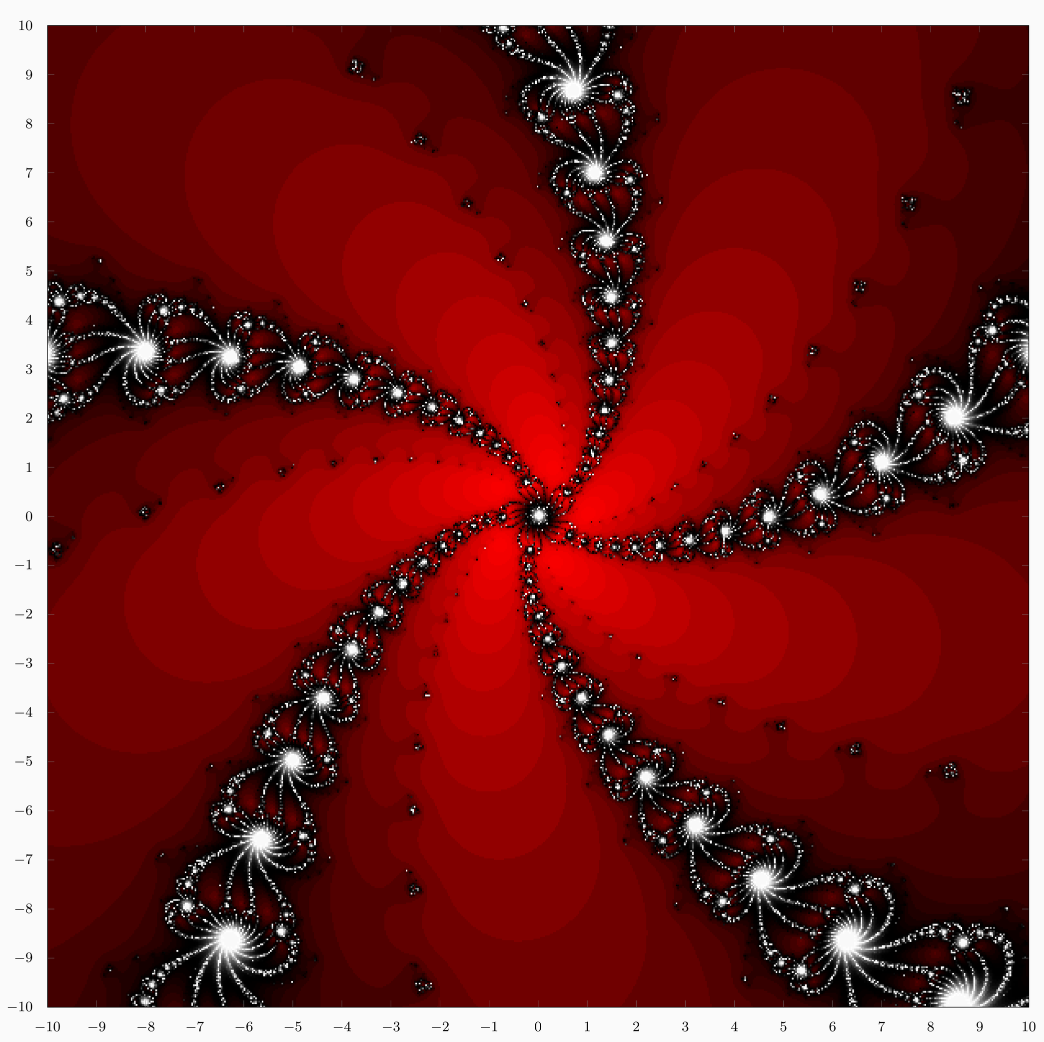
where $\alpha \in [0, 1]$ and $L_g(z) = \frac{g(z)g''(z)}{(g'(z))^2}$.

For $\alpha = 0$, we have Chebyshev's method and for $\alpha = \frac{1}{2}$ Halley's method. As α tends to ∞, the Cebyshev-Halley algorithms tend to Newton's method. The topic of the poster then arises naturally. Is the unbounded connected component of the Julia set of the Chebyshev-Halley maps applied to $f_{n,c}$ (for large enough α) homeomorphic to the Julia set of the map obtained by applying Newton's method to $f_{n,-1}$?

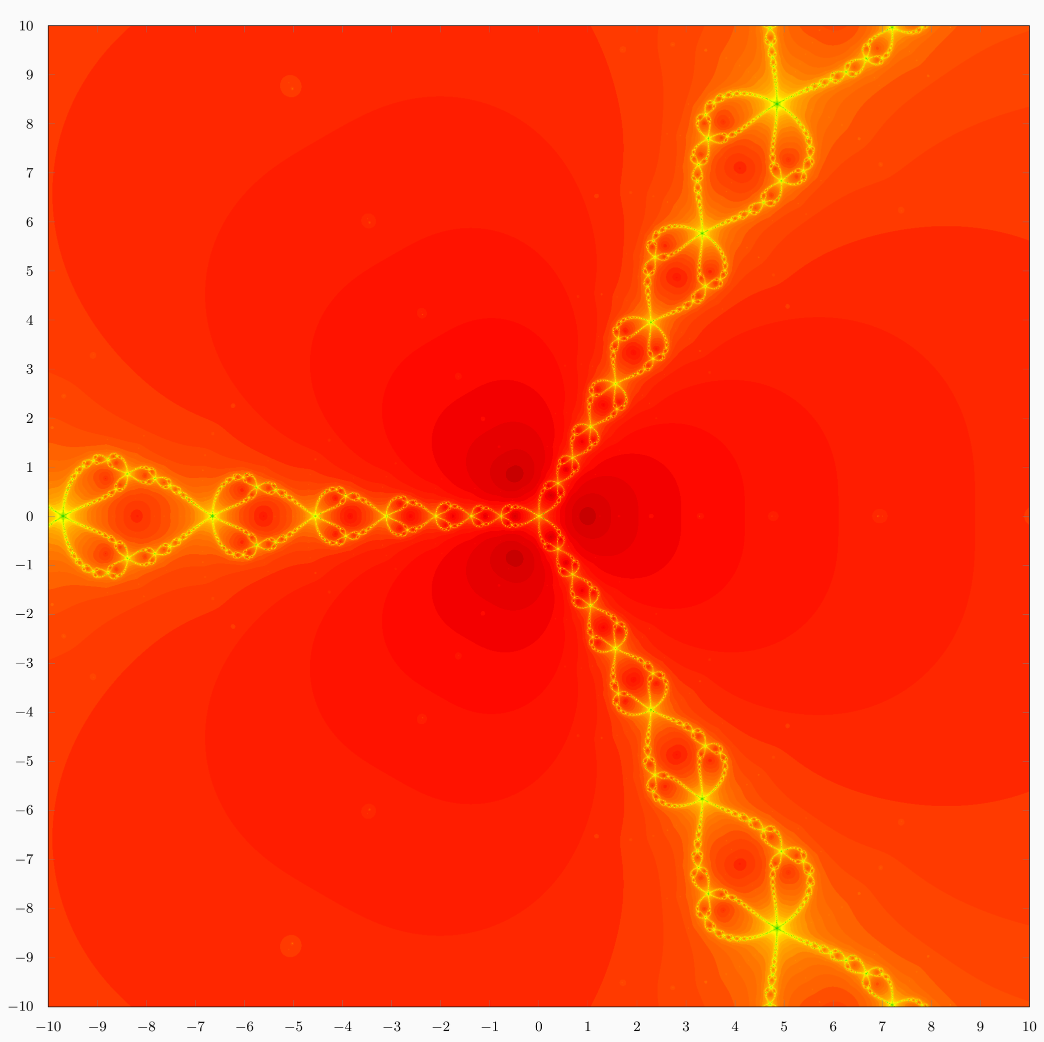
The maps obtained by applying the Cebyshev-Halley family to $f_{n,c}$ are all conjugated to the map obtained by applying the Cebyshev-Halley family to $f_{n,-1}$. By applying the Cebyshev-Halley method to $f_n(z) = z^n - 1$ we obtain the map:

$$O_{n,\alpha}(z) = \frac{(1-2\alpha)(n-1) + (2 - 4\alpha - 4n + 6\alpha n - 2\alpha n^2)z^n + (n-1)(1 - 2\alpha - 2n + 2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1-n) + (-\alpha - n + \alpha n)z^n)}.$$

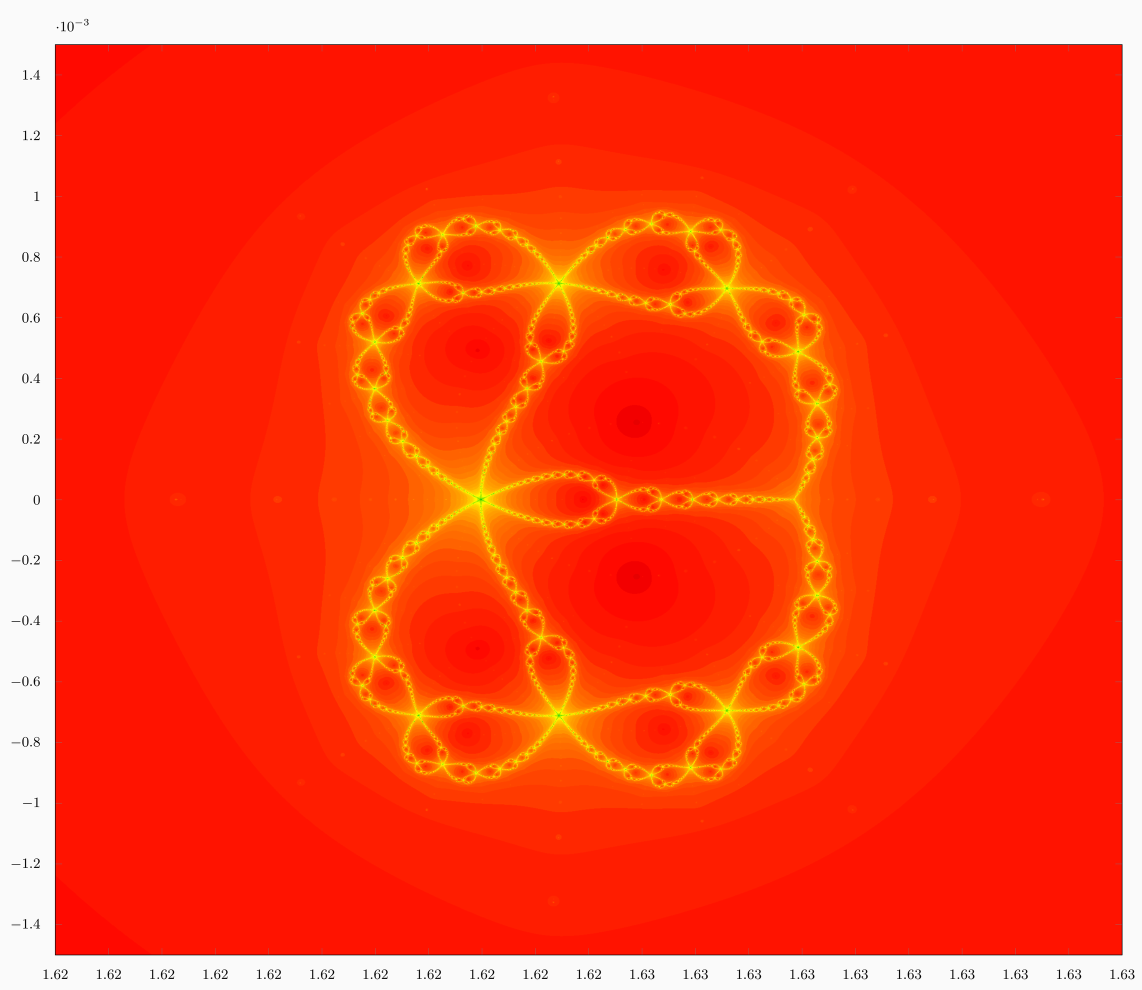
Dynamical plane of $O_{n,\alpha}$ for $n = 5$ and $\alpha = 1.5 + 1.5i$



Dynamical plane of $O_{n,\alpha}$ for $n = 3$ and $\alpha = 10$

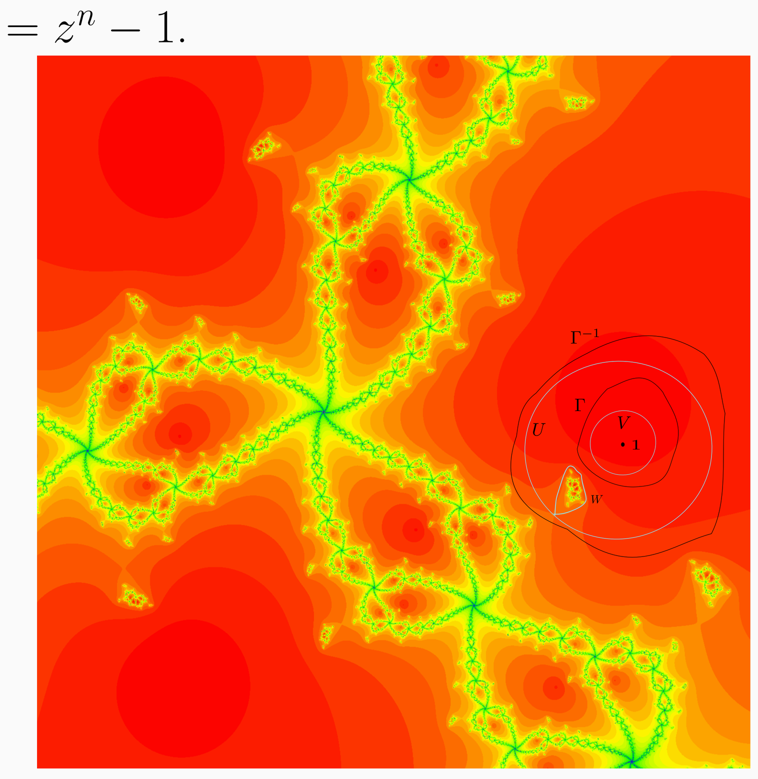


Component of the Julia set which lies in $A^*(1)$, for $n = 3$ and $\alpha = 10$



Theorem A

Fix $n \geq 2$ and assume that $A^*_{n,\alpha}(1)$ is infinitely connected for some $\alpha \in \mathbb{C}$. Then there exists an invariant Julia component Π (which contains $z = \infty$) which is a quasiconformal copy of the Julia set of N_{f_n}, where N_{f_n} is the map obtained by applying Newton's method to the polynomial $f_n(z) = z^n - 1$.

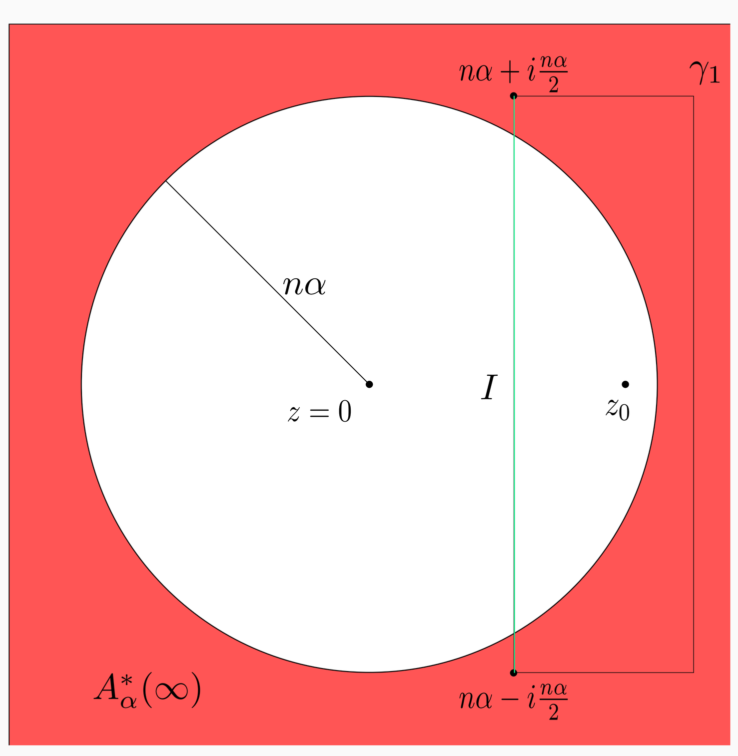


Sketch of the proof:

- construct a quasiregular f map in $A^*(1)$, and extend it by rotation map $I_\xi(z) = \xi z$ (where $\xi^n = 1$) to a quasiregular map F
- define μ_1 an F-invariant Beltrami coefficient in $A(1)$, and use it to construct μ which is I_ξ-invariant. Prove that μ is also F-invariant, and the corresponding quasiconformal map ϕ is unique and therefore $\phi \circ I_\xi = I_\xi \circ \phi$
- define $N_P = \phi \circ F \circ \phi^{-1}$ and use a criterion from [4] to conclude the proof.

Theorem B

Let $n \geq 2$. Then there exists $\alpha_0 > 0$ large enough such that for $\alpha > \alpha_0$, $\alpha \in \mathbb{R}$, $A^*_{n,\alpha}(1)$ is infinitely connected. Moreover, for $n = 2$, the statement is true for any $\alpha \in \mathbb{C}$ such that $|\alpha| > \alpha_0$.



Sketch of the proof:

- for $n = 2$, conjugate the map to a Blaschke product, then use specific numeric computations and the Schwarz Reflection Principle;
- for $n = 3$, conjugate the map by $M(z) = \frac{1}{z-1}$, obtaining $R_{n,\alpha} = M \circ O_{n,\alpha} \circ M^{-1}$
- prove the existence of a disk surrounding $z = \infty$ which is a subset of $A^*(\infty)$
- construct a closed curve which is a subset of $A^*(\infty)$ and separates the fixed point $z = 0$ and a preimage of it.

Bibliography

[1] *A. Cordero, J. Torregrosa, P. Vindel*, "Dynamics of a family of Chebyshev-Halley type methods", Appl. Math. Comput. 219, No. 16, 8568–8583.
[2] *B. Campos, J. Canela, P. Vindel*, "Convergence regions for the Chebyshev-Halley family", Commun. Nonlinear Sci. Numer. Simul. 56, 508-525.
[3] *B. Campos, J. Canela, P. Vindel*, "Connectivity of the Julia set for the Chebyshev-Halley family on degree n polynomials", Commun. Nonlinear Sci. Numer. Simul. 82, 105026.
[4] *L. Tan*, "Branch coverings and cubic Newton maps", Fund. Math. 154, no. 3, 207-260.