On geometric complexity of Julia sets - IV, Będlewo
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Typical absolute continuity for one-parameter families of dynamically defined measures

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joint work with Károly Simon, Boris Solomyak and Adam Śpiewak.

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ON ITERATED FUNCTION SYSTEMS WITH PLACE-DEPENDENT PROBABILITIES

BALÁZS BÁRÁNY

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where $N(m) = \left\lceil (m+1)\frac{\log 2}{-\log\kappa} \right\rceil$ according to Lemma 3.1. Therefore,

$$\begin{aligned}
&\sum_{\overline{\imath}\in\mathcal{S}^n}\sum_{\overline{\jmath}_1,\overline{\jmath}_2\in\mathcal{S}^{N(m)}}\sum_{\substack{l,p\in\mathcal{S}\\ l\neq p}}\int_{J_\delta(\boldsymbol{\lambda}_0)} G_{\frac{\operatorname{diam}X}{2^m}}(l\overline{\jmath}_1 1, p\overline{\jmath}_2 1,\boldsymbol{\lambda})\nu_{\boldsymbol{\lambda}}^2([\overline{\imath}l\overline{\jmath}_1]\times[\overline{\imath}p\overline{\jmath}_2])d\mathcal{L}_d(\boldsymbol{\lambda})\\
&\quad\leq \sum_{\overline{\imath}\in\mathcal{S}^n}\sum_{\overline{\jmath}_1,\overline{\jmath}_2\in\mathcal{S}^{N(m)}}\sum_{\substack{l,p\in\mathcal{S}\\ l\neq p}}\max_{\hbar_1,\hbar_2\in\mathcal{S}^{N(m)}}\\
&\qquad\times\int_{J_\delta(\boldsymbol{\lambda}_0)} G_{\frac{\operatorname{diam}X}{2^m}}(l\hbar_1 1, p\hbar_2 1,\boldsymbol{\lambda})\nu_{\boldsymbol{\lambda}}^2([\overline{\imath}l\overline{\jmath}_1]\times[\overline{\imath}p\overline{\jmath}_2])d\mathcal{L}_d(\boldsymbol{\lambda})\\
&\quad= \max_{\hbar_1,\hbar_2\in\mathcal{S}^{N(m)}}\int_{J_\delta(\boldsymbol{\lambda}_0)} G_{\frac{\operatorname{diam}X}{2^m}}(l\hbar_1 1, p\hbar_2 1,\boldsymbol{\lambda})\sum_{\overline{\imath}\in\mathcal{S}^n}\nu_{\boldsymbol{\lambda}}^2(A_{\overline{\imath}})d\mathcal{L}_d(\boldsymbol{\lambda})\\
&\quad\leq \sup_{\boldsymbol{\lambda}\in J_\delta(\boldsymbol{\lambda}_0)}\sum_{\overline{\imath}\in\mathcal{S}^n}\nu_{\boldsymbol{\lambda}}^2(A_{\overline{\imath}})\max_{\hbar_1,\hbar_2\in\mathcal{S}^{N(m)}}\int_{J_\delta(\boldsymbol{\lambda}_0)} G_{\frac{\operatorname{diam}X}{2^m}}(l\hbar_1 1, p\hbar_2 1,\boldsymbol{\lambda})d\mathcal{L}_d(\boldsymbol{\lambda})\\
&\quad\leq \frac{2C_1\operatorname{diam}X}{2^m}\sup_{\boldsymbol{\lambda}\in J_\delta(\boldsymbol{\lambda}_0)}\sum_{\overline{\imath}\in\mathcal{S}^n}\nu_{\boldsymbol{\lambda}}^2(A_{\overline{\imath}}),
\end{aligned}$$

Introduction

- Let $\mathcal{S} = \{f_1, \ldots, f_m\}$ be an iterated function system (IFS) of $C^{1+\alpha}$-maps on $\mathbb{R}$,
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- Let $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$ be the symbolic space and let $\sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots)$ the left-shift operator,
- Let $\Pi\colon \Omega \mapsto \Lambda$ be the natural projection

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For short, for $\omega \in \Omega_*$ let $f_\omega = f_{\omega_1} \circ \cdots \circ f_{\omega_n}$

Introduction

- Ruelle: If separation holds, i.e. $f_i(\Lambda) \cap f_j(\Lambda) = \emptyset$ then

$$\dim_H \Lambda = \dim_B \Lambda = s, \text{ where } P(s) = 0 \text{ and } P(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega\in\Omega_n} \|f'_\omega\|^s \right),$$

$$\dim_H \Pi_*\mu = \frac{h_\mu}{\chi_\mu}, \text{ where } h_\mu \text{ is the entropy and } \chi_\mu = -\int \log|f'_{\omega_1}(\Pi(\sigma\omega))| d\mu(\omega).$$

- $\overline{\dim}_B \Lambda \leq \min\{1, s\}$ and $\overline{\dim}_P \Pi_*\nu \leq \min\{1, \frac{h_\mu}{\chi_\mu}\}$ in general
- What about overlaps?

Introduction

Self-similar case ($f_i(x) = \rho_i x + t_i$) dimension

- Pollicott and Simon: $\{\lambda x, \lambda x + 1, \lambda x + 3\}$ of Leb.-a.e. $\lambda \in [1/4, 2/5]$,
- Hochman: under superexponential separation and Bernoulli μ,
- Jordan and Rapaport: ergodic measures,
- Varjú: $\{\lambda x, \lambda x + 1\}$ transcendental λ

Self-similar case absolute continuity

- Shmerkin: $\{\lambda x, \lambda x + 1\}$ except 0 dim of $\lambda \in [1/2, 1]$
- Varjú: $\{\lambda x, \lambda x + 1\}$ for some algebraic λ
- Shmerkin and Solomyak: uniform contraction and Bernoulli μ
- Saglietti, Shmerkin and Solomyak: general contracition and Bernoulli μ

Introduction

Self-conformal case

- Simon, Solomyak and Urbański: under transversality condition a.e. parameters
- Peres and Schlag: estimates on the dimension of exceptional parameters
- Hochman and Solomyak: rational maps (Furstenberg measure)

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Theorem (Simon, Solomyak and Urbański)**.** *Let* U *be a bounded open domain and let* $\mathcal{S}_\lambda = \{f_1^\lambda, \ldots, f_m^\lambda\}$ *be a parametrised family of conformal IFS such that*

- $0 < \gamma_1 \leq |(f_i^\lambda)'(x)| \leq \gamma_2 < 1$ *and* $\lambda \mapsto f_i^\lambda \in C^{1+\alpha}$ *is continuous,*
- *there exists* $C > 0$ *that for every* $\omega, \tau \in \Omega$ *with* $\omega_1 \neq \tau_1$

$$\mathcal{L}(\lambda \in U : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| < r) \leq Cr \text{ for all } r > 0.$$

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Then for every **fixed** *ergodic measure* μ

$$\dim_H(\Pi^\lambda)_*\mu = \min\{1, \frac{h_\mu}{\chi_\mu^\lambda}\} \text{ and } (\Pi^\lambda)_*\mu \ll \mathcal{L} \text{ if } \frac{h_\mu}{\chi_\mu^\lambda} > 1,$$

for Leb.-a.e. $\lambda \in U$.

E.g. $\{f_1(x) + \lambda_1, \ldots, f_m(x) + \lambda_m\}$ and $\|f_i'\| < 1/2$.

Parameter dependent measures

♣ Dimension maximizing measures/Equilibrium measures

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- Under the assumptions of Simon, Solomyak and Urbański:

$$\dim_H \Lambda_\lambda = s_\lambda, \text{ where } P_\lambda(s_\lambda) = 0 \text{ and } P_\lambda(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega\in\Omega_n} \|(f_\omega^\lambda)'\|^s \right).$$

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- Let $\phi_\lambda(\omega) = s_\lambda \log|(f_{\omega_1}^\lambda)'(\Pi_\lambda(\sigma\omega))|$, which is uniformly Hölder-continuous, hence by Bowen, there exists a unique ergodic shift-invariant measure μ_λ such that there exists $C > 0$ that for every $\lambda \in U$

$$C^{-1} \leq \frac{\mu_\lambda([\omega])}{\|(f_\omega^\lambda)'(x)\|^{s_\lambda}} \leq C \text{ for every } \omega \in \Omega_* \text{ and } \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}^\lambda} = s_\lambda.$$

Parameter dependent measures

♠ Place-dependent self-conformal measures/Chaos game

- The standard Chaos game: let $(p_1, \ldots, p_m)$ be a probability vector and
$$x_0 \to f_i(x_0) \text{ with probability } p_i$$
Its stationary measure is the self-conformal measure $\nu = \sum_{i=1}^m p_i (f_i)_* \nu$.

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- Let $p_i \colon \Lambda \mapsto (0, 1)$ be Hölder continuous maps such that $\sum_{i=1}^m p_i(x) \equiv 1$
$$x_0 \to f_i(x_0) \text{ with probability } p_i(x_0)$$
Fan and Lau: The stationary measure ν exists and unique, i.e. $\int g(x) d\nu(x) = \int \sum_{i=1}^m p_i(x) g(f_i(x)) d\nu(x)$.

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- In particular, ν is the push-forward of the Gibbs measure μ with respect to the potential $\phi(\omega) = \log p_{\omega_1}(\Pi(\sigma\omega))$, i.e. there exists $C > 0$ such that for all $\omega \in \Omega_*$ and $x \in \Lambda$

$$C^{-1} \leq \frac{\mu([\omega])}{p_{\omega_1}(f_{\omega_2} \circ \cdots \circ f_{\omega_n}(x)) \cdots p_{\omega_n}(x)} \leq C.$$

Setup and main results

Let U and I be compact intervals and let $\{f_i^\lambda\}_{i=1}^m$ be a parametrized IFS on I s. t.

(A1) The maps $x \mapsto f_i^\lambda(x)$ are $C^{2+\delta}$ on I (uniformly w.r.t $\lambda \in U$),

(A2) The maps $\lambda \mapsto f_i^\lambda(x)$ are $C^{1+\delta}$ on U (uniformly w.r.t $x \in I$),

(A3) The maps $\frac{\partial^2}{\partial x \partial \lambda} f_j^\lambda(x)$ are Hölder uniformly both in $x \in I$ and $\lambda \in U$,

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(A4) The IFS is uniformly hyperbolic, i.e. $0 < \gamma_1 \le |(f_j^\lambda)'(x)| \le \gamma_2 < 1$,

(T) Suppose that for every $\omega, \tau \in \Omega$ with $\omega_1 \neq \tau_1$

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| < \delta \Rightarrow |\frac{\partial}{\partial \lambda}\Pi_\lambda(\omega) - \frac{\partial}{\partial \lambda}\Pi_\lambda(\tau)| > \delta.$$

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Lemma (B., Simon, Solomyak, Śpiewak). *Under the conditions above*

$$\text{(T)} \Leftrightarrow \exists C > 0 : \mathcal{L}(\lambda \in U : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| < r) \leq Cr \ \forall \omega, \tau \in \Omega \text{ with } \omega_1 \neq \tau_1.$$

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Let μ_λ be a parametrized family of ergodic measures such that

(M0) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\lambda - \tau| < \delta$ then

$$e^{-\varepsilon|\omega|}\mu_\lambda([\omega]) \leq \mu_\tau([\omega]) \leq e^{\varepsilon|\omega|}\mu_\lambda([\omega]) \text{ for all } \omega \in \Omega_*,$$

(MC) The maps $\lambda \mapsto h_{\mu_\lambda}$, $\lambda \mapsto \chi^\lambda_{\mu_\lambda}$ are continuous.

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Theorem (B., Simon, Solomyak, Śpiewak). *If (A1)-(A4), (T) and (M0)-(MC) hold then*

$$\dim_H(\Pi_\lambda)_*\mu_\lambda = \min\left\{1, \frac{h_{\mu_\lambda}}{\chi^\lambda_{\mu_\lambda}}\right\} \text{ for Leb.-a.e. } \lambda \in U.$$

- This theorem was essentially proved by B. and Rams.

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Let μ_λ be a parametrized family of ergodic measures such that

(M) There exists $c, \theta > 0$ such that

$$e^{-c|\lambda-\tau|^\theta \cdot |\omega|}\mu_\lambda([\omega]) \leq \mu_\tau([\omega]) \leq e^{c|\lambda-\tau|^\theta \cdot |\omega|}\mu_\lambda([\omega]) \text{ for all } \omega \in \Omega_*.$$

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Theorem (B., Simon, Solomyak, Śpiewak). *If (A1)-(A4), (T) and (M)-(MC) hold then*

$$(\Pi_\lambda)_*\mu_\lambda \ll \mathcal{L} \textit{ for Leb.-a.e. } \lambda \in \left\{\lambda \in U : \frac{h_{\mu_\lambda}}{\chi^\lambda_{\mu_\lambda}} > 1\right\}.$$

- The proof is an adaptation of Peres and Schlag.

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Let $\phi^\lambda \colon \Omega \mapsto \mathbb{R}$ be a potential such that

(P1) The map $\omega \mapsto \phi^\lambda(\omega)$ is Hölder uniformly w.r.t. $\lambda \in U$,

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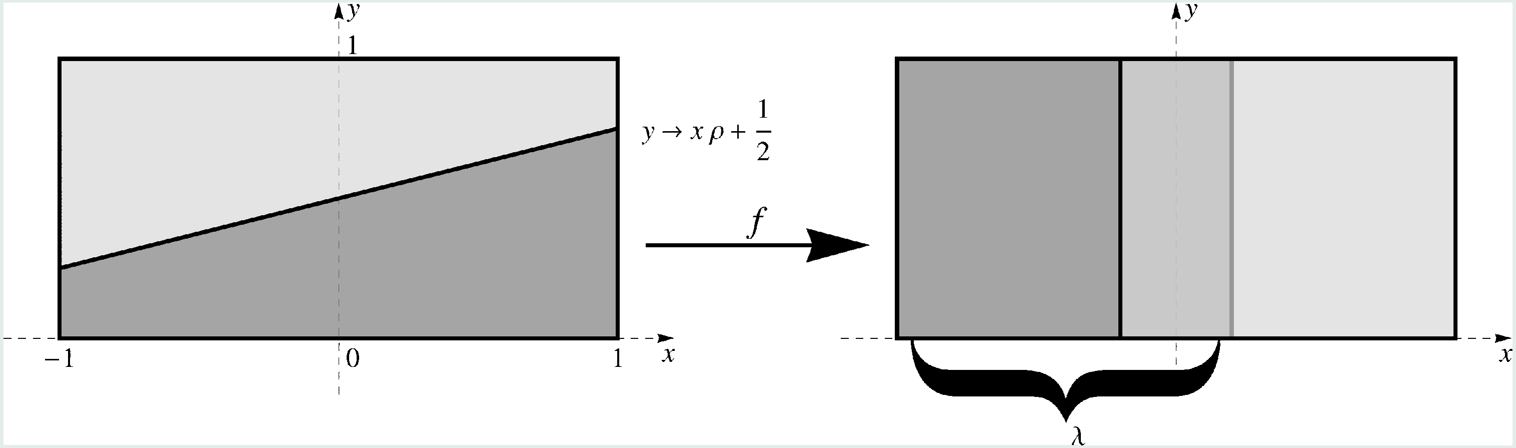
E.g., let $\{f_1(x), f_2(x)+t, f_3(x)\}$ be such that $f_i([0,1]) \subset [0,1]$, $f_1([0,1]) \cap f_3([0,1]) = \emptyset$ and $\|f_i'\| < 1/2$

Examples

♦ Place-dependent Bernoulli



$$f(x,y) = \begin{cases} (\lambda x - 1 + \lambda, \frac{2y}{1+2\rho x}) & \text{if } 0 \leq y < \rho x + 1/2 \\ (\lambda x + 1 - \lambda, \frac{2y-2\rho x-1}{1-2\rho x}) & \text{if } \rho x + 1/2 \leq y \leq 1 \end{cases}$$

$$\nu_{SBR} = \nu \times \mathcal{L}, \text{ where } \int g(x) d\nu(x) = \int (\frac{1}{2}+\rho x) g(\lambda x - 1 + \lambda) + (\frac{1}{2} - \rho x) g(\lambda x + 1 - \lambda) d\nu(x)$$

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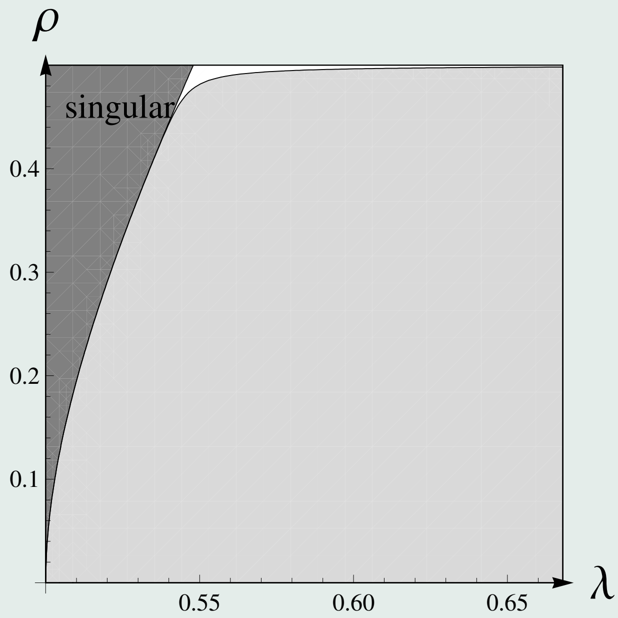

$$\nu_{SBR} \ll \mathcal{L}_2, \text{ in particular, } \nu \ll \mathcal{L}$$

Examples

♣ Blackwell measure of hidden Markov-chains

Let $p \in (0,1)$ and $\varepsilon \in (0,1)$ let $X_1, X_2, \dots,$ be the stationary Markov chain w.r.t.

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \text{ and i.i.d. noise } E_i \text{ with } \mathbb{P}(E_i = 1) = 1 - \mathbb{P}(E_i = 0) = \varepsilon.$$

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Let $Z_i = X_i + E_i \mod 2$ be the hidden Markov chain with entropy

$H(Z) = -\int \sum_{i=1}^{2} \log p_i(x) d\nu(x),$ where ν is the place-dependent prob. measure w.r.t

$$S_1(x) = \frac{x \cdot p(1-\varepsilon) + (1-x)\cdot(1-p)(1-\varepsilon)}{x\cdot(p(1-\varepsilon)+(1-p)\varepsilon) + (1-x)\cdot((1-p)(1-\varepsilon)+p\varepsilon)}$$

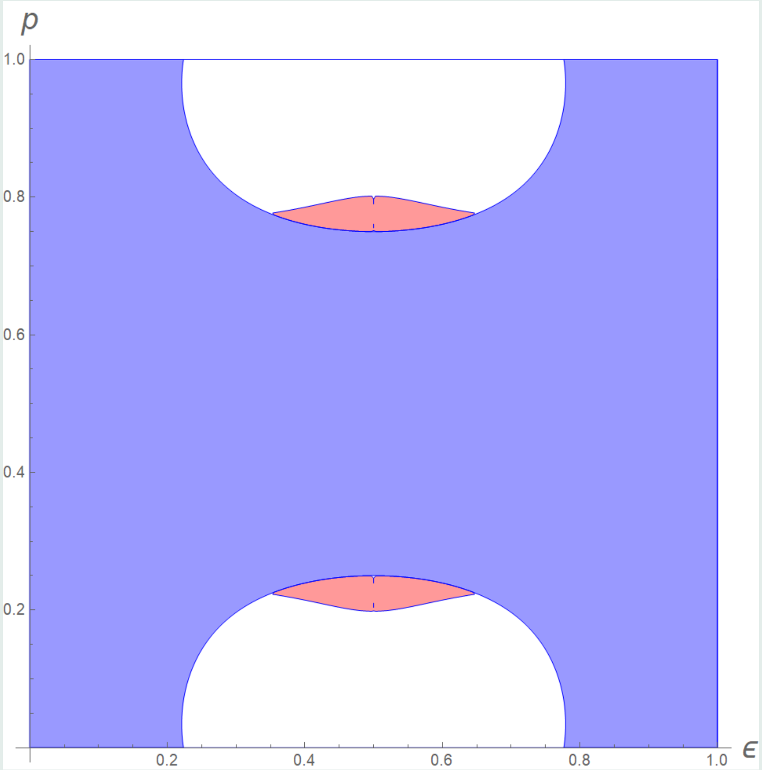
$$S_2(x) = \frac{x \cdot p\varepsilon + (1-x)\cdot(1-p)\varepsilon}{x\cdot((1-p)(1-\varepsilon)+p\varepsilon) + (1-x)\cdot(p(1-\varepsilon)+(1-p)\varepsilon)}$$

$$p_1(x) = x\cdot(p(1-\varepsilon)+(1-p)\varepsilon) + (1-x)\cdot((1-p)(1-\varepsilon)+p\varepsilon)$$

$$p_2(x) = x\cdot((1-p)(1-\varepsilon)+p\varepsilon) + (1-x)\cdot(p(1-\varepsilon)+(1-p)\varepsilon)$$

Examples

♣ Blackwell measure of hidden Markov-chains



- B., Pollicott, Simon: blue region of singularity
- B., Kolossváry: establish transversality in red region and a.c.

Outline of the proof: dimension

(M0) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\lambda - \tau| < \delta$ then

$$e^{-\varepsilon|\omega|}\mu_\lambda([\omega]) \leq \mu_\tau([\omega]) \leq e^{\varepsilon|\omega|}\mu_\lambda([\omega]) \text{ for all } \omega \in \Omega_*,$$

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- Hence there is a small neighbourhood V of λ_0 such that for

$$A^N_{\lambda_0} = \{\omega \in \Omega : e^{-n(h_{\mu_{\lambda_0}}+2\varepsilon)} \leq \mu_{\lambda_0}([\omega]) \leq e^{-n(h_{\mu_{\lambda_0}}-2\varepsilon)}$$
$$e^{-n(\chi^{\lambda_0}_{\mu_{\lambda_0}}+2\varepsilon)} \leq |(f^{\lambda_0}_\omega)'(x)| \leq e^{-n(\chi^{\lambda_0}_{\mu_{\lambda_0}}-2\varepsilon)} \text{ for } n \geq N\}$$

$\mu_\lambda(\bigcup_{N=1}^\infty A^N_{\lambda_0}) = 1$ for every $\lambda \in V$. (M0) remains valid for $\tilde{\mu}_\lambda = \mu_\lambda|_{A^N_{\lambda_0}}$.

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- **Enough**: for $\alpha < \min\{1, \frac{h_{\mu_{\lambda_0}}}{\chi_{\mu_{\lambda_0}}}\} - \varepsilon'$

$$\int_V \iint_{\Omega\times\Omega} \frac{1}{|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)|^\alpha} d\tilde{\mu}_\lambda(\omega) d\tilde{\mu}_\lambda(\tau) d\lambda < \infty$$

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From this moment, it is a usual transversality argument.

Outline of the proof: absolute continuity

- For a **fixed** measure, almost sure absolute continuity is proved by showing
$$\int_U \int_{\mathbb{R}} \underline{D}(\Pi_*^\lambda \mu_{\lambda_0}, x) d\Pi_*^\lambda \mu_{\lambda_0}(x) d\lambda \leq \liminf_{r \to 0} \frac{1}{2r} \int_U \int_{\mathbb{R}} \Pi_*^\lambda \mu_{\lambda_0}(B(x,r)) d\Pi_*^\lambda \mu_{\lambda_0}(x) d\lambda < \infty$$
- Transversality is used to obtain, roughly speaking,
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- **Problem in our case**: using the previous approach, we only obtain
$$\int_V \int_{\mathbb{R}} \Pi_*^\lambda \mu_\lambda(B(x,r)) d\Pi_*^\lambda \mu_\lambda(x) d\lambda$$
$$\lesssim r^{-\varepsilon} \int_V \int_{\mathbb{R}} \Pi_*^\lambda \mu_{\lambda_0}(B(x,r)) d\Pi_*^\lambda \mu_{\lambda_0}(x) d\lambda \leq Cr^{1-\varepsilon},$$
so we do not get the finiteness of the integral.

Outline of the proof: absolute continuity

The proof is an adaptation of the method of Peres and Schlag.

- **Sobolev energy**:

$$\mathcal{I}_\alpha(\nu) := \int_{\mathbb{R}} |\hat{\nu}(\xi)|^2 (1 + |\xi|)^{\alpha - 1} d\xi$$

- **Sobolev dimension**:

$$\dim_S(\nu) := \sup \{\alpha > 0 : \mathcal{I}_\alpha(\nu) < \infty\}$$

- if $\dim_S(\nu) > 1$, then $\hat{\nu} \in L^2(\mathbb{R})$

Outline of the proof: absolute continuity

- Instead of the usual energy integrals, one can rely on the decomposition obtained from the **Littlewood-Paley function**: $\psi : \mathbb{R} \to \mathbb{R}$ of Schwarz class, with $\widehat{\psi} \geq 0$ and

$$\mathrm{supp}(\widehat{\psi}) \subset \{\xi : \ 1 \leq |\xi| \leq 4\}, \qquad \sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1 \ \text{ for all } \xi \neq 0.$$

$$\mathcal{I}_\alpha(\nu_\lambda) \asymp \sum_{n=0}^{\infty} 2^{\alpha n} \int_\Omega \int_\Omega \psi\Big(2^n(\Pi_\lambda(\omega_1) - \Pi_\lambda(\omega_2))\Big)\, d\mu_\lambda(\omega_1)\, d\mu_\lambda(\omega_2)$$

- **Difficulties**: ψ is not non-negative, so using bounds on the measures to change $\lambda \to \lambda_0$ requires extra care

Outline of the proof: absolute continuity

- **Our strategy**:

$$\int_V \mathcal{I}_\alpha(\nu_\lambda) d\lambda$$

$$\approx \sum_{n=0}^{\infty} 2^{\alpha n} \int_{\mathbb{R}} \int_{\Omega} \int_{\Omega} \psi\big(2^n(\Pi^\lambda(\omega_1) - \Pi^\lambda(\omega_2))\big)\, d\mu_\lambda(\omega_1)\, d\mu_\lambda(\omega_2) d\lambda$$

$$\approx \sum_{n=0}^{\infty} 2^{\alpha n} \int_{\mathbb{R}} \int_{\Omega} \int_{\Omega} \psi\big(2^n(\Pi^\lambda(\omega_1) - \Pi^\lambda(\omega_2))\big) e_n(\omega_1, \omega_2, \lambda)\, d\mu_{\lambda_0}(\omega_1)\, d\mu_{\lambda_0}(\omega_2) d\lambda$$

We extended the proof of Peres and Schlag to apply transversality for the modified kernel $\psi(2^n \cdot) e_n(\cdot)$. This requires certain regularity of $e_n(\cdot)$, coming from condition (M).

Thank you for your attention!