Combinatorial structure of the renormalization attractor for multicritical circle maps

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Joint work with Michael Yampolsky

Multicritical circle maps

$S^1 := \mathbb{R}/\mathbb{Z}$

Definition: An analytic *multicritical circle map* is an orientation preserving analytic homeomorphism $f\colon S^1 \to S^1$ with several critical points.

Near a critical point:

$$f = \psi \circ q_d \circ \phi,$$

where ϕ and ψ are locally conformal and $q_d(x) = x^d$, $d \in 2\mathbb{N} + 1$.

- d – the critical exponent of a critical point.
  For simplicity, we will assume that $d = 3$, for each critical point.

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Special case: unicritical circle maps – the maps with a single critical point.

Golden-mean universality for unicritical circle maps

The golden-mean: $\rho_* := \frac{\sqrt{5}-1}{2} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = [1, 1, 1, \dots]$.

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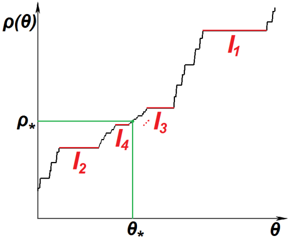
Universality: Let $f_\theta \colon S^1 \to S^1$ be a smooth family of unicritical circle maps with a fixed critical exponent $d \in 2\mathbb{N}+1$ and $\frac{\partial f_\theta}{\partial \theta}(x) > 0$, for all $x \in S^1$, $\theta \in \mathbb{R}$. Let θ_* be such that

$$\rho(f_{\theta_*}) = \rho_*.$$

Let $\{I_n\}$ be closed intervals consisting of parameters θ, for which

$$\rho(f_\theta) = p_n/q_n$$

and such that I_n accumulate to θ_*. Then



$$\lim_{n\to\infty} \frac{|I_n|}{|I_{n+1}|} = \delta > 1, \quad \delta = \delta(d)$$ is a universal constant, independent of a particular family f_θ. $\delta(3) \approx 2.83361$.

Universality via hyperbolicity of renormalization

- Renormalization $\mathcal{R}$ is a (nonlinear) operator on the space of circle maps: $f \mapsto \mathcal{R}f$.
- If $\rho(f) = [r_1, r_2, r_3, \dots]$, then $\rho(\mathcal{R}f) = [r_2, r_3, r_4, \dots]$.

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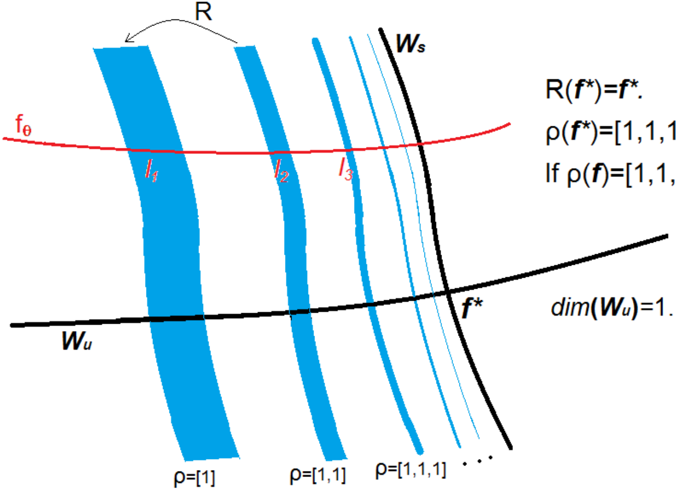

R(***f****)=***f****.

ρ(***f****)=[1,1,1,...].

If ρ(***f***)=[1,1,1,...], then ***f*** ϵ ***W***s.

dim(***W****u*)=1.

Rotation numbers of bounded type

- An irrational rotation number ρ is of type bounded by B, if $\rho(f) = [r_1, r_2, r_3, \dots]$, and $r_j \leq B$, for any $j \in \mathbb{N}$.

Equivalently $\left|\rho - \frac{p}{q}\right| \geq \frac{M}{q^2}$ for some $M = M(B)$ and any integers p and $q \neq 0$.

Renormalization Hyperbolicity Conjecture (RHC) for unicritical maps of type bounded by B

Renormalization Hyperbolicity Conjecture: The renormalization operator $\mathcal{R}$ satisfies the following properties:

(1) **(global horseshoe attractor)** If $\rho(f) = \rho(g) \notin \mathbb{Q}$ is of type bounde by B, then

$$\mathrm{dist}(\mathcal{R}^n f, \mathcal{R}^n g) \to 0.$$

Furthermore, there exists a compact, $\mathcal{R}$-invariant attracting set $\mathcal{A}_B$ on which the action of $\mathcal{R}$ is topologically conjugate to the right shift $\sigma\colon \{1, \dots, B\}^{\mathbb{Z}} \to \{1, \dots, B\}^{\mathbb{Z}}$:

$$\iota \circ \mathcal{R} \circ \iota^{-1} = \sigma,$$

and if $f = \iota^{-1}(\dots, r_{-k}, \dots, r_{-1}, r_0, r_1, \dots, r_k, \dots)$, then

$$\rho(f) = [r_0, r_1, \dots, r_k, \dots].$$

(2) **(hyperbolicity)** The set $\mathcal{A}_B$ is uniformly hyperbolic for the operator $\mathcal{R}$ with a one-dimensional unstable direction.

Results and observations

- RHC for unicritical circle maps has been proven by de Faria and de Melo, and also by Yampolsky in the late 1990's – early 2000's.

Observations:

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I.e., any two distinct elements of the attractor can be backwards renormalized a number of times until they will have different combinatorics.

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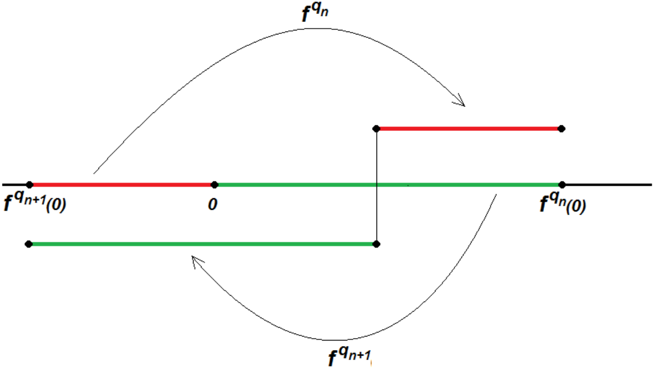
I.e., any two distinct elements of the attractor can be backwards renormalized a number of times until they will have different combinatorics.

Main result: (3) does not hold for multicritical circle maps with at least three critical points.

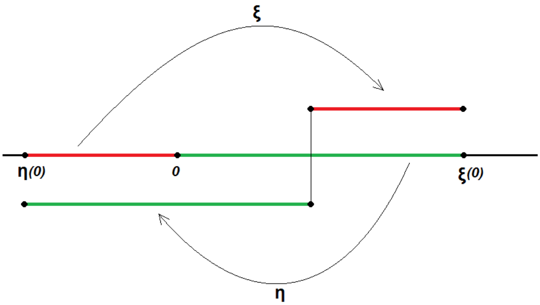
Renormalization of (uni)critical circle maps

If $\rho(f) = [r_1, r_2, \ldots]$, then $p_n/q_n = [r_1, \ldots, r_n]$.



$$\mathcal{R}^n f := (h \circ f^{q_{n+1}} \circ h^{-1}, h \circ f^{q_n} \circ h^{-1}),$$
where $h(x) = x/f^{q_n}(0)$.

Critical commuting pairs and circle maps



Can we "glue" a commuting pair into a circle?

The map

$$G_\zeta(x) = \begin{cases} \eta \circ \xi(x), & \text{if } x \in [\eta(0), 0], \\ \eta(x), & \text{if } x \in [0, \xi \circ \eta(0)] \end{cases}$$

projects to a smooth homeomorphism of the "circle" to itself with critical points.

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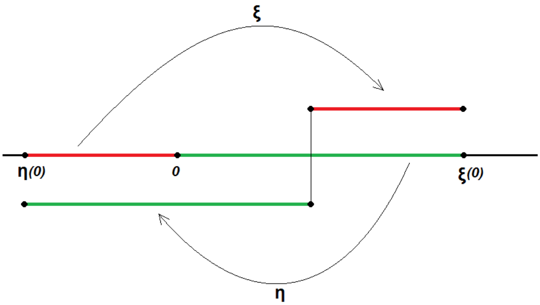

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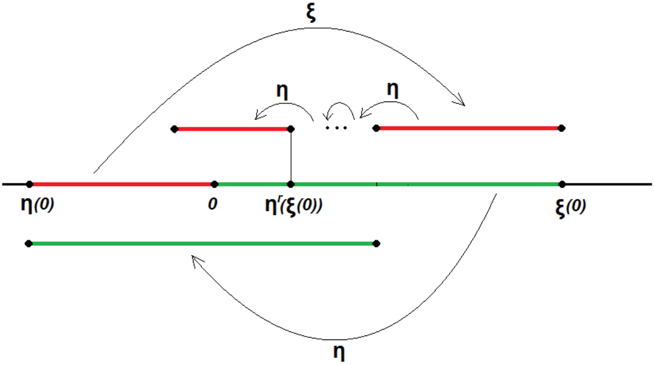
projects to a smooth homeomorphism of the "circle" to itself with critical points.

Difficulty: Lack of a canonical affine structure on this "circle".

Renormalization of critical commuting pairs

Commuting pair: $\zeta = (\eta, \xi)$.



$$\mathcal{R}\zeta := (h \circ \eta^r \circ \xi \circ h^{-1}, h \circ \eta \circ h^{-1}), \qquad \text{where } h(x) = x/\eta(0).$$

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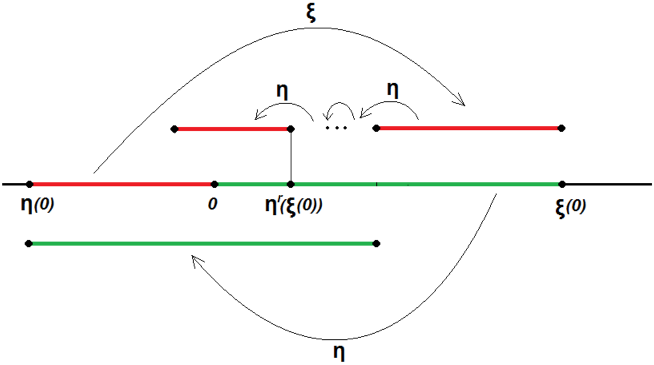

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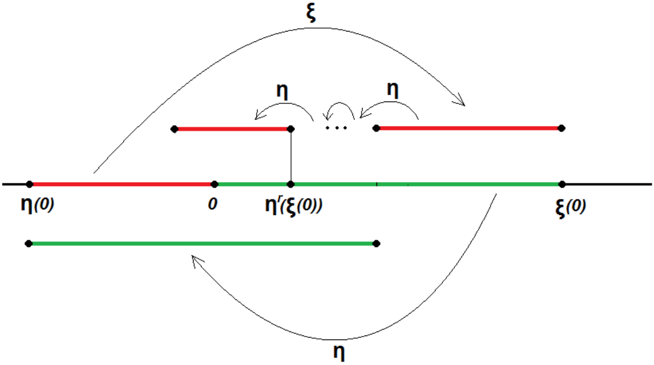

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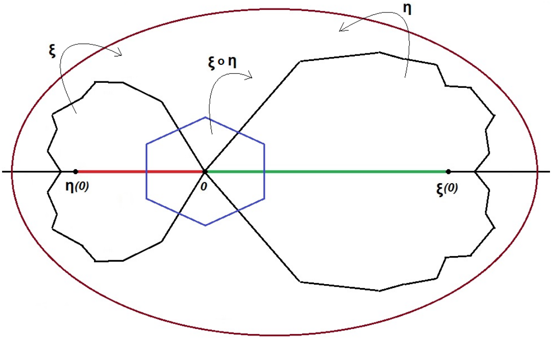
- r is the height of ζ, $\quad r = \chi(\zeta)$.
- $\rho(\zeta) = [r_0, r_1, r_2, \ldots]$, where $r_j := \chi(\mathcal{R}^j \zeta)$.

Complex a priori bounds for bounded type

Theorem (de Faria 1992): For any infinitely renormalizable ζ of type bounded by B, sufficiently high renormalizations $\mathcal{R}^n\zeta$ extend to *holomorphic (commuting) pairs* (analogues of polynomial-like maps) with definite moduli. The domains of the holomorphic pairs are $K_1(B)$-quasidisks of comparable sizes.



Any two such holomorphic pairs with the same rotation number are $K_2(B)$-qc conjugate.

Rigidity of McMullen's towers

- $p\mathcal{R}\zeta :=$ "Renormalization without rescaling".
- $\mathcal{H}$ – the space of holomorphic pairs, satisfying complex bounds.
- A limiting tower is an element of the product space $\mathcal{H}^{\mathbb{Z}}$,

$$\mathcal{T} = (\dots, H_{-n}, \dots, H_0, \dots, H_n, \dots),$$

such that for each $n \in \mathbb{Z}$, we have $p\mathcal{R}H_n|_{\mathbb{R}} = H_{n+1}|_{\mathbb{R}}$.

- Any element ζ of the attractor $\mathcal{A}_B$ is associated with a unique limiting tower $\mathcal{T}$, such that ζ is the restriction of H_0.

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Tower Rigidity Theorem (Yampolsky 2001):

(1) Any two limiting towers with the same combinatorics are qc conjugate.

(2) Any two qc conjugate limiting towers are affinely conjugate.

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Main result: (1) does not hold for multicritical circle maps with at least three critical points.

Results

Theorem (Yampolsky, G 2021): For any two analytic multicritical commuting pairs ζ_1 and ζ_2 with the same combinatorics and the same irrational rotation number of type bounded by B, we have

$$\mathrm{dist}_{C^0}(\mathcal{R}^n\zeta_1, \mathcal{R}^n\zeta_2) \to 0 \qquad \text{exponentially fast as} \quad n \to \infty.$$

The exponential convergence rate depends only on B and the number of critical points.

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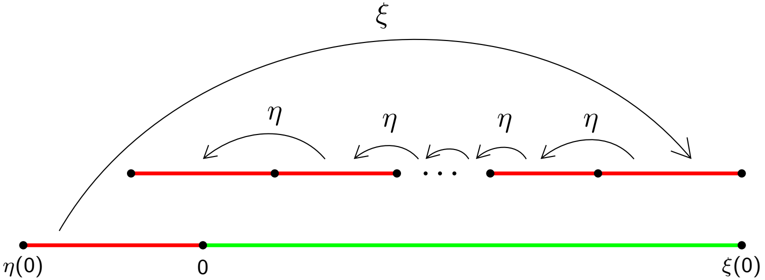
Theorem (Yampolsky, G 2021): Any two qc conjugate limiting towers with (bi-infinite) rotation numbers of bounded type are affinely conjugate.

Theorem (Yampolsky, G 2022): If the number of critical points is at least three, then there exist two limiting towers that have the same combinatorics, but are not topologically conjugate.
(The restrictions of these two towers to the real line will still be quasi-symmetrically conjugate.)

Combinatorics of multicritical commuting pairs



- If $\rho(\zeta) = [r_0, r_1, r_2, \dots]$, then the position of a critical point c can be encoded by a sequence

$$\omega(c) = (w_0, w_1, w_2, \dots),$$

where $w_j \in \{-1, 0, \dots, r_j\}$, and $w_{j+1} = 0$ iff $w_j = -1$. Then

$$\mathcal{R}[\omega(c)] = (w_1, w_2, \dots).$$

- Combinatorics of ζ consists of $\rho(\zeta)$ and an unordered list of sequences $\omega(c_1), \dots, \omega(c_k)$.

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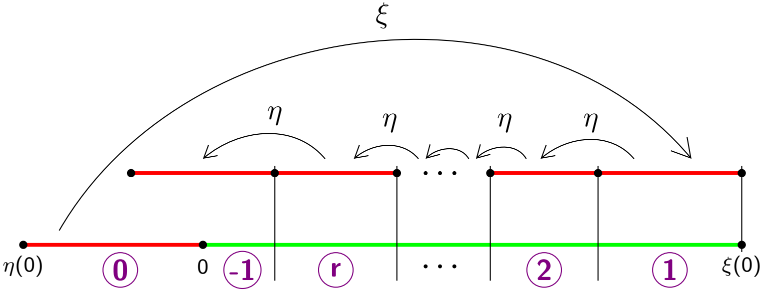

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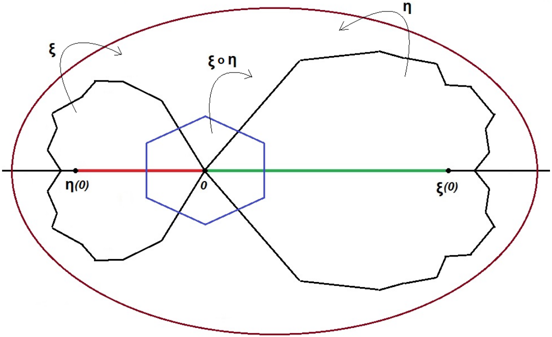
Complex a priori bounds for bounded type

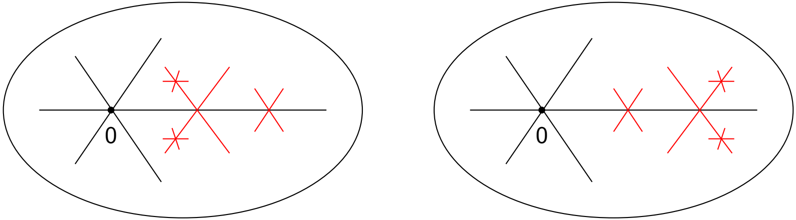
Theorem (Estevez, Smania, Yampolsky 2020): Complex bounds exist for any multicritical commuting pair ζ with $\rho(\zeta)$ of bounded type.

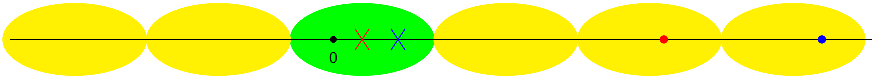


Attention! Two multicritical holomorphic commuting pairs with the same (real) combinatorics might not be qc (even topologically!) conjugate.

Complex a priori bounds for bounded type

Important: In the multicritical case, holomorphic pair extensions of $\mathcal{R}^n\zeta$ will have critical points both on $\mathbb{R}$ and on $\mathbb{C}\setminus\mathbb{R}$.





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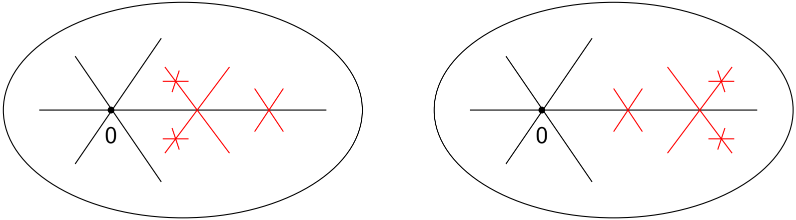
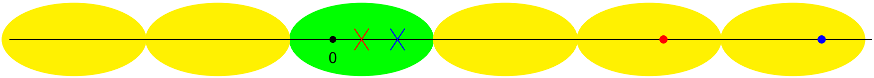




Lemma: For any two multicritical holomorphic commuting pairs H_1 and H_2 with the same (real) combinatorics, there exists $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$, the holomorphic pairs $\mathcal{R}^n H_1$ and $\mathcal{R}^n H_2$ are qc conjugate.

Multicritical limiting towers

- $\mathcal{T}_1$, $\mathcal{T}_2$ – two towers with $k > 1$ critical points and the same "bi-infinite" combinatorics

$$\rho = (\ldots, r_{-n}, \ldots, r_0, \ldots, r_n, \ldots), \qquad \text{where } r_j \leq B;$$

$$\omega(c_j) = (\ldots, w^j_{-n}, \ldots, w^j_0, \ldots, w^j_n, \ldots), \qquad \text{for } j = 1, \ldots, k-1.$$

- Two critical points c_i and c_j have asymptotically non-equal histories, if the sequences

$$(\ldots, w^i_{-3}, w^i_{-2}, w^i_{-1}) \qquad \text{and} \qquad (\ldots, w^j_{-3}, w^j_{-2}, w^j_{-1})$$

differ at infinitely many positions.

Lemma: If the critical points of $\mathcal{T}_1$ (and hence, also of $\mathcal{T}_2$) have pairwise asymptotically non-equal histories, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are qc conjugate.

Construction of non-conjugate towers

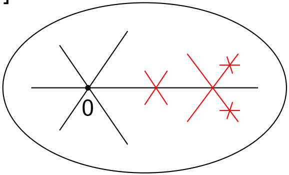
- For $n \in \mathbb{N} \cup \{\infty\}$, consider the sequences of length $2n$

$$\overline{a_n} = \underbrace{(1, \dots, 1)}_{2n}, \quad \overline{b_n} = \underbrace{(2, 3, 2, 3 \dots, 2, 3)}_{2n}, \quad \overline{c_n} = \underbrace{(3, 2, 3, 2 \dots, 3, 2)}_{2n}.$$

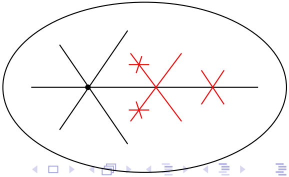
- Take two multicritical commuting pairs ζ^1 and ζ^2 with the same rotation number

$$\rho = [5, 5, 5, 5, 5, \dots]$$

and two non-zero critical points:



$$\omega(c_1^1) = (4, \overline{a_1}, \overline{b_1}, \dots, 4, \overline{a_n}, \overline{b_n}, \dots),$$
$$\omega(c_2^1) = (5, \overline{a_1}, \overline{c_1}, \dots, 5, \overline{a_n}, \overline{c_n}, \dots),$$



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Obtain two non-conjugate limiting towers $\mathcal{T}_1$ and $\mathcal{T}_2$ with the same combinatorics

$$\rho = (\dots, 5, 5, 5, \dots), \quad \omega(c_1) = (a_\infty, \underline{1}, b_\infty), \quad \omega(c_2) = (a_\infty, \underline{1}, c_\infty).$$