

DYNAMICS ON THE BOUNDARY OF INVARIANT FATOU COMPONENTS

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On geometric complexity of Julia sets IV
August 19, 2022

INTRODUCTION

- $f: \mathbb{C} \rightarrow \mathbb{C}$ transcendental entire function (TEF)
- $\mathcal{F}(f)$, **Fatou set**; $\mathcal{J}(f)$, **Julia set**
- $\mathcal{K}(f)$, **bounded-orbit set**; $\mathcal{BU}(f)$, **bungee set**; $\mathcal{I}(f)$, **escaping set**
- U invariant **Fatou component**

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$$f: U \rightarrow U$$

- $U \subset \mathcal{F}(f)$, stable dynamics
- Well-understood: classification of Fatou components, normal forms...

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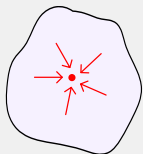
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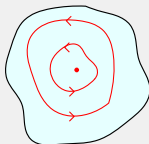
Goal: Understand the dynamics of $f: \partial U \rightarrow \partial U$, specially when U is unbounded

DYNAMICS OF $f: U \rightarrow U$

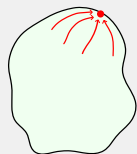
CLASSIFICATION OF INVARIANT FATOU COMPONENTS



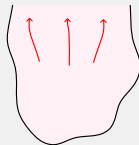
1. $f|_U^n \rightarrow z_0 \in U$
Attracting basin
 $|f'(z_0)| < 1$



3. $f|_U \sim e^{2\pi i \theta} z$, $\theta \notin \mathbb{Q}$
Siegel disk



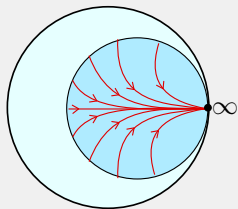
2. $f|_U^n \rightarrow z_0 \in \partial U$
Parabolic basin
 $f'(z_0) = 1$



4. f transcendental,
 $f|_U^n \rightarrow \infty$
Baker domain

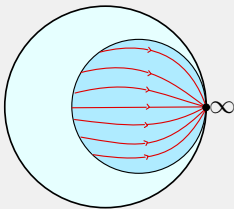
DYNAMICS OF $f: U \rightarrow U$

CLASSIFICATION OF BAKER DOMAINS



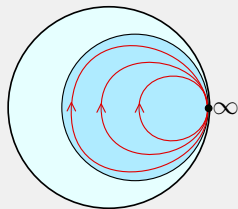
1. Doubly parabolic

$$f \sim \text{id}_{\mathbb{C}} + 1$$



2. Hyperbolic

$$f \sim \lambda \text{id}_{\mathbb{H}}, \lambda > 0$$



3. Simply parabolic

$$f \sim \text{id}_{\mathbb{H}} \pm 1$$

DYNAMICS OF $f: \partial U \rightarrow \partial U$

- $\partial U \subset \mathcal{J}(f)$
- Periodic points are dense in $\mathcal{J}(f)$
- $\partial \mathcal{I}(f) = \mathcal{J}(f)$, and (TEF) $\mathcal{I}(f) \cap \mathcal{J}(f) \neq \emptyset$

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Question: Are periodic/escaping boundary points dense in ∂U ?

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- U bounded Fatou component $\Rightarrow \partial U \subset \mathcal{K}(f)$



Image by W. Bergweiler, N. Fagella, L. Rempe

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- U Siegel D, ∂U Jordan curve \Rightarrow no periodic boundary points

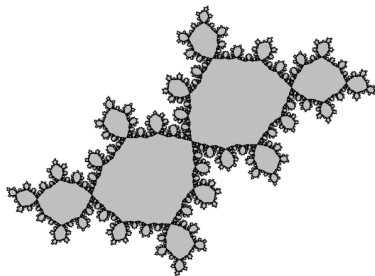


Image by A. Chéritat

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- For some hyperbolic/simply parabolic BD, ∂U is a Jordan curve, $\partial U \subset \mathcal{I}(f)$

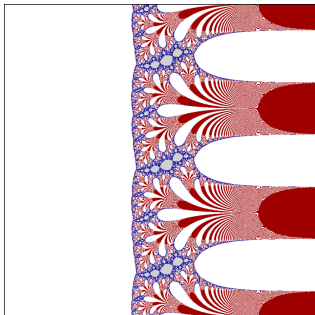


Image by N. Fagella, C. Henriksen

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POSITIVE ANSWERS:

- Existence of periodic boundary points on basins of rational maps.¹ Also for bounded basins of TEF \rightsquigarrow The proof does not work if $\infty \in \partial U$

¹Fatou. *Sur les équations fonctionnelles*

²Przytycki, Zdunik. *Density of periodic sources in the boundary of a basin of attraction for iteration of holomorphic maps: geometric coding trees technique*

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- Density of periodic boundary points for basins of rational maps² \rightsquigarrow For TEF???

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DYNAMICS OF $f: \partial U \rightarrow \partial U$

TYPICAL BEHAVIOUR OF BOUNDARY ORBITS:

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⁴Barański, Fagella, Jarque, Karpińska. *Escaping points in the boundaries of Baker domains*

⁵Doering, Mañé. *The dynamics of inner functions*

DYNAMICS OF $f: \partial U \rightarrow \partial U$

TYPICAL BEHAVIOUR OF BOUNDARY ORBITS:

1.- Some hyperbolic/simply parabolic BD $\implies \omega_U(\partial U \cap \mathcal{I}(f)) = 1$ ^{3,4}

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If U unbounded, $\omega_U(\partial U \cap \mathcal{BU}(f)) = 1$

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Question: U unbounded, type 2. Are there periodic/escaping boundary points in ∂U ? Are they dense in ∂U ?

(sets of zero ω_U -measure)

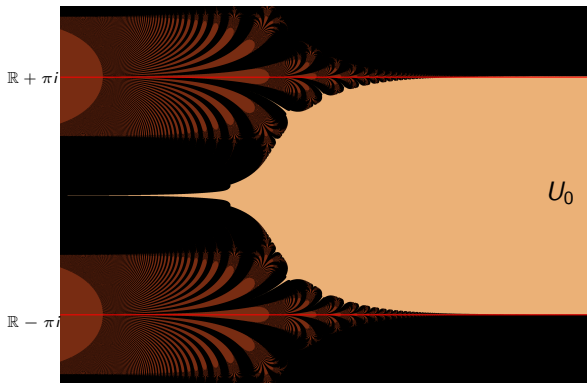
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FIRST STEP: studying the model $f(z) = z + e^{-z}$

- Doubly parabolic BD of deg. 2, $U_k \subset S_k := \{(2k - 1)\pi \leq \operatorname{Im} z \leq (2k + 1)\pi\}$

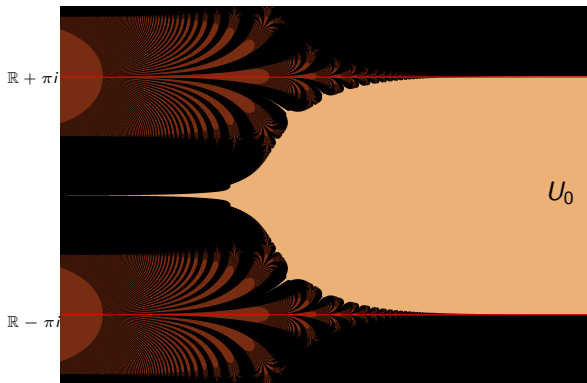


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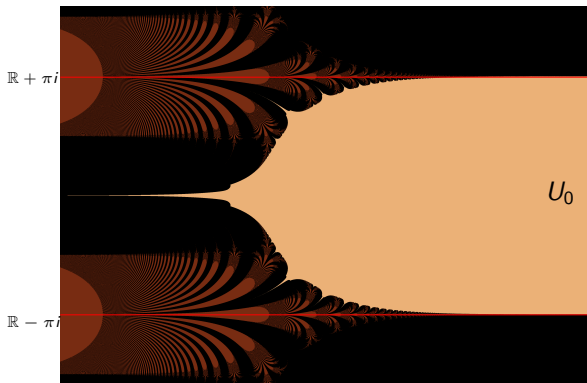
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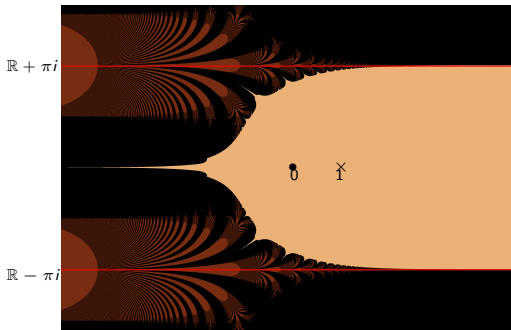
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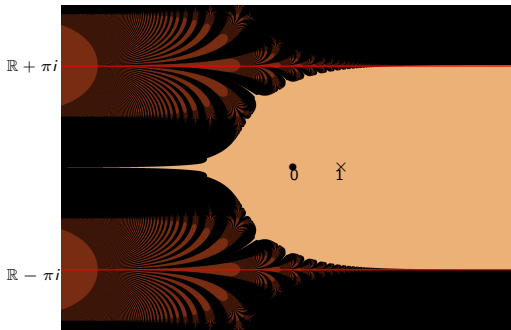
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- 3.- $f|_{\partial U}$ **recurrent** \Rightarrow existence of boundary points with dense orbit in ∂U

GENERALIZATION

MAIN RESULT

f transcendental entire function, U invariant Fatou component s.t.

- 1.- ∞ is accessible from U ,
- 2.- $\exists \Omega$ simply connected domain s.t. $\overline{U} \subset \Omega$ and $\overline{P(f)} \cap \Omega \subset U$,
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$$Cl_{\mathbb{C}}(\varphi, e^{i\theta}) = \{z \in \mathbb{C}: \exists \{w_n\}_n \subset \mathbb{D}, w_n \rightarrow e^{i\theta}, \varphi(w_n) \rightarrow z\}$$

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THEOREM: f TEF, U invariant FC (not hyp./simply p. BD), ∞ accessible.

- $\infty \in \overline{Cl_{\mathbb{C}}(\varphi, e^{i\theta})}$, for all $e^{i\theta} \in \partial \mathbb{D}$ ⁸
- $\overline{\{e^{i\theta} : \varphi^*(e^{i\theta}) = \infty\}} = \partial \mathbb{D}$ ⁹

⁸Baker, Weinreich. *Boundaries which arise in the dynamics of entire functions*

⁹Baker, Domínguez. *Boundaries of unbounded Fatou components of entire functions*

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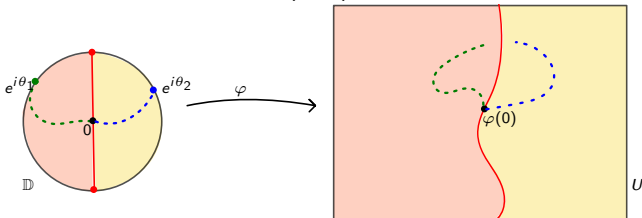
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STRUCTURE ON THE BOUNDARY:

$$\partial U = \bigsqcup_{e^{i\theta} \in \partial \mathbb{D}} Cl_{\mathbb{C}}(\varphi, e^{i\theta})$$

\rightsquigarrow each cluster set is either an unbounded continuum or the union of two unbounded continua

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DEFINITION: Let U be an invariant Fatou component. For $z_0 \in \partial U$, f is *locally surjective* if, for any suf. small nbh N of z_0 , $f(N \cap U) = f(N) \cap U$.

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$\exists \Omega$ simply connected domain s.t. $\overline{U} \subset \Omega$, $\overline{P(f)} \cap \Omega \subset U \Rightarrow$ local surjectivity _{11/14}

3.- Periodic boundary points are dense in ∂U

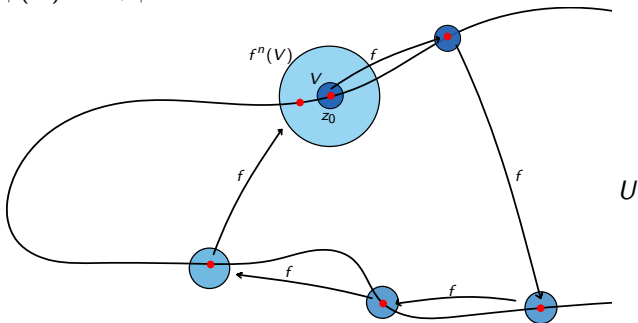
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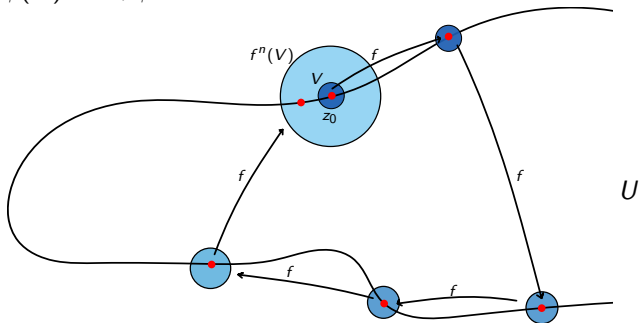
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- **Local expansion+inverses well-defined:** a nbh V of z_0 satisfies $V \subset f^n(V)$, for some n , i.e. $\phi(V) \subset V$, ϕ branch of f^{-n}



3.- Periodic boundary points are dense in ∂U

- $f|_{\partial U}$ recurrent \Rightarrow existence of boundary points with dense orbit in ∂U
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- **Schwarz lemma+local surjectivity:** \exists fixed point for ϕ in ∂U (periodic pt for f)

4.- Approximating periodic points by **escaping** points

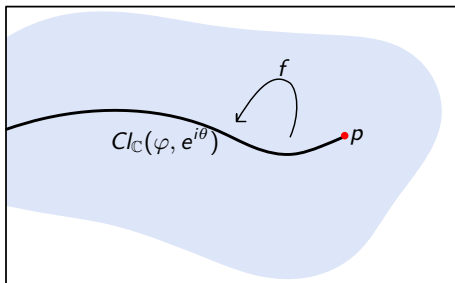
LEMMA: Let $p \in Cl_{\mathbb{C}}(\varphi, e^{i\theta}) \subset \partial U$, p periodic. If $z \in Cl_{\mathbb{C}}(\varphi, e^{i\theta}) \setminus \{p\}$, then z is escaping.

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Idea of the proof: Assume p is fixed.

$f(Cl_{\mathbb{C}}(\varphi, e^{i\theta})) \subset Cl_{\mathbb{C}}(\varphi, e^{i\theta}) + f$ is locally expanding

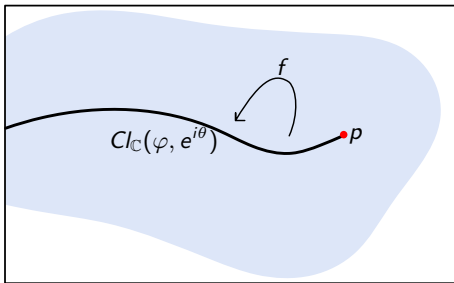


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Consequence: Escaping boundary points are dense in ∂U

Thank you for your attention!!!