

# *A Priori Bounds* and Degeneration of Herman Rings

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## Rotation domains

Let  $f \in \text{Rat}_d$ . A maximal invariant domain  $U \subset \hat{\mathbb{C}}$  is a **rotation domain** if  $f|_U$  is conjugate to a rigid rotation. There are 2 types:

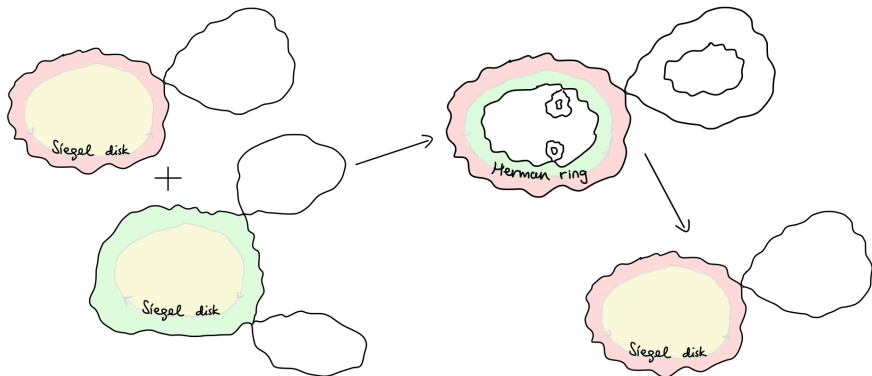
- 1  $U$  is simply connected, i.e. a **Siegel disk**;
- 2  $U$  is an annulus, i.e. a **Herman ring**.

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- 1  $U$  is simply connected, i.e. a **Siegel disk**;
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The two can be converted into one another via quasiconformal surgery. (Shishikura '87)



## Bounded type rotation domains

Assume from now on that  $\alpha \in (0;1)$  is an irrational number of **bounded type**,

i.e. there is some  $B \in \mathbb{N}$  such that  $\sup_n a_n \leq B$  where  $\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$ .

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### Theorem (Zhang '11)

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Applying Shishikura's surgery, we have:

### Corollary

*Every boundary component of an invariant Herman ring of a map  $f \in \text{Rat}_d$  with rotation number  $\alpha$  and modulus  $M$  is a  $K(d; B; M)$ -quasicircle containing a critical point.*

$H_{d_0; d_1}$  ;

Denote by  $H_{d_0; d_1}$  ; the space of all maps  $f \in \text{Rat}_{d_0+d_1-1}$  such that

- (I) the only non-repelling periodic points are superattracting fixed points 0 and 1 of criticalities  $d_0 - 2$  and  $d_1 - 2$  respectively;
- (II)  $f$  has an invariant Herman ring  $H$  of rotation number  $\theta$  ;
- (III)  $H$  separates 0 and 1 ;
- (IV) every critical point of  $f$  other than 0 and 1 lies on  $\partial H$ .

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### Proposition

$H_{d_0; d_1}$  consists of all rational maps that can be obtained from Shishikura surgery out of a pair of polynomials  $P_0, P_1$  such that for  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ ;

- $\deg(P_\theta) = d_\theta$ ;
- $P_\theta$  has an invariant Siegel disk  $Z_\theta$ ;
- $\text{rot}(Z_0) = \theta$  and  $\text{rot}(Z_1) = 1 - \theta$ ;
- all free critical points of  $P_\theta$  lie in  $\partial Z_\theta$ .



It turns out that for  $H_{d_0; d_1}$ , we can remove the dependence on the modulus.

### Theorem (WRL)

*The boundary components of the Herman ring of every rational map in  $H_{d_0; d_1}$  are  $K(d_0; d_1; B)$ -quasicircles.*

## Definition

A **Herman curve**  $\mathbf{H}$  of a rational map  $f$  is a forward invariant Jordan curve where

- 1  $f|_{\mathbf{H}}$  is conjugate to an irrational rotation, and
- 2  $\mathbf{H}$  is not contained in the closure of any rotation domain.

Additionally,  $\mathbf{H}$  is called a **Herman quasicircle** if it is a quasicircle.

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Trivial example:

There is a unique  $\alpha \in \mathbb{T}$  such that the unit circle  $\mathbb{T}$  is a Herman curve of rotation number  $\alpha$  for the map

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Question: Can non-trivial Herman curves exist?

# Degeneration of Herman rings

Consider the limit space

$$H_{d_0; d_1}^{\text{at}} := \overline{H_{d_0; d_1} \cap nH_{d_0; d_1}}; \quad \text{Rat}_{d_0+d_1-1}$$

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Every  $f \in H_{d_0; d_1}^{\otimes}$  has a Herman quasicircle of rotation number  $\theta$ .

0

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Herman ring of  $f \in H_{4;3}$

Herman quasicircle of  $f \in H_{4;3}^{\otimes}$

Let  $f \in H_{d_0; d_1}^{\otimes}$ . Endow its Herman quasicircle  $\mathbf{H}$  with the *combinatorial metric*, i.e. the pullback of the normalized Euclidean metric under the linearization of  $f$ .

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### Theorem (WRL)

*Given any prescribed combinatorics, there exists a rational map in  $H_{d_0; d_1}^{\otimes}$  that realizes such combinatorics.*

# Existence

Let  $f \in H_{d_0; d_1}^{\otimes}$ . Endow its Herman quasidisk  $\mathbf{H}$  with the *combinatorial metric*, i.e. the pullback of the normalized Euclidean metric under the linearization of  $f$ .

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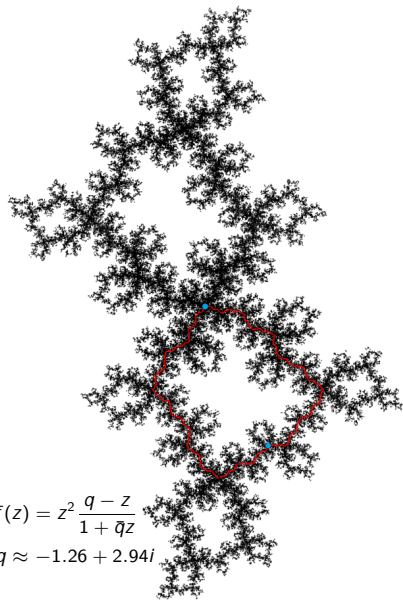
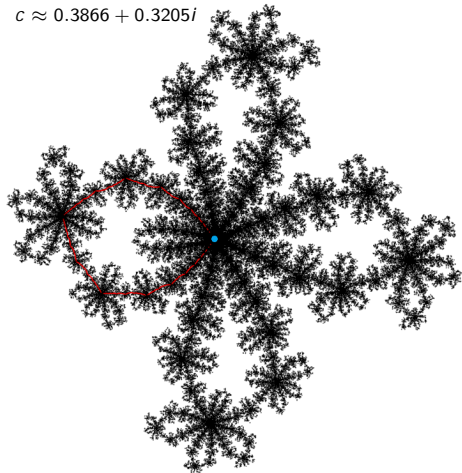
Sketch of proof:

|    |  |   |  |
|----|--|---|--|
| 1. | Thurston-type result for Herman rings (Wang '12) | ) | $\exists f_1 \in H_{d_0; d_1}^{\otimes}$ having a Herman ring with combinatorics similar to the chosen one;                    |
| 2. | QC deformation                                   | ) | $\exists$ a normalized family $\{f_t\}_{0 < t < 1}$ in $H_{d_0; d_1}^{\otimes}$ with the same combinatorics but mod $\epsilon$ |
| 3. | <i>a priori</i> bounds                           | ) | $\exists f \in H_{d_0; d_1}^{\otimes}$ such that $f_t \rightarrow f$ subsequentially as $t \rightarrow 0$ .                    |

# Non-trivial examples of golden mean Herman curves

$$f(z) = cz^2 \frac{z^3 - 5z^2 + 10z - 10}{5z - 1},$$

$$c \approx 0.3866 + 0.3205i$$



$$f(z) = z^2 \frac{q - z}{1 + \bar{q}z}$$

$$q \approx -1.26 + 2.94i$$

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To prove *a priori* bounds, it is sufficient to find some  $K = K(d_0; d_1 ; B) > 0$  such that every interval  $I \subset H$  satisfies  $W_{10}(I) < K$ .

## Amplification

Our goal is reduced to showing:

### Theorem

*There is some  $K > 0$  and  $0 < \epsilon < 1$  depending only on  $d_0; d_1; B$  such that if*

*there is an interval  $I \subset H$  with length  $|I| < \epsilon$  and width  $W_{10}(I) \leq K$ ,*

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The proof of such uses the near-degenerate regime<sup>1</sup>. Many of our steps are inspired by Kahn-Lyubich '05, Kahn '06, and D. Dudko-Lyubich '22.

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Main challenges:

- Both sides of  $H$  are dynamically nontrivial (unlike the boundary of Siegel disks);
- Lack of positive entropy (unlike primitively renormalizable quadratic maps);
- Intervals in  $H$  are not perfectly invariant (unlike little Julia sets in PL renorm.);
- Arbitrary number of critical points and combinatorics.

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- 4 Is every Herman curve a limit of degenerating Herman rings?
  - ) For  $H_{d_0; d_1}^{\otimes}$ , this follows from combinatorial rigidity. (in progress)



Thank you!