A Priori Bounds and Degeneration of Herman Rings

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Rotation domains

Let $f \in \mathrm{Rat}_d$. A maximal invariant domain $U \subset \hat{\mathbb{C}}$ is a rotation domain if $f|_U$ is conjugate to a rigid rotation. There are 2 types:

1. U is simply connected, i.e. a Siegel disk;
2. U is an annulus, i.e. a Herman ring.

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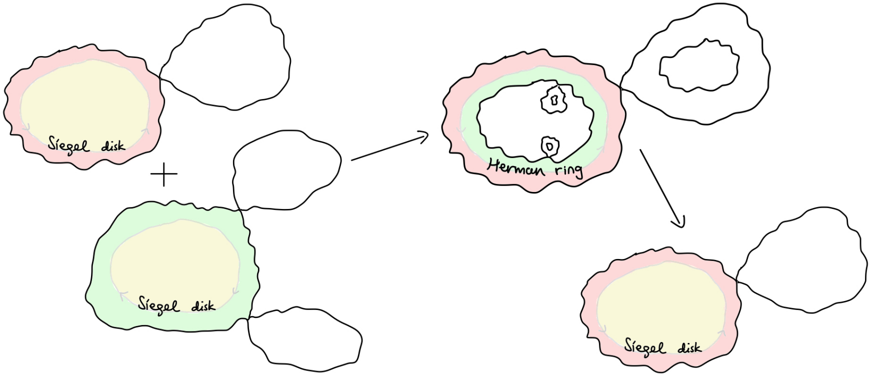
2. U is an annulus, i.e. a Herman ring.

The two can be converted into one another via quasiconformal surgery. (Shishikura '87)



Bounded type rotation domains

Assume from now on that $\theta \in (0,1)$ is an irrational number of bounded type, i.e. there is some $B \in \mathbb{N}$ such that $\sup_n a_n \leq B$ where $\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$.

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Applying Shishikura's surgery, we have:

Corollary

Every boundary component of an invariant Herman ring of a map $f \in Rat_d$ with rotation number θ and modulus μ is a $K(d,B,\mu)$-quasicircle containing a critical point.

$\mathcal{H}_{d_0,d_\infty,\theta}$

Denote by $\mathcal{H}_{d_0,d_\infty,\theta}$ the space of all maps $f \in \mathrm{Rat}_{d_0+d_\infty-1}$ such that

(I) the only non-repelling periodic points are superattracting fixed points 0 and ∞ of criticalities $d_0 \geq 2$ and $d_\infty \geq 2$ respectively;

(II) f has an invariant Herman ring $\mathbb{H}$ of rotation number θ;

(III) $\mathbb{H}$ separates 0 and ∞;

(IV) every critical point of f other than 0 and ∞ lies on $\partial\mathbb{H}$.

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Proposition

$\mathcal{H}_{d_0,d_\infty,\theta}$ consists of all rational maps that can be obtained from Shishikura surgery out of a pair of polynomials P_0, P_∞ such that for $\star \in \{0, \infty\}$,

- *$deg(P_\star) = d_\star$;*
- *$P_\star$ has an invariant Siegel disk $Z_\star$;*
- *$rot(Z_0) = \theta$ and $rot(Z_\infty) = 1 - \theta$;*
- *all free critical points of $P_\star$ lie in $\partial Z_\star$.*

A priori bounds

It turns out that for $\mathcal{H}_{d_0,d_\infty,\theta}$, we can remove the dependence on the modulus μ.

Theorem (WRL)

The boundary components of the Herman ring of every rational map in $\mathcal{H}_{d_0,d_\infty,\theta}$ *are* $K(d_0, d_\infty, B)$*-quasicircles.*

Herman curves

Definition

A Herman curve $\mathbf{H}$ of a rational map f is a forward invariant Jordan curve where

1. $f|_{\mathbf{H}}$ is conjugate to an irrational rotation, and
2. $\mathbf{H}$ is not contained in the closure of any rotation domain.

Additionally, $\mathbf{H}$ is called a Herman quasicircle if it is a quasicircle.

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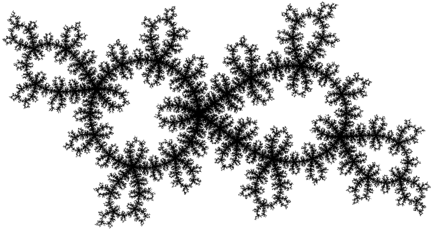
Trivial example:
There is a unique $\zeta_\theta \in \mathbb{T}$ such that the unit circle $\mathbb{T}$ is a Herman curve of rotation number θ for the map

$$f_\theta(z) = \zeta_\theta z^2 \frac{z-3}{1-3z}.$$



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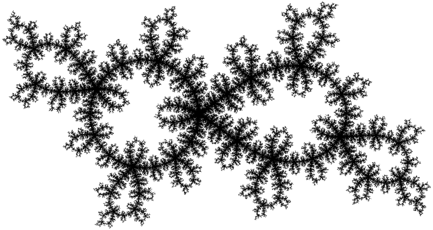

Question: Can non-trivial Herman curves exist?

Degeneration of Herman rings

Consider the limit space

$$\mathcal{H}^{\partial}_{d_0,d_\infty,\theta} := \overline{\mathcal{H}_{d_0,d_\infty,\theta}} \setminus \mathcal{H}_{d_0,d_\infty,\theta} \subset \mathrm{Rat}_{d_0+d_\infty-1}.$$

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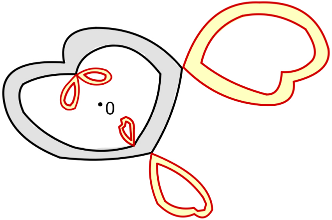
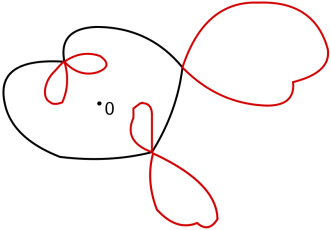


Herman ring of $f \in \mathcal{H}_{4,3,\theta}$



Herman quasicircle of $f \in \mathcal{H}^{\partial}_{4,3,\theta}$

Existence

Let $f \in \mathcal{H}^{\partial}_{d_0,d_\infty,\theta}$. Endow its Herman quasicircle $\mathbf{H}$ with the *combinatorial metric*, i.e. the pullback of the normalized Euclidean metric under the linearization of f.

The combinatorics of $f \in \mathcal{H}^{\partial}_{d_0,d_\infty,\theta}$ is determined by the criticality and the relative combinatorial position of its free critical points along $\mathbf{H}$.

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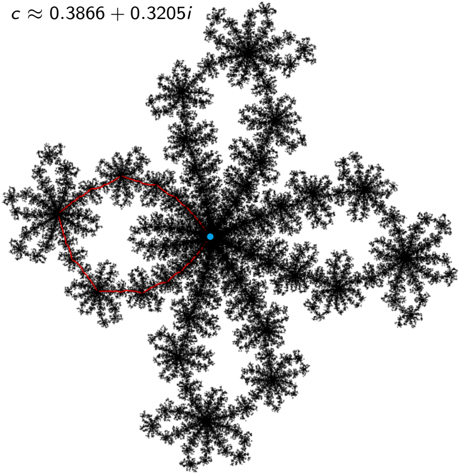
Sketch of proof:

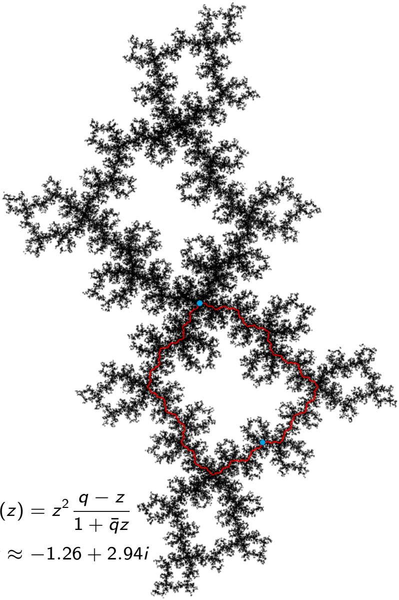
1.	Thurston-type result for Herman rings (Wang '12)	$\Rightarrow$	$\exists\ f_1 \in \mathcal{H}_{d_0,d_\infty,\theta}$ having a Herman ring with combinatorics similar to the chosen one;
2.	QC deformation	$\Rightarrow$	$\exists$ a normalized family $\{f_t\}_{0<t\leq 1}$ in $\mathcal{H}_{d_0,d_\infty,\theta}$ with the same combinatorics but mod$\to 0$
3.	*a priori bounds*	$\Rightarrow$	$\exists\ f \in \mathcal{H}^{\partial}_{d_0,d_\infty,\theta}$ such that $f_t \to f$ subsequentially as $t \to 0$.

Non-trivial examples of golden mean Herman curves

$$f(z) = cz^2 \, \frac{z^3 - 5z^2 + 10z - 10}{5z - 1},$$

$c \approx 0.3866 + 0.3205i$





$$f(z) = z^2 \frac{q - z}{1 + \bar{q}z}$$

$q \approx -1.26 + 2.94i$

How to prove *a priori bounds*?

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I = an interval in H of (combinatorial) length $|I| < 0.1$.
$10I$ = the interval of length $10|I|$ having the same midpoint as I.
$W_{10}(I)$ = the extremal width of curves connecting I and $H \setminus 10I$.

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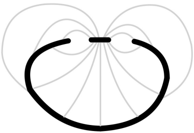

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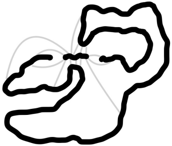

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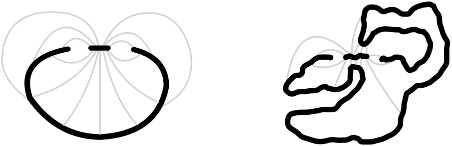

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To prove *a priori bounds*, it is sufficient to find some $K = K(d_0, d_\infty, B) > 0$ such that every interval $I \subset H$ satisfies $W_{10}(I) < K$.

Amplification

Our goal is reduced to showing:

Theorem

There is some $K > 0$ and $0 < \epsilon < 1$ depending only on d_0, d_∞, B such that if

$$\text{there is an interval } I \subset H \text{ with length } |I| < \epsilon \text{ and width } W_{10}(I) \geq K,$$

then

$$\text{there is another interval } J \subset H \text{ with length } |J| < \epsilon \text{ and width } W_{10}(J) \geq 2\, W_{10}(I).$$

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The proof of such uses the near-degenerate regime[1]. Many of our steps are inspired by Kahn-Lyubich '05, Kahn '06, and D. Dudko-Lyubich '22.

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Main challenges:

- Both sides of H are dynamically nontrivial (unlike the boundary of Siegel disks);
- Lack of positive entropy (unlike primitively renormalizable quadratic maps);
- Intervals in H are not perfectly invariant (unlike little Julia sets in PL renorm.);
- Arbitrary number of critical points and combinatorics.

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$\Rightarrow$ For $\mathcal{H}^{\partial}_{d_0,d_\infty,\theta}$, this follows from combinatorial rigidity. (in progress)

Thank you!