Generating holomorphic functions with critical orbit relation

K. Mamayusupov

National University of Uzbekistan,
Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences

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Introduction

Denote $f^{\circ n}(z)$ n-th iterate of a map f, i.e. $f^{\circ 0}(z) = z$, $f^{\circ 1}(z) = f(z)$, $f^{\circ 2}(z) = f(f(z))$, etc.

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Let $\mathrm{Per}_1(\lambda)$ be the set of conformal conjugacy classes of maps, in the moduli space $\mathcal{M}_2$ of quadratic rational maps, with a fixed point of multiplier $\lambda \in \mathbb{C}$. For $\lambda = 0$, $\mathrm{Per}_1(0) = \{c \in \mathbb{C} : z^2 + c\}$.

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In $\mathcal{M}_d$ for any $d \geq 2$ postcritically finite maps form a Zariski dense subset. Some subvarieties intersecting $\mathcal{M}_d$ are special.

Consider $f_t(z) = \lambda z/(z^2 + tz + 1)$ with $t \in \mathbb{C}$ for each $\lambda \neq 0 \in \mathbb{C}$, with marked critical points at ± 1. Denote the space by $\mathrm{Per}_1(\lambda)^{cm}$ which is a double cover of $\mathrm{Per}_1(\lambda)$.

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Our main theorem is the following.

Theorem

For each $\lambda \neq 0$ which is not a root of unity, in the family $f_t(z) = \lambda z/(z^2 + tz + 1)$ all critical orbit relations are realized except $(0, 0)$ and $(n, 1)$ for each $n \geq 1$.

Let $f_t(z)$ for $t \in \mathcal{X}$ be a holomorphic family of rational functions of degree at least 2.
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Definition (Critical orbit relation)

A **critical orbit relation** is a triple (n, m, t) with non-negative integers n and m such that for the critical points $c_1(t)$ and $c_2(t)$ we have

$$f_t^{\circ n}(c_1(t)) = f_t^{\circ m}(c_2(t)).$$

Let us mark critical points $c_1(t), c_2(t), \dots, c_{2d-2}(t)$ (pass to a branched cover).
A point $t = t_0$ belongs to stability locus if the Julia sets $J(f_t)$ move holomorphically in a neighborhood of t_0.
Alternatively, a point $t = t_0$ belongs to stability locus if the sequence

$$\{t \mapsto f_t^{\circ n}(c_i(t))\}$$

forms a normal family for each i on some neighborhood of t_0.
A point $t = t_0$ belongs to the **bifurcation locus** if the stability fails at t_0.

Setup

Assume the bifurcation locus is not empty and $\#\{\textit{orbit of } c_j\} \geq 3$ persists in $\mathcal{X}$ and c_i is active for $i \neq j$.

Lemma

Then there are infinitely many parameters $t \in \mathcal{X}$ such that $c_i(t)$ and $c_j(t)$ have critical orbit relations.

Proof.

Montel's theorem. □

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In fact there are infinitely many parameters $(n, 0, t)$ such that $f_t^{\circ n}(c_i(t)) = c_j(t)$.

Proof. Consider two preimages $c_j^0 \neq c_j^1$ of c_j and apply Montel's theorem with with the triple c_j^0, c_j^1, c_j which is persistent.

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3. $a_d z^d + a_{d-1} z^{d-1} + \ldots + a_1 z + a_0,\ a_k \in \mathbb{C},\ d \geq 3$.

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4. $f(z) = \frac{p(z)}{q(z)}$ rational functions.
5. $f_a(z) = z^2 \dfrac{z + a - 1}{(a+1)z - 1}.$

Cubic polynomials, $z^3 - 3a^2z + b$

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• Two distinct cubics $z^3 - 3a^2z + b$ and $z^3 - 3a'^2z + b'$ are affine conjugate if and only if $a' = -a$ and $b' = -b$, the conjugacy is $z \mapsto -z$. This conjugacy interchanges the markings of critical points $\pm a$.

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Thus the moduli space, consisting of all affine conjugacy classes of cubics with marked critical point, can be identified with coordinates $(a^2, b^2) \in \mathbb{C}^2$.

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$$\left(p^{\circ n}(a) - p^{\circ m}(-a)\right)\left(p^{\circ n}(-a) - p^{\circ m}(a)\right) = 0.$$

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- $p^{\circ(n-i)}(a) \neq p^{\circ(m-i)}(-a)$ for all $1 \leq i \leq \min\{n, m\}$.

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• Every critical orbit relation of the form $(n, 0)$ is minimal.

• As the critical orbit relation is symmetric with respect to n and m, it suffices to consider only the cases of $n \geq m$.

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• The problem maybe reduced to computing the resultant of two polynomials $p^{\circ n}(z) - p^{\circ m}(-z)$ and $z^2 - a^2$. The resultant is a polynomial on the parameters a, b.

• Equivalently, one can also find the Gröbner basis of $\{p^{\circ n}(z) - p^{\circ m}(-z), z^2 - a^2\}$.

The main idea

Lemma (Key-Lemma)

There exist sequences $\{A_n(a,b)\}_{n\geq 0}$ *and* $\{B_n(a,b)\}_{n\geq 0}$ *of polynomials of parameters* a, b *such that if* z *is a critical point of* $p(z)$ *then for all* $n \geq 0$ *the relation* $p^{\circ n}(z) = A_n(a,b)z + B_n(a,b)$ *holds.*

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Proof.

As $p^{\circ 0}(z) = z$, set $A_0(a,b) = 1$ and $B_0(a,b) = 0$.

- Recurrently define polynomials $A_n(a,b)$ and $B_n(a,b)$ with

$$A_{n+1}(a,b) = A_n(a,b)(a^2A_n^2(a,b) + 3B_n^2(a,b) - 3a^2),$$
$$B_{n+1}(a,b) = B_n^3(a,b) + 3a^2B_n(a,b)(A_n^2(a,b) - 1) + b,$$

such that $A_{n+1}(a,b)z + B_{n+1}(a,b) = p\big(A_n(a,b)z + B_n(a,b)\big)$.

The above formulas are obtained by substituting $z^2 = a^2$, $z^3 = a^2 z$ into the expansion of $(A_n(a,b)z + B_n(a,b))^3 - 3a^2(A_n(a,b)z + B_n(a,b)) + b$ and combining common terms. $\square$

It is easy to see from the recurrence relations that $\deg_a A_n(a,b) = \deg A_n(a,b) = 3^n - 1$ for $n \geq 1$, $\deg_a B_n(a,b) = 3^n - 3$ and $\deg B_n(a,b) = 3^n - 2$ for $n \geq 1$.

Lemma

There exist sequences $\{\tilde{A}_n(x,y)\}_{n\geq 0}$ *and* $\{\tilde{B}_n(x,y)\}_{n\geq 0}$ *of polynomials such that for every* $n \geq 0$ *one has* $A_n(a,b) = \tilde{A}_n(a^2,b^2)$ *and* $B_n(a,b) = b\tilde{B}_n(a^2,b^2)$.

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Our main theorem is the following.

Theorem

Except $(1,1)$ *all critical orbit relations are realized. In particular, there are infinitely many cubic polynomials with critical orbit relations.*

The proof is split into three separate cases.

Case of (n, n)

Proof. By the Key-Lemma we have $p^{\circ n}(z) - p^{\circ n}(-z) = A_n(a, b)z + B_n(a, b) - (-A_n(a, b)z + B_n(a, b)) = 2A_n(a, b)z$ for $n \geq 1$. It implies that the critical orbit relation reduces to $A_n(a, b) = 0$.
As $A_1 = -2a^2$, it vanishes if $a = 0$. In this case both critical points collide so the critical orbit relation is $(0, 0)$. This means that there is no cubic polynomial with an exact critical orbit relation $(1, 1)$.

Case of (n, n)

Set $P_{n,n}(a,b) = A_n(a,b)/A_1(a,b)$. We have

$$P_{n,n}(a,b) = A_{n-1}(a,b)/A_1(a,b)(a^2 A_{n-1}^2(a,b) + 3B_{n-1}^2(a,b) - 3a^2).$$

Set

$$\tilde{P}_{n,n}(a,b) = a^2 A_{n-1}^2(a,b) + 3B_{n-1}^2(a,b) - 3a^2,$$

or we can write

$$\tilde{P}_{n,n}(a,b) = a^2 \tilde{A}_{n-1}^2(a^2,b^2) + 3b^2 \tilde{B}_{n-1}^2(a^2,b^2) - 3a^2.$$

This implies that for $n \geq 2$ we can write

$$P_{n,n}(a,b) = P_{n-1,n-1}(a,b) \cdot \tilde{P}_{n,n}(a,b).$$

Case of (n, n)

Proposition

For $n \geq 1$ set

$$Q_{n,n}(x,y) = x\tilde{A}^2_{n-1}(x,y) + 3y\tilde{B}^2_{n-1}(x,y) - 3x$$

then $\tilde{P}_{n,n}(x,y) = Q_{n,n}(x^2,y^2)$. Moreover,
$\deg_a P_{n,n}(a,b) = \deg P_{n,n}(a,b) = 3^n - 3$ *and*
$\deg_a \tilde{P}_{n,n}(a,b) = \deg \tilde{P}_{n,n}(a,b) = 2 \cdot 3^{n-1}$ *for $n \geq 1$.*

Case of $n > m$ for $m = 0$ and $m = 1$

For $z = \pm a$ we have that $p^{\circ n}(z) - p^{\circ m}(-z) = A_n(a,b)z + B_n(a,b) - (-A_m(a,b)z + B_m(a,b)) = (A_n(a,b) + A_m(a,b))z + B_n(a,b) - B_m(a,b)$. Solving the critical orbit relation for z (equating the latter to zero) we obtain

$$z = \frac{B_m(a,b) - B_n(a,b)}{A_n(a,b) + A_m(a,b)}.$$

Since the obtained z is a critical point, it satisfies the equation $z^2 - a^2 = 0$. Set
$P_{n,m}(a,b) = a^2(A_n(a,b) + A_m(a,b))^2 - (B_n(a,b) - B_m(a,b))^2$.

Recall that $A_0 = 1$, $B_0 = 0$ and $A_1 = -2a^2$, $B_1 = b$.
For $n \geq 1$ we have that $P_{n,0} = a^2(A_n(a,b)+1)^2 - B_n^2(a,b)$. Set

$$\tilde{P}_{n,0}(a,b) = P_{n,0}(a,b) = a^2(\tilde{A}_n(a^2,b^2)+1)^2 - b^2\tilde{B}_n^2(a^2,b^2).$$

Note that the critical orbit relation $(n,0)$ is exact (minimal). An easy calculation shows that

$$P_{n,1} = \left(a^2(A_{n-1}+1)^2 - B_{n-1}^2\right)^2 \cdot \left(a^2(A_{n-1}-2)^2 - B_{n-1}^2\right).$$

For $n \geq 1$ set

$$\tilde{P}_{n,1}(a,b) = a^2(A_{n-1}(a,b)-2)^2 - B_{n-1}^2(a,b),$$

or we can write it as

$$\tilde{P}_{n,1}(a,b) = a^2(\tilde{A}_{n-1}(a^2,b^2)-2)^2 - b^2\tilde{B}_{n-1}^2(a^2,b^2)$$

then the above implies that

$$P_{n,1} = P_{n-1,0}^2 \cdot \tilde{P}_{n,1}.$$

Proposition

For $n \geq 1$ set

$$\begin{aligned} Q_{n,0}(x,y) =& x(\tilde{A}_n(x,y)+1)^2 - y\tilde{B}_n^2(x,y), \\ Q_{n,1}(x,y) =& x(\tilde{A}_{n-1}(x,y)-2)^2 - y\tilde{B}_{n-1}^2(x,y) \end{aligned}$$

then $\tilde{P}_{n,0}(x,y) = Q_{n,0}(x^2,y^2)$ and $\tilde{P}_{n,1}(x,y) = Q_{n,1}(x^2,y^2)$. Moreover, $\deg_a \tilde{P}_{n,0}(a,b) = \deg \tilde{P}_{n,0}(a,b) = 2 \cdot 3^n$ and $\deg_a \tilde{P}_{n,1} = \deg \tilde{P}_{n,1} = 2 \cdot 3^{n-1}$ for $n \geq 1$.

Case of $n > m \geq 2$

Set

$$
\begin{aligned}
\tilde{P}_{n,m}(a.b) = &\Big(a^2(A_{n-1}^2 - A_{n-1}A_{m-1} + A_{m-1}^2) \\
&+ B_{n-1}^2 + B_{n-1}B_{m-1} + B_{m-1}^2 - 3a^2\Big)^2 \\
&- a^2\Big((2A_{n-1} - A_{m-1})B_{n-1} + (A_{n-1} - 2A_{m-1})B_{m-1}\Big)^2,
\end{aligned}
$$

then we have that

$$P_{n,m}(a,b) = P_{n-1,m-1}(a,b) \cdot \tilde{P}_{n,m}(a,b).$$

Proposition

Let $n > m \geq 2$ and set

$$Q_{n,m}(x,y) = \Big(x(\tilde{A}^2_{n-1}(x,y) - \tilde{A}_{n-1}(x,y)\tilde{A}_{m-1}(x,y) + \tilde{A}^2_{m-1}(x,y)) + y\tilde{B}^2_{n-1}(x,y) + y\tilde{B}_{n-1}(x,y)\tilde{B}_{m-1}(x,y) + y\tilde{B}^2_{m-1}(x,y) - 3x\Big)^2 - xy\Big((2\tilde{A}_{n-1}(x,y) - \tilde{A}_{m-1}(x,y))\tilde{B}_{n-1}(x,y) + (\tilde{A}_{n-1}(x,y) - 2\tilde{A}_{m-1}(x,y))\tilde{B}_{m-1}(x,y)\Big)^2,$$

then $\tilde{P}_{n,m}(x,y) = Q_{n,m}(x^2,y^2)$. *Moreover,*
$\deg P_{n,m}(a,b) = \deg_a P_{n,m}(a,b) = 2 \cdot 3^n$ *and*
$\deg \tilde{P}_{n,m}(a,b) = \deg_a \tilde{P}_{n,m}(a,b) = 4 \cdot 3^{n-1}$.

All three cases ((n,n), (n,m) for $n > m$ and $m = 0$ and $m = 1$, (n,m) for $n > m \geq 2$) have been considered in the above three propositions.
For each case the zero level of polynomials $\tilde{P}_{n,m}(a,b)$ corresponds to exactly (n,m) critical orbit relation.
Denote $\mathrm{Crit}(n,m) = \{(a,b) : \tilde{P}_{n,m}(a,b) = 0\}$.
The degree counts show that all but $(1,1)$ critical orbit relations are realized so that there are infinitely many cubic polynomials with critical orbit relations.

Corollary

In the moduli space of cubics of the form $z^3 - 3a^2z + b$ *with coordinates* $x = a^2$ *and* $y = b^2$ *the exact (minimal) critical orbit relation* (n, m) *corresponds to the set* $\{(x, y) \in \mathbb{C}^2 : Q_{n,m}(x, y) = 0\}$, *where* $Q_{n,m}(x, y)$ *is defined above. It is never empty, except for the relation* $(1, 1)$.

Denote $\mathcal{S}_{n,m} = \{(x, y) \in \mathbb{C}^2 : Q_{n,m}(x, y) = 0\}$ the affine algebraic curve in $\mathbb{C}^2$. It seems that each curve $\mathcal{S}_{n,m}$, except $\mathcal{S}_{1,1}$ (which is an empty set), is irreducible. These curves are analogous to those defined by Milnor.

Some examples

Here are some examples of these special curves in $\mathbb{C}^2$.
$\mathcal{S}_{0,0} = \{x = 0\}$, $\mathcal{S}_{1,0} = \{x(2x-1)^2 - y = 0\}$,
$\mathcal{S}_{2,0} = \{x(8x^4 - 6x^2 + 6xy - 1)^2 - y(12x^3 - 3x + y + 1)^2 = 0\}$,
$\mathcal{S}_{2,1} = \{4x(1+x)^2 - y = 0\}$, $\mathcal{S}_{2,2} = \{4x^3 - 3x + 3y = 0\}$, and
$\mathcal{S}_{3,3} = \{64x^9 - 96x^7 + 528x^6y + 36x^5 - 288x^4y + 108x^3y^2 + 72x^3y + 27x^2y - 18xy^2 - 18xy - 3x + 3y^3 + 6y^2 + 3y = 0\}$.
The curves $\mathcal{S}_{0,0}$, $\mathcal{S}_{1,0}$, $\mathcal{S}_{2,1}$, and $\mathcal{S}_{2,2}$ can be identified with the complex plain $\mathbb{C}$ as these are graphs of polynomials.

Corollary

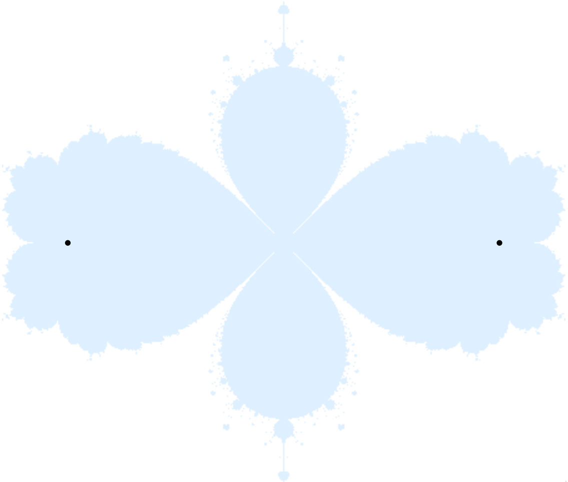
The degree of the curve $\mathcal{S}_{n,m}$ is a half of the degree of the polynomial $\tilde{P}_{(n,m)}(a,b)$.

Table: The degree row of $\mathcal{S}_{n,m}$ for $n \geq 2$.

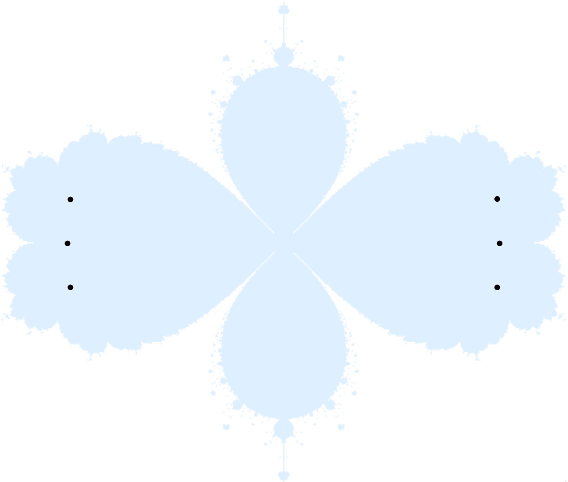
	m						
	0	1	2	.	.	$n-1$	n
n	3^n	3^{n-1}	$2 \cdot 3^{n-1}$	.	.	$2 \cdot 3^{n-1}$	3^{n-1}

In Table 1 we list degrees of $\mathcal{S}_{n,m}$ for $n \geq 2$ in a row.

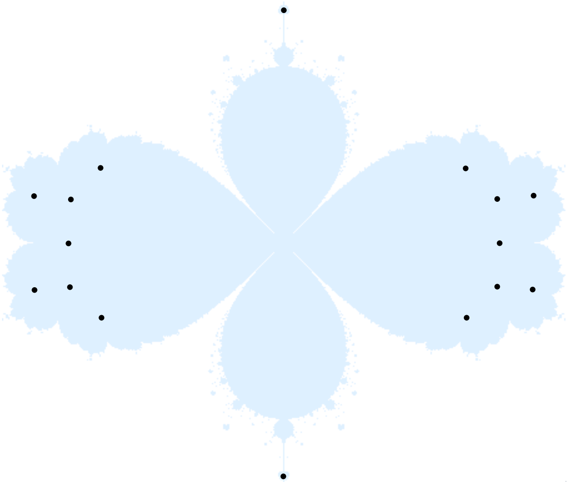
Parameter space of $z^3 + az^2 + z$ with COR $(0, 0)$



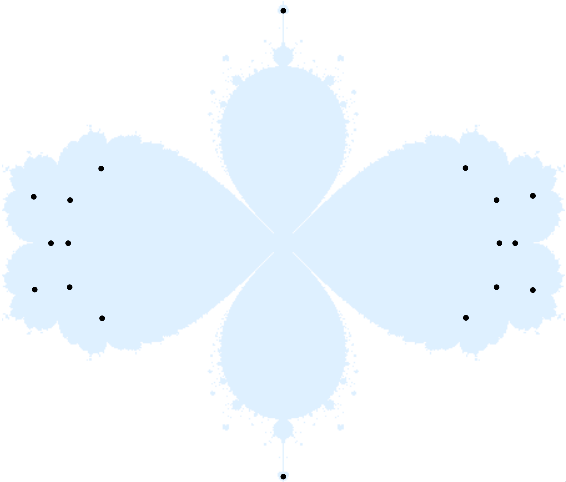
Parameter space of $z^3 + az^2 + z$ with COR $(1, 0)$



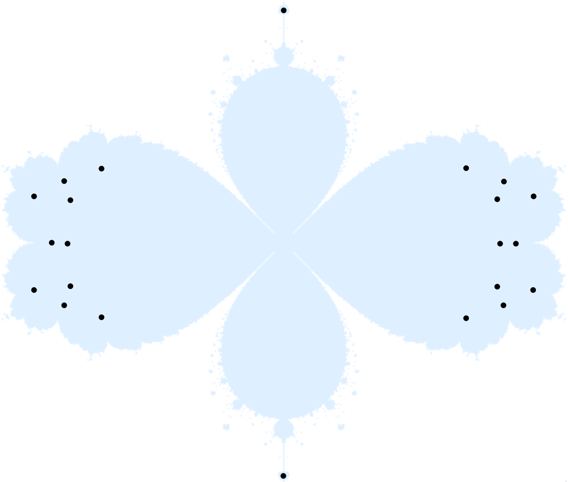
Parameter space of $z^3 + az^2 + z$ with COR $(2, 0)$



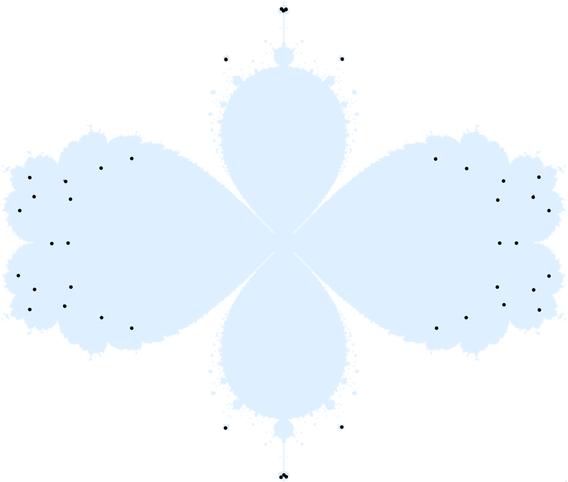
Parameter space of $z^3 + az^2 + z$ with COR $(2, 1)$



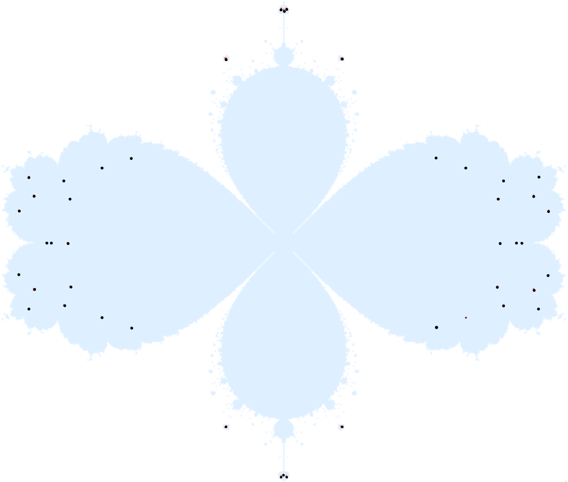
Parameter space of $z^3 + az^2 + z$ with COR $(2, 2)$



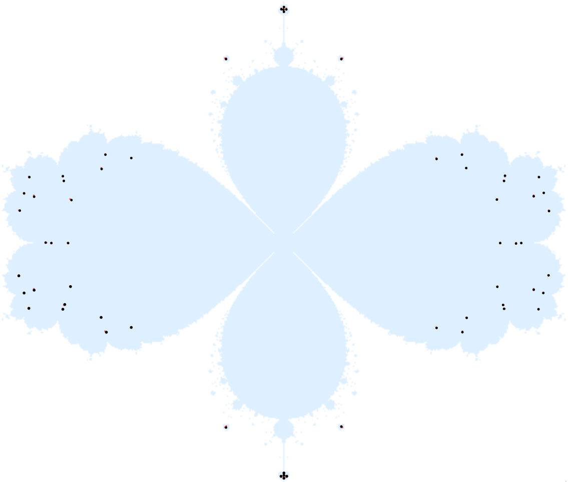
Parameter space of $z^3 + az^2 + z$ with COR $(3, 0)$



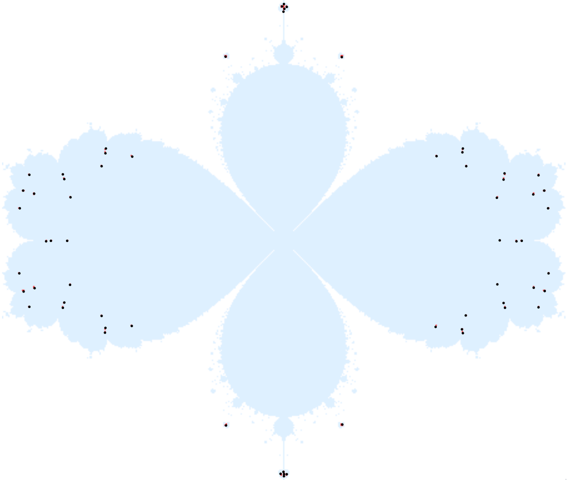
Parameter space of $z^3 + az^2 + z$ with COR $(3, 1)$



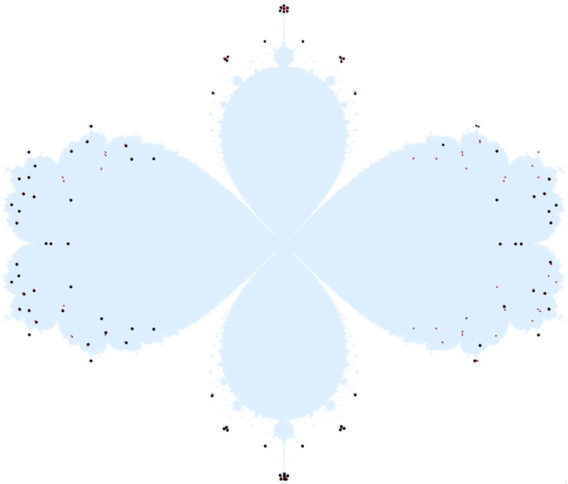
Parameter space of $z^3 + az^2 + z$ with COR $(3, 2)$



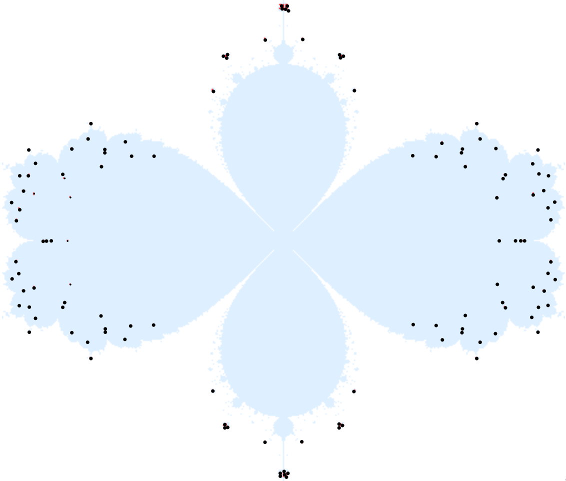
Parameter space of $z^3 + az^2 + z$ with COR $(3, 3)$



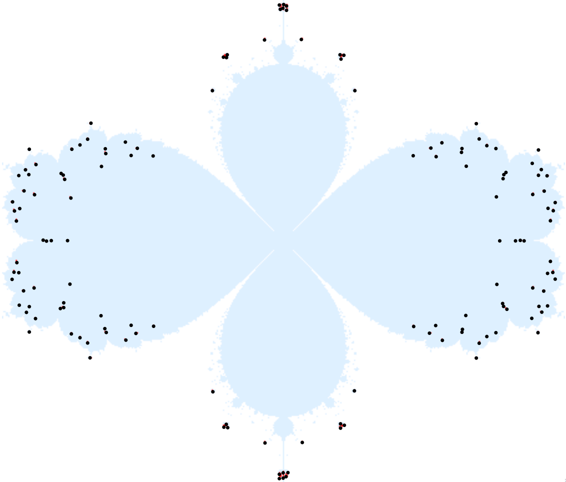
Parameter space of $z^3 + az^2 + z$ with COR $(4, 0)$



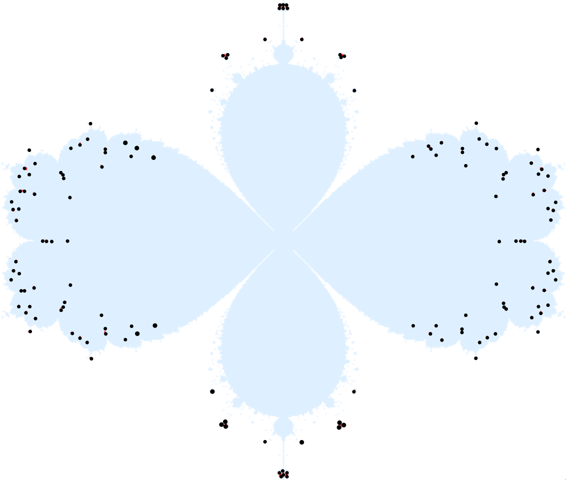
Parameter space of $z^3 + az^2 + z$ with COR $(4, 1)$



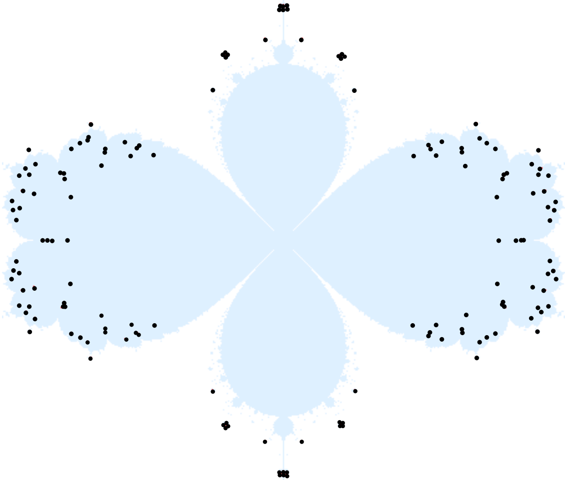
Parameter space of $z^3 + az^2 + z$ with COR $(4, 2)$



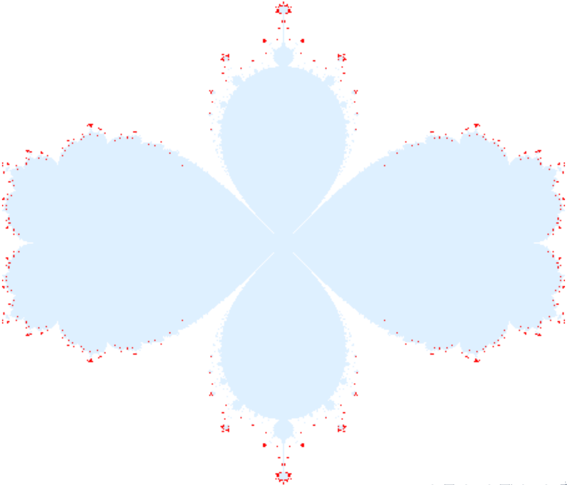
Parameter space of $z^3 + az^2 + z$ with COR $(4, 3)$



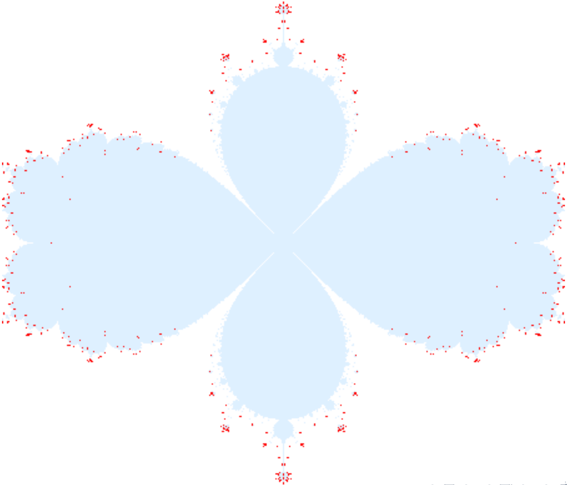
Parameter space of $z^3 + az^2 + z$ with COR $(4, 4)$



COR just $(6, 0)$



COR just $(6, 5)$



The space $\lambda z/(z^2 + tz + 1)$

Now consider the space of functions $f_t(z) = \lambda z/(z^2 + tz + 1)$ with a fixed point at the origin with multiplier $\lambda \neq 0 \in \mathbb{C}$ for each $t \in \mathbb{C}$.
Each f_t has critical points at ± 1.
The map $z \mapsto -z$ conjugates f_t to f_{-t} and interchanges the two critical points.
The critical orbit relation (n, m) becomes

$$\left(f_t^{\circ n}(1) - f_t^{\circ m}(-1)\right)\left(f_t^{\circ n}(-1) - f_t^{\circ m}(1)\right) = 0.$$

The main idea

Lemma

There exist sequences $\{A_n(t)\}_{n\geq 0}$, $\{B_n(t)\}_{n\geq 0}$, $\{C_n(t)\}_{n\geq 0}$ *and* $\{D_n(t)\}_{n\geq 0}$ *of polynomials of* t *such that if* z *is a critical point of* f_t *then for all* $n \geq 0$ *the equality* $f_t^{\circ n}(z) = \frac{A_n(t)z+B_n(t)}{C_n(t)z+D_n(t)}$ *holds.*

The main idea

Lemma

There exist sequences $\{A_n(t)\}_{n\geq 0}$, $\{B_n(t)\}_{n\geq 0}$, $\{C_n(t)\}_{n\geq 0}$ and $\{D_n(t)\}_{n\geq 0}$ of polynomials of t such that if z is a critical point of f_t then for all $n \geq 0$ the equality $f_t^{\circ n}(z) = \frac{A_n(t)z+B_n(t)}{C_n(t)z+D_n(t)}$ holds.

$$
\begin{aligned}
A_{n+1}(t) =& \lambda(A_n(t)D_n(t) + B_n(t)C_n(t));\\
B_{n+1}(t) =& \lambda(A_n(t)C_n(t) + B_n(t)D_n(t));\\
C_{n+1}(t) =& 2(A_nB_n + C_nD_n) + t(A_nD_n + B_nC_n);\\
D_{n+1}(t) =& A_n^2 + B_n^2 + C_n^2 + D_n^2 + t(A_nC_n + B_nD_n),
\end{aligned}
$$

with $A_0(t) = 1$ and $B_0(t) = 0$, $C_0(t) = 0$ and $D_0(t) = 1$.

Lemma

If λ is not a root of unity then $\deg_t A_n = 2^n - 2$, $\deg_t B_n = 2^n - 1$, $\deg_t C_n = 2^n - 1$, *and* $\deg_t D_n = 2^n$ *with the leading coefficients* a_n, b_n, c_n, *and* d_n *(polynomials of λ) respectively that satisfy* $a_{n+1} = \lambda(a_n d_n + b_n c_n)$, $b_{n+1} = \lambda b_n d_n$, $c_{n+1} = 2c_n d_n + a_n d_n + b_n c_n$, *and* $d_{n+1} = d_n(d_n + b_n)$ *for every* $n \geq 2$.

The explicit expression for d_n is as follows.

$$\begin{aligned} d_n =& (1+\lambda)^{2^{n-3}}(1+\lambda+\lambda^2)^{2^{n-4}} \cdots (1+\lambda+\lambda+\cdots+\lambda^{n-2})^{2^0} \\ & (1+\lambda+\lambda+\cdots+\lambda^{n-1}) \end{aligned}$$

for every $n \geq 3$.

Our main theorem is the following.

Theorem

For each $\lambda \neq 0$ *which is not a root of unity, in the family* $f_t(z) = \lambda z/(z^2 + tz + 1)$ *all critical orbit relations are realized except* $(0, 0)$ *and* $(n, 1)$ *for each* $n \geq 1$.

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Define polynomials as follows.

$$\begin{aligned} P_{n,m}(t) =& (A_n D_m + A_m D_n - B_n C_m - B_m C_n)^2 \\ & - (A_n C_m - A_m C_n - B_n D_m + B_m D_n)^2 \quad \textit{and} \\ P_{n,n}(t) =& A_n D_n - B_n C_n. \end{aligned}$$

The critical orbit relation (n, m) is equivalent to $P_{n,m}(t) = 0$.

Case of (n, n)

Set $\tilde{P}_{n,n} = A_{n-1}^2 - B_{n-1}^2 - C_{n-1}^2 + D_{n-1}^2$.

Proposition

For all $n \geq 1$, $P_{n,n}(t) = \lambda P_{n-1,n-1}(t) \cdot \tilde{P}_{n,n}(t)$ holds and if λ is not a root of unity then $\deg_t \tilde{P}_{n,n}(t) = 2^n$ for all $n \geq 2$.

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It yields that if (n, n) relation is minimal then $\tilde{P}_{n,n}(t) = 0$.
As $\tilde{P}_{1,1}(t) = 2$ there is no critical orbit relation of $(1, 1)$.

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As $\tilde{P}_{1,1}(t) = 2$ there is no critical orbit relation of $(1, 1)$.
The case of (n, n) factors as following.

Corollary

$P_{n,n}(t) = \lambda^{n-1} \tilde{P}_{1,1}(t) \tilde{P}_{2,2}(t) \cdots \tilde{P}_{n-1,n-1}(t) \tilde{P}_{n,n}(t)$ *holds for all $n \geq 1$.*

Cases of $(n, 0)$ and $(n, 1)$

Note that $A_1(t) = \lambda$, $B_1(t) = 0$, $C_1(t) = t$, $D_1(t) = 2$.
We have $P_{n,0}(t) = (A_n(t) + D_n(t))^2 - (B_n(t) + C_n(t))^2$ and the following holds.

Proposition

$P_{n+1,1}(t) = -\lambda^2 P_{n,0}^2(t)$ holds for all $n \geq 1$ with $\deg_t P_{n,0}(t) = 2^{n+1}$ for all $n \geq 2$.

Cases of $(n, 0)$ and $(n, 1)$

Note that $A_1(t) = \lambda,\ B_1(t) = 0,\ C_1(t) = t,\ D_1(t) = 2$.
We have $P_{n,0}(t) = (A_n(t) + D_n(t))^2 - (B_n(t) + C_n(t))^2$ and the following holds.

Proposition

$P_{n+1,1}(t) = -\lambda^2 P_{n,0}^2(t)$ *holds for all* $n \geq 1$ *with* $\deg_t P_{n,0}(t) = 2^{n+1}$ *for all* $n \geq 2$.

It implies that $(n, 1)$ for $n \geq 2$ critical orbit relations do not exist. Combining with the above we conclude that critical orbit relations $(n, 1)$ for $n \geq 1$ do not exist.

Case of $n > m \geq 2$

Set
$$\tilde{P}_{n,m}(t) = \left((A_{n-1}+B_{n-1})B_{m-1} + (C_{n-1}+D_{n-1})C_{m-1}\right)^2 - \left((A_{n-1}+B_{n-1})A_{m-1} + (C_{n-1}+D_{n-1})D_{m-1}\right)^2.$$

Proposition

$P_{n,m}(t) = \lambda^2 P_{n-1,m-1}(t) \cdot \tilde{P}_{n,m}(t)$ holds with $\deg_t \tilde{P}_{n,m}(t) = 2^n + 2^m$ for all $n > m \geq 2$.

Case of $n > m \geq 2$

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Proposition

$P_{n,m}(t) = \lambda^2 P_{n-1,m-1}(t) \cdot \tilde{P}_{n,m}(t)$ holds with $\deg_t \tilde{P}_{n,m}(t) = 2^n + 2^m$ for all $n > m \geq 2$.

The case of $(n+k, k)$, $n \geq 1$, $k \geq 2$, factors as following.

Corollary

$P_{n+k,k}(t) = -\lambda^{2k} \tilde{P}^2_{n,0}(t) \tilde{P}_{n+2,2}(t) \tilde{P}_{n+3,3}(t) \cdots \tilde{P}_{n+k,k}(t)$ holds for all $k \geq 2$.

$a_d z^d + a_{d-1} z^{d-1} + \ldots + a_1 z + a_0$, $a_k \in \mathbb{C}$, $d \geq 3$.

Let $p(z)$ be a family of polynomial of degree $d \geq 3$ with at least two distinct simple critical points. By changing coordinates we can put its two critical points to ± 1.
Critical orbit relation is $(p^{\circ n}(-1) - p^{\circ m}(1))(p^{\circ n}(1) - p^{\circ m}(-1)) = 0$.
For every $z \in \mathbb{C}$ by Taylor's formula we obtain the following

$$p(z) = p(1) + p'(1)(z-1) + \frac{p''(1)}{2!}(z-1)^2 + \cdots + \frac{p^d(1)}{d!}(z-1)^d.$$

As $z = 1$ is a simple critical point we get

$$\begin{aligned} p(z) - p(1) =& \frac{p''(1)}{2!}(z-1)^2 + \cdots + \frac{p^d(1)}{d!}(z-1)^d \\ =& (z-1)^2\left(\frac{p''(1)}{2!} + \cdots + \frac{p^d(1)}{d!}(z-1)^{d-2}\right). \end{aligned}$$

Plug in into the above equality $z = p^{\circ(n-1)}(-1)$ and obtain
$p^{\circ n}(-1) - p(1) = (p^{\circ(n-1)}(-1) - 1)^2 \cdot g_1$, where
$g_1 = \frac{p''(1)}{2!} + \frac{p'''(1)}{3!}(p^{\circ(n-1)}(-1) - 1) + \cdots + \frac{p^d(1)}{d!}(p^{\circ(n-1)}(-1) - 1)^{d-2}$.
If we write the Taylor's formula about $z = -1$ then we obtain

$$p(z) - p(-1) = p'(-1)(z+1) + \frac{p''(-1)}{2!}(z+1)^2 + \cdots + \frac{p^d(-1)}{d!}(z+1)^d.$$

Now plug in $z = p^{\circ(n-1)}(1)$ and obtain
$p^{\circ n}(1) - p(-1) = (p^{\circ(n-1)}(1) + 1)^2 \cdot g_2$, where
$g_2 = \frac{p''(-1)}{2!} + \frac{p'''(-1)}{3!}(p^{\circ(n-1)}(1) + 1) + \cdots + \frac{p^d(-1)}{d!}(p^{\circ(n-1)}(1) + 1)^{d-2}$.

The critical orbit relations $(n, 1)$ and $(n-1, 0)$ are related to each other as following $(p^{\circ n}(-1) - p(1))(p^{\circ n}(1) - p(-1)) = (p^{\circ(n-1)}(-1) - 1)^2(p^{\circ(n-1)}(1) + 1)^2 g_1 g_2$.
Thus the exact (minimal) critical orbit relation $(n, 1)$ becomes $g_1 \cdot g_2 = 0$, set $\tilde{P}_{n,1} = g_1 \cdot g_2$.
For the cubic family $p(z) = z^3 - 3a^2 z + b$, we obtain $g_1 g_2 = (p^{\circ(n-1)}(a) - 2a)(p^{\circ(n-1)}(-a) + 2a) = B_{n-1}^2 - a^2(A_{n-1} - 2)^2$, which coincides with the previous result.

Rational function $f(z) = \frac{p(z)}{q(z)}$

Consider $f(z) = \frac{p(z)}{q(z)}$ a rational function with two distinct simple critical points at ± 1 (after a coordinate change). Taylor development at $z = 1$ is

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \cdots + \frac{f^k(1)}{k!}(z-1)^k + \cdots .$$

Then $f(z) - f(1) = (z-1)^2\left(\frac{f''(1)}{2!} + \cdots + \frac{f^k(1)}{k!}(z-1)^{k-2} + \cdots\right)$. Now plug in $z = f^{\circ(n-1)}(-1)$ into the above and obtain $f^{\circ n}(-1) - f(1) = (f^{\circ(n-1)}(-1) - 1)^2 \cdot g_1$, where $g_1 = \frac{f''(1)}{2!} + \frac{f'''(1)}{3!}(f^{\circ(n-1)}(-1) - 1) + \cdots + \frac{f^k(1)}{k!}(f^{\circ(n-1)}(-1) - 1)^{k-2} + \cdots .$

Similarly, Taylor development at $z = -1$ is

$$f(z) = f(-1) + f'(-1)(z+1) + \frac{f''(-1)}{2!}(z+1)^2 + \cdots + \frac{f^k(-1)}{k!}(z+1)^k + \cdots$$

Then $f(z) - f(-1) = (z+1)^2\big(\frac{f''(-1)}{2!} + \cdots + \frac{f^k(-1)}{k!}(z+1)^{k-2} + \cdots\big)$. Now plug in $z = f^{\circ(n-1)}(1)$ into the above and obtain $f^{\circ n}(1) - f(-1) = (f^{\circ(n-1)}(1) + 1)^2 \cdot g_2$, where $g_2 = \frac{f''(-1)}{2!} + \frac{f'''(-1)}{3!}(f^{\circ(n-1)}(1) + 1) + \cdots + \frac{f^k(-1)}{k!}(f^{\circ(n-1)}(1) + 1)^{k-2} + \cdots$. Analogously, the exact critical orbit relation $(n, 1)$ reduces to $g_1 g_2 = 0$.

Cubic rational functions $f_a(z) = z^2 \frac{z+a-1}{(a+1)z-1}$

Let $f_a(z) = z^2 \dfrac{z+a-1}{(a+1)z-1}$. The map $h(z) = 1/z$ conjugates f_a with f_{-a}. Consider parameters $a \neq 0$ such that there exists a pair of non-negative integers n and m with

$$f_a^{\circ n}(z) = f_a^{\circ m}(w),$$

where z and w are critical points of f_a which are roots of

$$2(a+1)z^2 + (a^2-4)z - 2(a-1) = 0.$$

If $a = -1$, $f_{-1}(z) = -z^2(z-2)$ is a polynomial. Its critical point $4/3$ converges to the parabolic fixed point at $z = 1$ and do not make any orbit relations with ∞.

The main idea

Lemma

There exist sequences $\{A_n(a)\}_{n\geq 0}$, $\{B_n(a)\}_{n\geq 0}$, $\{C_n(a)\}_{n\geq 0}$ *and* $\{D_n(a)\}_{n\geq 0}$ *of polynomials of a such that if z is a critical point of* f_a *then for all* $n \geq 0$ *the equality* $f_a^{\circ n}(z) = \frac{A_n(a)z+B_n(a)}{C_n(a)z+D_n(a)}$ *holds.*

$$\begin{aligned} A_{n+1}(a) =& (a^4 - 4a^2 + 12)A_n^2(A_n + (a-1)C_n) \\ &+ 12(a+1)^2 A_n B_n^2 - (a-1)(a^2-4)C_n \\ &+ 4(a-1)(a+1)^2 B_n^2 C_n - 2(a+1)D_n(t) \\ &+ A_n^2\big(-6(a+1)(a^2-4)B_n - 2(a+1)(a^2-4)D_n\big); \end{aligned}$$

$$
\begin{aligned}
B_{n+1}(a) =& -2(a-1)(a^2-4)A_n^3 + 2(a-1)A_n^2\big(6(a+1)B_n \\
& -(a-1)((a^2-4)C_n - 2(a+1)D_n)\big) \\
& + 8(a-1)^2(a+1)A_nB_nC_n + 4(a+1)^2B_n^2((a-1)D_n + B_n); \\
C_{n+1}(a) =& (a+1)C_n^2\big((a^4-4a^2+12)A_n - 2(a^2-4)((a+1)B_n - 3D_n)\big) \\
& -(a^4-4a^2+12)C_n^3 + 4(a+1)^3A_nD_n^2 \\
& -4(a+1)^2C_nD_n((a^2-4)A_n - 2(a+1)B_n + 3D_n); \\
D_{n+1}(a) =& -2(a-1)C_n^2\big((a+1)(a^2-4)A_n - 2(1+a)^2B_n \\
& -(a^2-4)C_n\big) + 4(a^2-1)(2(a+1)A_n \\
& -3C_n)C_nD_n + 4(a+1)^3B_nD_n^2 - 4(a+1)^2D_n^3,
\end{aligned}
$$

with $A_0(a) = 1$ and $B_0(a) = 0$, $C_0(a) = 0$ and $D_0(a) = 1$.

Lemma

$a^{(3^n-3)/2}$ divides each of $A_n(a)$, $B_n(a)$, $C_n(a)$, $D_n(a)$, $(a+1)^{2^{n+1}-1}$ divides each of $C_n(a)$, $D_n(a)$ and $\deg A_n(a) = 2\cdot 3^n - 2$, $\deg B_n(a) = 2\cdot 3^n - 3$, $\deg C_n(a) = 2\cdot 3^n - 3$, and $\deg D_n(a) = 2\cdot 3^n - 4$ with the leading coefficients a_n, b_n, c_n, and d_n respectively that satisfy for all $n \geq 1$ recurrence relations $a_{n+1} = a_n^2(a_n + c_n)$, $b_{n+1} = -2a_n^2(a_n + c_n)$, $c_{n+1} = a_n c_n^2$, and $d_{n+1} = -2a_n c_n^2$ with $a_1 = -1$, $b_1 = 2$, $c_1 = 4$, and $d_1 = -4$.

Set

$$
\begin{aligned}
P_{n,m}(a) = & \big((a^2-4)A_nB_n + 2(a-1)A_n^2 - 2(a+1)B_n^2\big)\big((a^2-4)C_mD_m \\
& + 2(a-1)C_m^2 - 2(a+1)D_m^2\big) + 2(a+1)B_mD_m\big((a^2-4)A_nD \\
& + (a^2-4)B_nC_n + 4(a-1)A_nC_n - 4(a+1)B_nD_n\big) \\
& + 2\big((a-1)A_m^2 - (a+1)B_m^2\big)\big((a^2-4)C_nD_n + 2(a-1)C_n^2 \\
& - 2(a+1)D_n^2\big) + A_m\big(-2(a-1)C_m((a^2-4)A_nD_n \\
& + (a^2-4)B_nC_n + 4(a-1)A_nC_n - 4(a+1)B_nD_n) \\
& + (a^2-4)B_m((a^2-4)C_nD_n + 2(a-1)C_n^2 - 2(a+1)D_n^2) \\
& - D_m((a^4+8)A_nD_n - 2(a+1)B_n((a^2-4)D_n + 4(a-1)C_n) \\
& + 2(a-1)(a^2-4)A_nC_n)\big) + B_mC_m\big(2(a+1)D_n((a^2-4)B_n \\
& + 4(a-1)A_n) - C_n((a^4+8)B_n + 2(a-1)(a^2-4)A_n)\big).
\end{aligned}
$$

and

$$P_{n,n}(a) = A_n(a)D_n(a) - B_n(a)C_n(a).$$

The critical orbit relation (n, m) is equivalent to $P_{n,m}(a) = 0$.

Theorem (Main)

In the family $f_a(z) = z^2 \dfrac{z + a - 1}{(a+1)z - 1}$ *all critical orbit relations are realized except* $(1, 1)$.

Case of (n, n)

Set

$$\begin{aligned}\tilde{P}_{n,n}(a) =& 2(a^2-1)A_{n-1}^3((a^2-4)D_{n-1}+4(a-1)C_{n-1}) \\ &+ A_{n-1}^2\big(D_{n-1}(2(a-1)(a^2-4)^2C_{n-1} \\ &+ (a+1)(a^4-16a^2+24)B_{n-1}) + (-5a^4+12a^2-16)D_{n-1}^2 \\ &+ 2(a-1)(a^2-4)C_{n-1}\big(3(a+1)B_{n-1}+2(a-1)C_{n-1}\big)\big) \\ &+ B_{n-1}\big(-2(a+1)^2B_{n-1}^2\big((a^2-4)C_{n-1}-4(a+1)D_{n-1}\big) \\ &- (a-1)\big((a^2-4)C_{n-1}-4(a+1)D_{n-1}\big)\big((a^2-4)C_{n-1}D_{n-1} \\ &+ 2(a-1)C_{n-1}^2 - 2(a+1)D_{n-1}^2\big) \\ &+ B_{n-1}(-2(a+1)(a^2-4)^2C_{n-1}D_{n-1} \\ &+ 4(a+1)^2(a^2-4)D_{n-1}^2 + (-5a^4+12a^2-16)C_{n-1}^2)\big)\end{aligned}$$

$$
\begin{aligned}
cont. &+ A_{n-1}\big(-(a-1)((a^2-4)D_{n-1}+4(a-1)C_{n-1}) \\
&((a^2-4)C_{n-1}D_{n-1}+2(a-1)C_{n-1}^2-2(a+1)D_{n-1}^2) \\
&+(a+1)B_{n-1}^2((a^4-16a^2+24)C_{n-1} \\
&-6(a-2)(a+1)(a+2)D_{n-1})+B_{n-1}(2(a-1)(a^2-4)^2C_{n-1}^2 \\
&-2(a+1)(a^2-4)^2D_{n-1}^2+(a^6-10a^4+64a^2-64)C_{n-1}D_{n-1})\big).
\end{aligned}
$$

Proposition

For all $n \geq 2$, $P_{n,n}(a) = 4(a+1)^2 P_{n-1,n-1}(a) \cdot \tilde{P}_{n,n}(a)$ *holds with* $\deg \tilde{P}_{n,n}(a)/(a^{2\cdot 3^{n-1}}(a+1)^{2\cdot 3^{n-1}-2}) = 2 \cdot 3^{n-1}$.

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It yields that if (n,n) relation is minimal then $\tilde{P}_{n,n}(a) = 0$.
As $\tilde{P}_{1,1}(a) = -a^2(a^2+8)$ there is no critical orbit relation of $(1,1)$.
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The case of (n,n) factors as following.

Corollary

$P_{n,n}(a) = 4^n(a+1)^{2n}\tilde{P}_{1,1}(a)\tilde{P}_{2,2}(a)\cdots\tilde{P}_{n-1,n-1}(a)\tilde{P}_{n,n}(a)$ *holds for all* $n \geq 2$.

Cases of $(n, 0)$ and $(n, 1)$

Set

$$\begin{aligned}\tilde{P}_{n+1,1}(a) =& 2(a^2-1)A_n^2 - 2(a+1)^2B_n^2 + A_n\big(2(a-1)(a^2+2)C_n \\ & - (5a^2+4)D_n + (a-2)(a+1)(a+2)B_n\big) \\ & + (a+1)B_n\big((a-1)(a^2+4)C_n - 2(a^2+2)D_n\big) \\ & - (a-1)\big((a^2-4)C_nD_n + 2(a-1)C_n^2 - 2(a+1)D_n^2\big), \\ P_{n,0}(a) =& 4(a^2-1)(A_n^2+D_n^2) - 4\big((a+1)B_n + (a-1)C_n\big)^2 \\ & + 2(a^2-4)(A_n-D_n)\big((a+1)B_n + (a-1)C_n\big) \\ & + (a^4+8)A_nD_n.\end{aligned}$$

Note that $A_1(a) = -a^4 + 6a^2 + 4$, $B_1(a) = 2(a-1)(a^2+2)$, $C_1(a) = 4(a+1)^3$, $D_1(a) = -4(a+1)^2$.

Proposition

$P_{n+1,1}(a) = 16a^2(1+a)^4 P_{n,0}^2(a)\tilde{P}_{n+1,1}(a)$ with $\deg P_{n,0}(a)/(a^{3^n+1}(a+1)^{3^n-1}) = 3^n + 1$ holds for all $n \geq 1$.

Some examples

$$P_{1,0}(a) = a^4 + 12a^2 + 68,$$
$$P_{2,0}(a) = 9a^{10} + 116a^8 + 1932a^6 + 10896a^4 + 35984a^2 + 10112,$$
$$\tilde{P}_{2,1}(a) = 9a^4 + 56a^2 + 16,$$
$$\tilde{P}_{2,2}(a) = a^6 + 8a^4 + 56a^2 + 16,$$
$$P_{3,0}(a) = 13689a^{28} + 179100a^{26} + 6874588a^{24} + 94460304a^{22} + 1225422576a^{20} + 10841205568a^{18} + 76505084288a^{16} + 392572421632a^{14} + 1527281530112a^{12} + 4123190390784a^{10} + 7458475134976a^8 + 6466193604608a^6 + 2436690755584a^4 + 369190502400a^2 + 14524874752,$$
$$\tilde{P}_{3,1}(a) = 169a^{10} + 1616a^8 + 8432a^6 + 28608a^4 + 19200a^2 + 1024,$$

Some examples

$\tilde{P}_{3,2}(a) = 13689a^{24} + 248832a^{22} + 4837752a^{20} + 54139104a^{18} + 492002832a^{16} + 3000822272a^{14} + 14056360704a^{12} + 43529908736a^{10} + 93937358848a^{8} + 87954415616a^{6} + 34018004992a^{4} + 5179047936a^{2} + 202375168,$

$\tilde{P}_{3,3}(a) = 1521a^{18} + 15080a^{16} + 316912a^{14} + 2485344a^{12} + 16203168a^{10} + 58029440a^{8} + 151108096a^{6} + 126720256a^{4} + 31131648a^{2} + 1409024,$

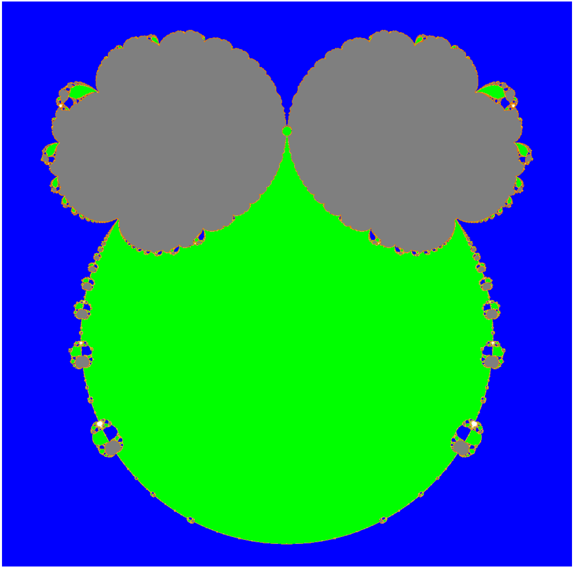
$\tilde{P}_{4,1}(a) = 423801a^{28} + 6546872a^{26} + 104315104a^{24} + 1161470304a^{22} + 10072950592a^{20} + 66979653504a^{18} + 335044251904a^{16} + 1279180918784a^{14} + 3481356134400a^{12} + 6537205153792a^{10} + 6880747786240a^{8} + 3452825862144a^{6} + 770545090560a^{4} + 60874031104a^{2} + 687865856.$

Table: Degree of $\tilde{P}_{n,m}(a)$ for (n, m) up to $(6, 6)$.

n \ m	0	1	2	3	4	5	6
1	4	0					
2	10	4	6				
3	28	10	24	18			
4	82	28	60	72	54		
5	244	82	168	180	216	162	
6	730	244	492	504	540	648	486

Proposition

For all $n \geq 2$ *one has* $\deg \tilde{P}_{(n,1)}(a) = \deg \tilde{P}_{(n-1,0)}(a) = 3^{n-1} + 1$ *and* $\deg \tilde{P}_{(n,n)}(a) = 2 \cdot 3^{n-1}$ *for* $n \geq 2$*. For all* $n \geq 3$ *and* $2 \leq k \leq n-1$*,* $\deg \tilde{P}_{(n,k)}(a) = 2 \cdot 3^{n-1} + 2 \cdot 3^{k-1}$ *holds.*



Open problems

We need to study the irreducibility of obtained polynomials $\tilde{P}_{n,m}$.
Another research direction is to study the distribution of functions with critical orbit relations in the moduli space.

Thank you.