Wandering Domains in Transcendental Dynamics

Topology and Dynamics

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— joint work with **Lasse Rempe** and **James Waterman** —

On geometric complexity of Julia sets - IV

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INTRODUCTION

Wandering domains of
transcendental entire functions

Classification of Fatou components

For an entire function $f : \mathbb{C} \to \mathbb{C}$, we define

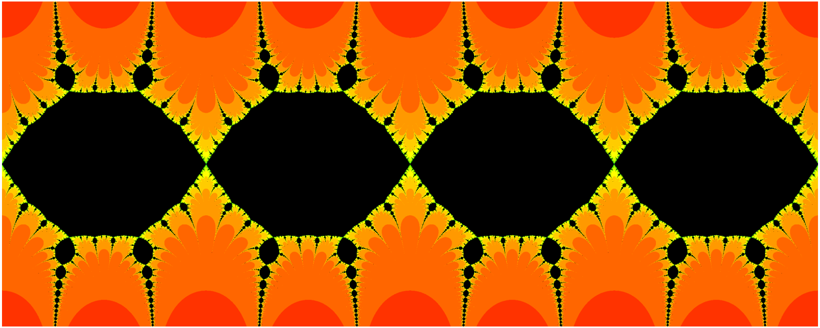
- the **Fatou set** $F(f)$ is the largest open set on which the iterates $\{f^n\}$ form a normal family;
- the **Julia set** $J(f)$ is given by $J(f) := \mathbb{C} \setminus F(f)$.

The connected components U of $F(f)$ are called **Fatou components** and can be

- **periodic**: $\exists p \in \mathbb{N}$ such that $f^p(U) \subseteq U$;
- **preperiodic**: $\exists q \in \mathbb{N}$ such that $f^q(U)$ lies in a periodic Fatou component, i.e., $\exists p, q \in \mathbb{N}$ such that
$$f^{p+q}(U) \subseteq f^q(U);$$
- **wandering domain**: $f^m(U) \cap f^n(U) \neq \emptyset$ if and only if $m = n$.

[Ber93] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 151-188

Example



$$f(z) = z + \sin z$$

Existence of wandering domains

Theorem (Baker, 1976)

There exists a transcendental entire function with a wandering domain.

Theorem (Sullivan, 1985)

Rational functions (in one complex dimension) do not have wandering domains.

Theorem (Astorg, Buff, Dujardin, Peters and Raissy, 2016)

There exists a polynomial skew product $P : \mathbb{C}^2 \to \mathbb{C}^2$ *with a wandering domain.*

[Bak76] I. N. Baker, *An entire function which has wandering domains*, J. Austral. Math. Soc. Ser. A **22** (1976), no. 2, 173-176

[Sul85] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the FatouJulia problem on wandering domains*, Ann. of Math. (2) **122** (1985) 401-418

[ABDPR16] M. Astorg, X. Buff, R. Dujardin, H. Peters and J. Raissy, *A two-dimensional polynomial mapping with a wandering Fatou component*, Ann. of Math. (2) **184** (2016) 263-313

Limit functions

According to the **accumulation set** of their orbit, we can partition $\mathbb{C}$ into the sets:

- $I(f) := \{z \in \mathbb{C} : \ f^n(z) \to \infty \text{ as } z \to \infty\}$ (**escaping set**);
- $K(f) := \{z \in \mathbb{C} : \ \exists R > 0, \ |f^n(z)| < R \text{ for all } n \in \mathbb{N}\}$ (**bounded orbits**);
- $BU(f) := \mathbb{C} \setminus (I(f) \cup K(f))$ (**bungee set**).

Every Fatou component is necessarily contained in one of these sets, thus we say:

- U is an **escaping** wandering domain if $U \subseteq I(f)$;
- U is a **bounded-orbit** wandering domain if $U \subseteq K(f)$;
- U is an **oscillating** wandering domain if $U \subseteq BU(f)$.

Question

Do bounded-orbit wandering domains exist?

[Ere89] A. E. Eremenko, *On the iteration of entire functions*, Dynamical systems and ergodic theory (Warsaw, 1986), Banach Center Publications 23 (PWN, Warsaw, 1989), 339-345

[OS16] J. W. Osborne and D. J. Sixsmith, *On the set where the iterates of an entire function are neither escaping nor bounded*, Ann. Acad. Sci. Fenn. Math. 41 (2016) 561-578

Multiply connected wandering domains

Baker's first example of a wandering domain was **multiply connected**.

Theorem (Baker, 1984)

Let U be a multiply connected Fatou component of a transcendental entire function f. Then U is a wandering domain, and has the following properties:

(i) *each $U_n := f^n(U)$ is bounded and multiply connected;*

(ii) *there exists $N \in \mathbb{N}$ such that U_n and 0 lie in a bounded complementary component of U_{n+1} for all $n \geq N$;*

(iii) $\operatorname{dist}(U_n, 0) \to +\infty$ *as $n \to \infty$.*

[Bak75] I. N. Baker, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Ser. AI Math. **1** (1975) 277-283

[Bak84] I. N. Baker, *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc. (3) **49** (1984) 563576

More examples

- Baker (1976) constructed an infinite product with a wandering domain that is **multiply connected** and **escapes** to infinity
- Herman (1984) studied the first example of a **simply connected** wandering domain given by the formula $f(z) = z + \sin z + 2\pi$, which is a lift of a holomorphic self-map of the punctured plane
- Eremenko and Lyubich (1987) used approximation theory to produce the first **oscillating** wandering domain
- Kisaka and Shishikura (2008) constructed entire functions with multiply connected wandering domains of **different eventual connectivity** using quasiconformal surgery
- Bishop (2015) obtained the first wandering domain for a function in the Eremenko–Lyubich class $\mathcal{B}$ (with a **bounded set of singular values**) using quasiconformal folding

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Connections with other problems in transcendental dynamics

Theorem (Bishop, 2018)

*There exists a transcendental entire function f with a multiply connected **wandering domain** such that $\dim_H J(f) = 1$.*

Theorem (Benini, Rippon and Stallard, 2016)

*Let f, g be transcendental entire functions with no fast escaping simply connected **wandering domain**. Then if $f \circ g = g \circ f$, we have $J(f) = J(g)$.*

Theorem (MP, Rempe and Waterman, 2022)

*There exists a transcendental entire function with an unbounded **wandering domain** that is a counterexample to Eremenko's conjecture.*

[Bis18] C. J. Bishop, *A transcendental Julia set of dimension 1*, Invent. Math. **212** (2018), 407-460

[BRS16] A. M. Benini, P. J. Rippon and G. M. Stallard, *Permutable entire functions and multiply connected wandering domains*, Adv. Math. **287** (2016), 451-462

[MRW22] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering Lakes of Wada, and maverick points*, preprint arXiv:2108.10256

PART I: TOPOLOGY

Which open sets arise as a wandering domain of some transcendental entire function?

Theorem (Zdunik, 1991)

Let P be a polynomial and let U be an invariant basin of attraction of P. Then either

- *∂U is an analytic curve, or*
- $\dim_H(\partial U) > 1$.

In fact, in the first case either

- *∂U is a* ***circle*** *and P is conjugated to a power map, or*
- *∂U is an* ***arc of a circle*** *and P is conjugated to a Chebyshev polynomial.*

[Zdu91] A. Zdunik, *Harmonic measure versus Hausdorff measures on repellers for holomorphic maps*, Trans. Amer. Math. Soc. **326** (1991), no. 2, 633–652

[Prz06] F. Przytycki, *On the hyperbolic Hausdorff dimension of the boundary of a basin of attraction for a holomorphic map and of quasirepellers*, Bull. Pol. Acad. Sci. Math. **54** (2006), no. 1, 41–52

Fatou's question

In the context of rational functions, Fatou asked the following:

Question (Fatou, 1920)

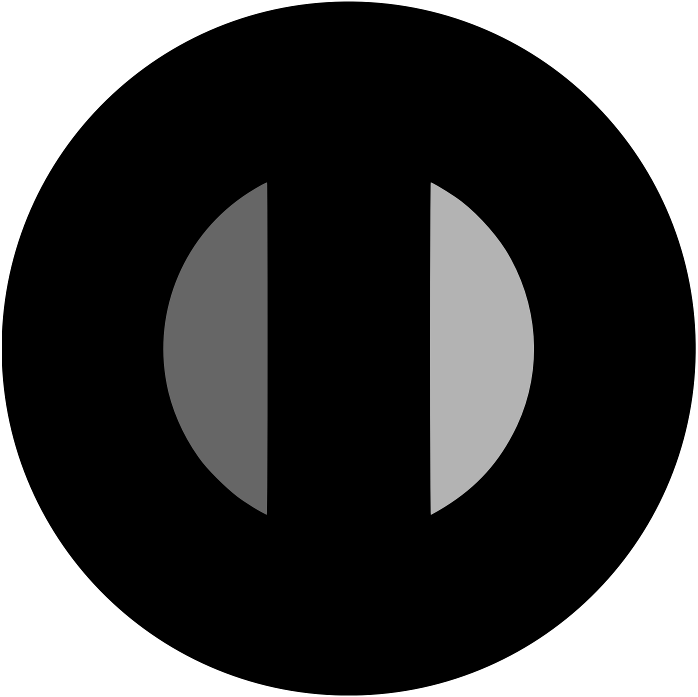
If f has more than two Fatou components, can two of these components share the same boundary?

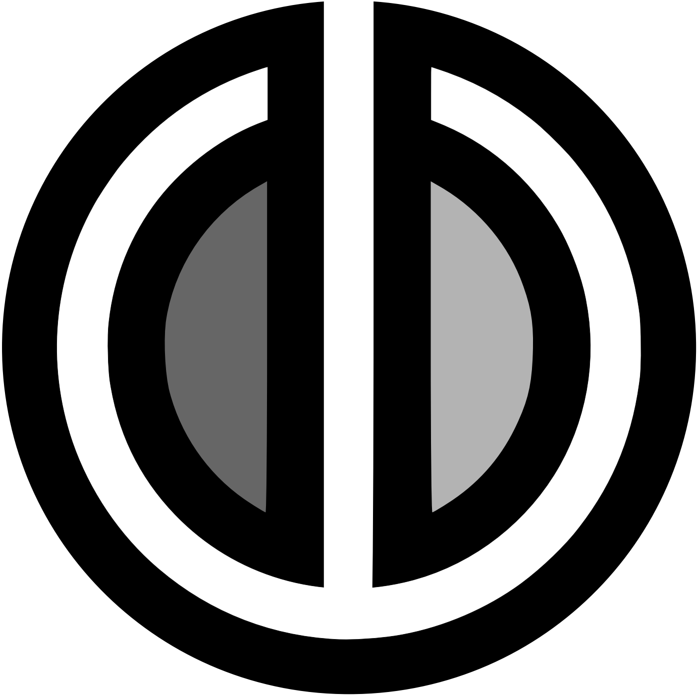
This question makes equal sense for transcendental entire functions, whose study Fatou initiated in 1926.

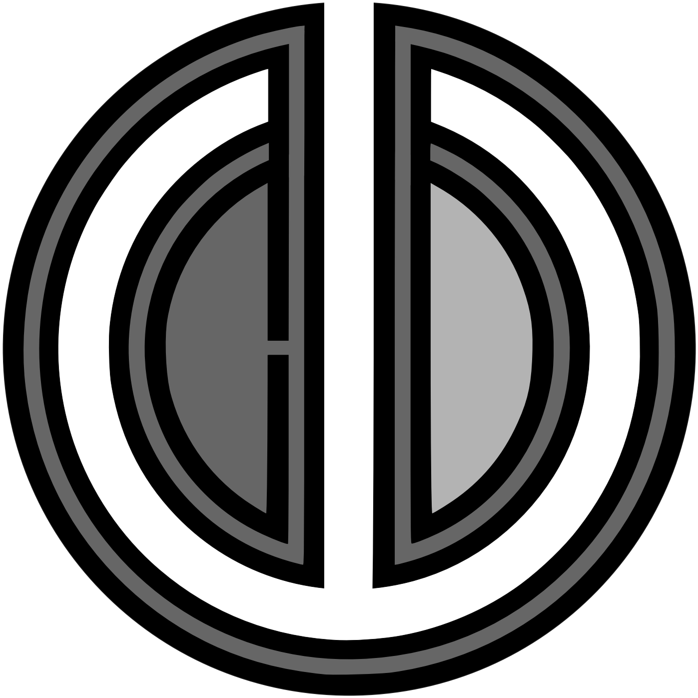
One situation in which this happens in a strong way is when the Fatou components form **Lakes of Wada**.

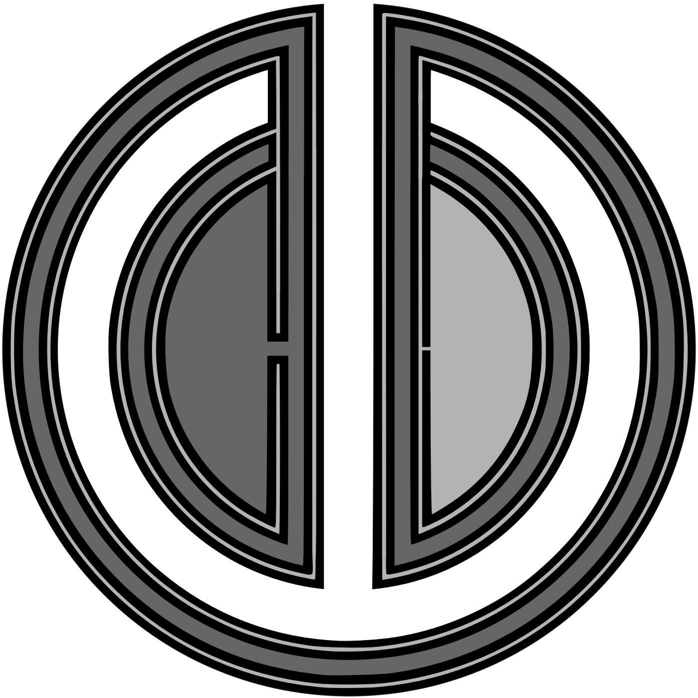
[Fat20] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), 33–94

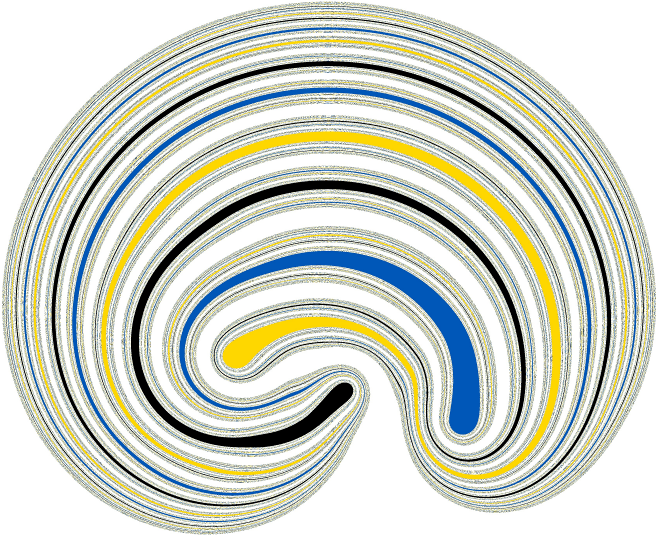
[Fat26] P. Fatou, *Sur l'itération des fonctions transcendantes Entiéres*, Acta Math. **47** (1926), no. 4, 337–370











A Lakes of Wada continuum

(Picture produced by James Waterman following the algorithm in [Cou06])

[Cou06] Y. Coudene, *Pictures of hyperbolic dynamical systems*, Notices Amer. Math. Soc. **53** (2006), no. 1, 8–13

Lakes of Wada

Definition

We say that a continuum $X \subseteq \mathbb{C}$ is a **Lakes of Wada continuum** if X is the common boundary of 3 disjoint domains $U_1, U_2, U_3 \subseteq \mathbb{C}$, that is,

$$X = \partial U_1 = \partial U_2 = \partial U_3.$$

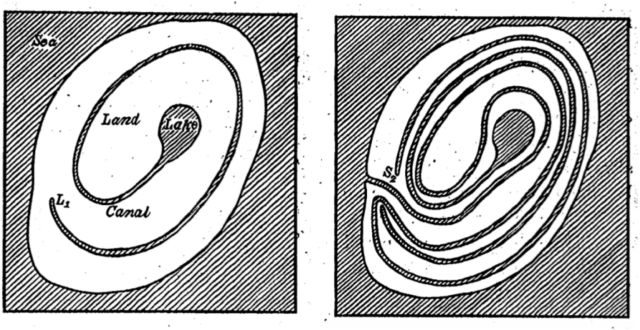
The existence of such continua was established by Brouwer in 1910.

Note that by adding new lakes subsequently, one may obtain Lakes of Wada continua that are the common boundary of **infinitely many** domains.

[Bro10] L. E. J. Brouwer, *Zur Analysis Situs* (German), Math. Ann. **68** (1910), no. 3, 422–434

Yoneyama's paper

In 1917, Yoneyama described how to obtain such continua using the previous iterative construction. These are the original pictures from his paper:



In that paper, he said that he learnt of this construction from Mr. Wada, a Japanese mathematician at Kyoto University working in analysis and topology.

[Yon17] K. Yoneyama, *Theory of continuous set of points*, Tohoku Math. J. **12** (1917), 43–158

Takeo Wada (1882–1944)

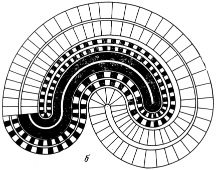
By courtesy of the Mathematics Library at Kyoto University.

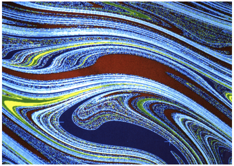
[Ish17] Y. Ishii, *Dynamics of polynomial diffeomorphisms of* $\mathbb{C}^2$*: combinatorial and topological aspects*, Arnold Math. J. 3 (2017), no. 1, 119–173

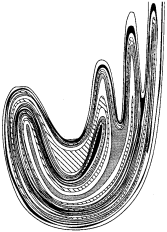
Lakes of Wada in Dynamical Systems

Lakes of Wada continua may appear pathological, but they occur naturally in the study of Dynamical Systems in two real dimensions:

- **Perturbations of Anosov diffeomorphisms of the torus** (Plykin)
- **Forced damped pendulum** (Kennedy and Yorke)
- **Hénon maps in $\mathbb{R}^2$** (Hubbard and Oberste-Vorth)







(Pictures from [Fig 1, Ply74], [Plate II, KY91] and [Figure 8.10, HO95], respectively.)

[Ply74] R. V. Plykin, *Sources and sinks of A-diffeomorphisms of surfaces*, Mat. Sb. (N.S.) **94**(136) (1974), 243–264, 336

[KY91] J. Kennedy and J. A. Yorke, *Basins of Wada*, Nonlinear science: the next decade (Los Alamos, NM, 1990), vol. 51, Physica D. Nonlinear Phenomena, no. 1-3, MIT Press, 1991, pp. 213–225

[HO95] J. H. Hubbard and R. W. Oberste-Vorth, *Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials*, Real and complex dynamical systems (Hillerød, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 464, Kluwer Acad. Publ., Dordrecht, 1995, pp. 89–132

Question (Fatou, 1920)

If f has more than two Fatou components, can two of these components share the same boundary?

The following result answers the analogue of Fatou's question for transcendental entire functions:

Theorem (MP, Rempe and Waterman, 2022)

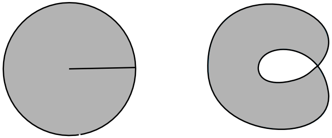
There exists a transcendental entire function with a bounded Fatou component whose boundary is a Lakes of Wada continuum.

The Fatou components of the function in the theorem are **wandering domains**; we may choose the function to have **infinitely many** wandering domains sharing the same boundary.

[MRW22] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering Lakes of Wada, and maverick points*, preprint arXiv:2108.10256

Boc Thaler's result

Let $U \subseteq \mathbb{C}$ be a bounded simply connected domain. Then, U is **regular** if $\mathrm{int}(\overline{U}) = U$.



Theorem (Boc Thaler, 2021)

Let U be a bounded simply connected domain such that U is regular and $\mathbb{C} \setminus \overline{U}$ is connected. Then there exists a transcendental entire function f for which U is a wandering domain.

Boc Thaler proved that being regular is a necessary condition.

Question (Boc Thaler, 2021)

Is it true that the closure of any bounded simply connected Fatou component of an entire function has a connected complement?

Our Lakes of Wada wandering domains provide a negative answer to this question.

[Boc21] L. Boc Thaler, *On the geometry of simply connected wandering domains*, Bull. Lond. Math. Soc. **53** (2021), no. 6, 1663–1673

Main result

Our result on the wandering Lakes of Wada is a corollary of the following result. Recall that a compact set $K \subseteq \mathbb{C}$ is **full** if $\mathbb{C} \setminus K$ has no bounded component (i.e. $K = \text{fill}(K)$).

Theorem (MP, Rempe and Waterman, 2022)

Let $K \subseteq \mathbb{C}$ be a full compact set. Then there exists a transcendental entire function f such that

(i) $\partial K \subseteq J(f)$;

(ii) $f^n(K) \cap f^m(K) = \emptyset$ *for* $n \neq m$;

(iii) *every connected component of* $\text{int}(K)$ *is a wandering domain of* f;

(iv) $f^n|_K \to \infty$ *uniformly as* $n \to \infty$.

This is a generalization of the result by Boc Thaler, and poses the following question.

Question (MP, Rempe and Waterman, 2022)

Suppose that U is a bounded simply connected Fatou component of a transcendental entire function, and let $K = \text{fill}(\overline{U})$. Is it true that $\partial U = \partial K$?

[MRW22] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering Lakes of Wada, and maverick points*, preprint arXiv:2108.10256

Sketch of the proof I

The proof uses the following classical result from approximation theory:

Theorem (Runge, 1885)

Let $K \subseteq \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. Suppose that $\varepsilon > 0$ and the function $\phi : K \to \mathbb{C}$ is analytic on a neighbourhood of K. Then, there exists a polynomial f such that

$$|f(z) - \phi(z)| \leqslant \varepsilon, \quad \text{for } z \in K.$$

To prove his result, Boc Thaler used (a modification of) a version of Runge's theorem due to Eremenko and Lyubich where you can prescribe the image of a finite number of points, but it turns out that this is not necessary.

[EL87] A. E. Eremenko and M. Yu. Lyubich, *Examples of entire functions with pathological dynamics*, J. London Math. Soc. (2) **36** (1987), no. 3, 458–46

[Gai87] D. Gaier, *Lectures on complex approximation*. Translated from the German by Renate McLaughlin. Birkhäuser Boston, Inc., Boston, MA, 1987. xvi+196 pp.

Sketch of the proof II

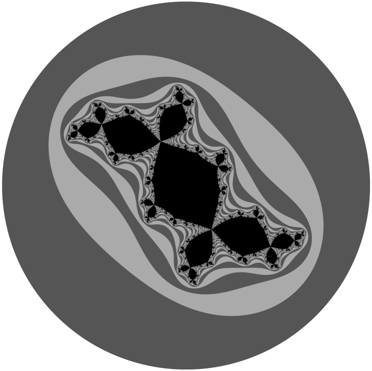
Let $K \subseteq \mathbb{C}$ be a **full compact set**.

We choose sequences (K_j) and (L_j) of compact sets with the following properties:

(a) $L_j \subseteq \text{int}(K_{j-1})$ and
$K_j \subseteq \text{int}(L_j)$ for all $j \geqslant 1$;

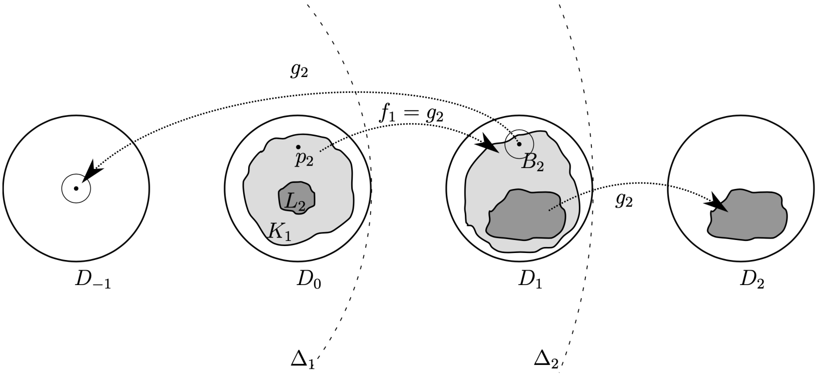
(b) $\bigcap K_j = \bigcap L_j = K$;

(c) Each of the sets L_j and K_j is the fill of a finite union of pairwise disjoint simple closed curves.



We choose a sequence (p_j) with $p_j \in K_{j-1} \setminus L_j$ for all $j \geqslant 1$ so that every point of ∂K is an accumulation point of (p_j).

Sketch of the proof III



The function f is constructed as the limit of a sequence of polynomials (f_n) obtained from Runge's theorem.

Meromorphic functions

For a **meromorphic function** $f : \mathbb{C} \to \widehat{\mathbb{C}}$, we define

- the **Fatou set** $F(f)$ is the largest open set on which the iterates $\{f^n\}$ **are defined and** form a normal family;
- the **Julia set** $J(f)$ is given by $J(f) := \mathbb{C} \setminus F(f)$.

We can then define wandering domains in the same way.

Observe that for general meromorphic functions, multiply connected Fatou components are not special and need not be wandering domains!

[Ber93] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 151-188

Characterisation of bounded Fatou components

For meromorphic functions, we are able to give a complete characterisation of which **bounded** sets are Fatou and Julia components.

Theorem (MP, Rempe and Waterman, 2022)

Let $U \subset \mathbb{C}$ be a bounded domain. The following are equivalent:

(a) *U is regular;*

(b) *there is a meromorphic function f such that U is a Fatou component of f.*

Note that Boc Thaler proved that wandering domains of transcendental entire functions are regular but actually this holds more generally for every Fatou component which is not equal to the whole Fatou set.

[MRW22b] D. Martí-Pete, L. Rempe and J. Waterman, *Bounded Fatou and Julia components of meromorphic functions*, preprint arXiv:2204.11781

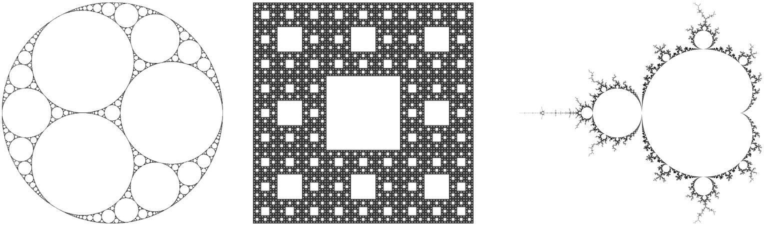
Characterisation of bounded Julia components

Theorem (MP, Rempe and Waterman, 2022)

Let $X \subset \mathbb{C}$ be a continuum. The following are equivalent:

(a) *X has empty interior;*

(b) *there is a meromorphic function f such that X is a Julia component of f.*



Note that here X is a **component** of $J(f)$ rather than just a **subset**.

[MRW22b] D. Martí-Pete, L. Rempe and J. Waterman, *Bounded Fatou and Julia components of meromorphic functions*, preprint arXiv:2204.11781

PART II: DYNAMICS

Internal dynamics and
dynamics on the boundaries of
wandering domains

Internal dynamics

Benini, Evdoridou, Fagella, Rippon and Stallard used inner functions to classify the internal dynamics of **simply connected** wandering domains in terms of the following quantities:

- the distances between the iterates of **two points** (contracting, semi-contracting or eventually isometric)
- the distance of the iterates of a point to the **boundary** (away, bungee or converging)

which give rise to 9 cases (escaping) and 6 cases (oscillating) and they proved that all of these are realised.

Bergweiler, Rippon and Stallard proved that **multiply connected** wandering domains of entire functions contain large absorbing annuli (see Ferreira for meromorphic functions).

[BEFRS22] A. M. Benini, V. Evdoridou, N. Fagella, P. J. Rippon, G. M. Stallard, *Classifying simply connected wandering domains*, to appear in Math. Ann. (published online)

[ERS22] V. Evdoridou, P. J. Rippon, G. M. Stallard, *Oscillating simply connected wandering domains*, to appear in Ergodic Theory Dynam. Syst. (published online)

[BRS13] W. Bergweiler, P. J. Rippon and G. M. Stallard, *Multiply connected wandering domains of entire functions*, Proc. London Math. Soc. **107**, 1261-1301

[Fer21] G. R. Ferreira, *Multiply connected wandering domains of meromorphic functions: the pursuit of uniform internal dynamics*, preprint arXiv:2111.05007

Maverick points

Recall that for a transcendental entire function f, we can define the sets

- $I(f) := \{z \in \mathbb{C}\colon f^n(z) \to \infty$ as $n \to \infty\}$ (**escaping set**);
- $K(f) := \{z \in \mathbb{C}\colon \exists R > 0,\ |f^n(z)| < R$ for all $n \in \mathbb{N}\}$ (**bounded orbits**);
- $BU(f) := \mathbb{C} \setminus (I(f) \cup K(f))$ (**bungee set**).

If U is a wandering domain of f, by normality U is contained in one of these 3 sets.

Definition

Let f be a transcendental entire function with a wandering domain U. We say that $z \in \partial U$ is a **maverick point** of U if f there is a sequence (n_k) such that

$$f^{n_k}(z) \to w \in \widehat{\mathbb{C}} \quad \text{as } k \to \infty,$$

but w is not a limit function of $(f^{n_k}(U))$.

Note that multiply connected wandering domains have no maverick points.

[MRW22] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering Lakes of Wada, and maverick points*, preprint arXiv:2108.10256

[BEFRS22] A. M. Benini, V. Evdoridou, N. Fagella, P. J. Rippon, G. M. Stallard, *The Denjoy–Wolff set for holomorphic sequences, non-autonomous dynamical systems and wandering domains*, preprint arXiv:2203.06235

Most boundary points are not maverick

Theorem (Rippon and Stallard, 2011)

Let f be a transcendental entire function and let U be an escaping wandering domain. Then the set of non-escaping points in ∂U has harmonic measure 0 with respect to U.

Theorem (Osborne and Sixsmith, 2016)

Let f be a transcendental entire function and let U be an oscillating wandering domain. Then the set of points whose ω-limit set

$$\omega(z,f) = \bigcap_{n\in\mathbb{N}} \overline{\{f^k(z)\colon k>n\}}$$

differs with the ω-limit set of points in U has 0 harmonic measure with respect to U.

Theorem (MP, Rempe and Waterman, 2022)

Let f be a transcendental entire function and let U be a wandering domain. The set of maverick points in ∂U has harmonic measure 0 with respect to U.

[RS11] P. J. Rippon and G. M. Stallard, *Boundaries of escaping Fatou components*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2807–2820

[OS16] J. W. Osborne and D. J. Sixsmith, *On the set where the iterates of an entire function are neither escaping nor bounded*, Ann. Acad. Sci. Fenn. Math. **41** (2016), 561-578

[MRW22] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering Lakes of Wada, and maverick points*, preprint arXiv:2108.10256

Rippon's question

Question (Rippon, 2019)

Let f be a transcendental entire function. If U is a bounded escaping wandering domain of f, is $\partial U \subseteq I(f)$?

The following result gives a negative answer to this question:

Theorem (MP, Rempe and Waterman, 2022)

There exists a transcendental entire function f with an escaping or oscillating wandering domain U such that the set of maverick points contains a continuum of positive Lebesgue measure.

[HL19] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory, Fiftieth Anniversary Edition*, Problem Books in Mathematics, Springer, Cham, 2019

[MRW22] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering Lakes of Wada, and maverick points*, preprint arXiv:2108.10256

The general result

Theorem (MP, Rempe and Waterman, 2022)

Let $K \subseteq \mathbb{C}$ be a full compact set. Let $Z_{\mathrm{I}}, Z_{\mathrm{BU}} \subset K$ be disjoint finite or countably infinite sets such that no connected component of $\mathrm{int}(K)$ *contains more than one point of $Z_{\mathrm{I}} \cup Z_{\mathrm{BU}}$. Then there exists a transcendental entire function f such that*

(i) $\partial K \subseteq J(f)$;

(ii) $f^n(K) \cap f^m(K) = \emptyset$ *for* $n \neq m$;

(iii) *every connected component of* $\mathrm{int}(K)$ *is a wandering domain of* f;

(iv) $Z_{\mathrm{I}} \subset I(f)$ *and* $Z_{\mathrm{BU}} \subset BU(f)$.

[MRW22] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering Lakes of Wada, and maverick points*, preprint arXiv:2108.10256

Summarizing...

- Wandering domains can be constructed using a wide range of **techniques** and they are related to many **problems** in transcendental dynamics.
- For transcendental entire functions, **simply connected** wandering domains appear to be much more diverse in terms of their topology and dynamics than multiply connected wandering domains.
- For **meromorphic functions**, we have a complete characterisation of which sets are *bounded* wandering domains.
- There are many open questions regarding the set of **maverick points** and, more generally, the relationship between internal dynamics and boundary dynamics of wandering domains.

Thank you for your attention!

Dziękuję bardzo za uwagę!