Symmetries of Julia Sets

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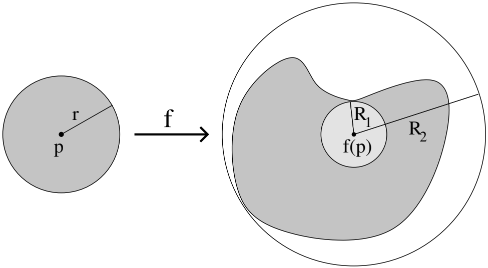
Quasisymmetric deformations

$f\colon\ X \to Y$ quasisymmetry if $\exists\, \eta\colon\ [0,\infty) \to [0,\infty)$ distortion function

$$\frac{d_Y(f(o),f(p))}{d_Y(f(o),f(q))} \leq \eta\left(\frac{d_X(o,p)}{d_X(o,q)}\right).$$



$$R_2/R_1 \leq Const.$$

Carpet Julia set

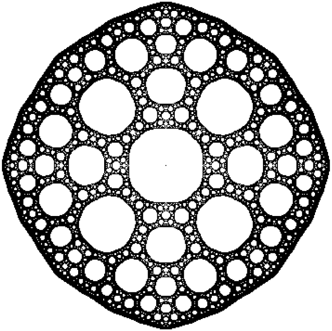


Figure: Sierpiński carpet Julia set, $f(z) = z^2 - \frac{1}{16z^2}$.

Carpet Julia sets

Theorem (M. Bonk, M. Lyubich, S. M.'2016)

$f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *post-critically finite rational*
$\mathcal{J}$ *Sierpiński carpet*
$\xi\colon \mathcal{J} \to \mathcal{J}$ *quasisymmetry*
$\Rightarrow \xi \in$ *Möb.*

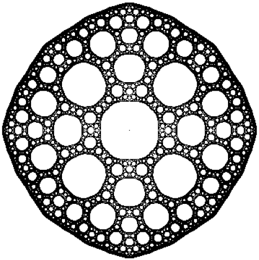


Figure: $f(z) = z^2 - \frac{1}{16z^2}$.

Quasisymmetric non-equivalence

Corollary

f post-critically finite
$\mathcal{J}$ Sierpiński carpet
$\Rightarrow QS(\mathcal{J})$ finite.

Corollary

f post-critically finite
G Kleinian group
$\Rightarrow \mathcal{J} \not\sim_{qs} \Lambda(G)$.

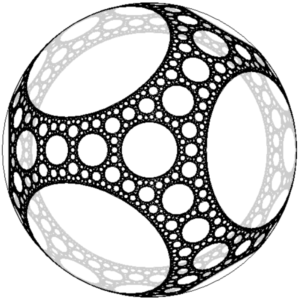


Figure: Sierpiński carpet limit set $\Lambda(\pi_1(M^3))$.
http://www.math.harvard.edu/ctm/gallery/

The Sierpiński carpet

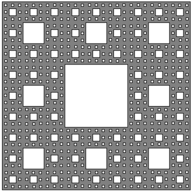


Figure: The Sierpiński carpet S_3.

Theorem (M. Bonk, S. M.'2013)

$\xi: \ S_3 \to S_3$ quasisymmetry
$\Rightarrow \xi$ isometry.

Corollary

$\Rightarrow \mathcal{S}_3 \not\sim_{qs} \Lambda(G), \mathcal{J}(-1/16)$.

Tree-like Julia set

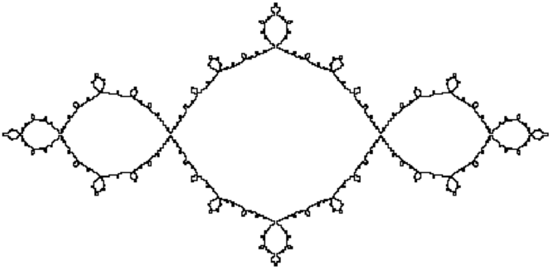


Figure: Basilica Julia set, $f(z) = z^2 - 1$.

$T\colon\ \mathcal{J} \to \mathcal{J}$ Thompson's group action; $\quad \iota\colon\ \mathcal{J} \to \mathcal{J}$ inversion.

Thompson's group T

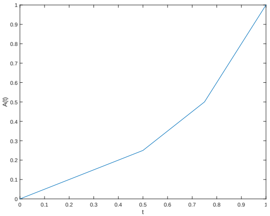


Figure: A.

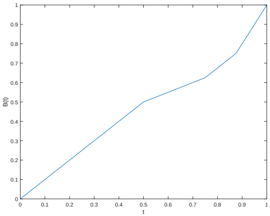


Figure: B.

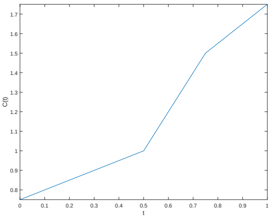
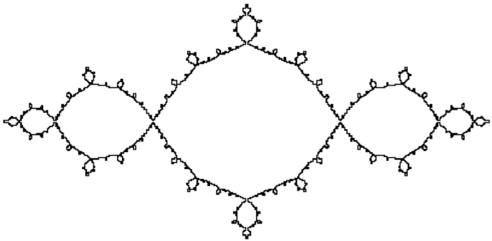


Figure: C.

$\hat{T}$ group action

$$A = \begin{cases} f^{-2}, & 0 \le t \le 1/2, \quad f^{-2}(1) = 1, \\ f^{-4} \circ f^{4}, & 1/2 \le t \le 3/4, \quad f^{-4} \colon (0,1) \to (1/4, 1/2), \\ f^{2}, & 3/4 \le t \le 1. \end{cases}$$

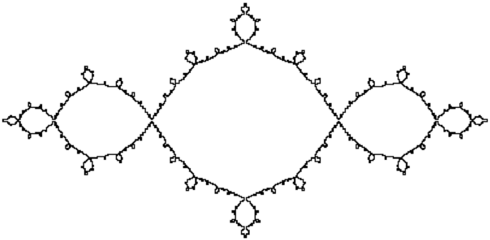
$$\iota = \begin{cases} f, & L, \\ f^{-1}, & R. \end{cases}$$



$\hat{T}$-closure

Theorem (M. Lyubich, S. M.'2018)

$\forall\,\eta$ *distortion function* $\exists\,\eta'$
$\forall\,\xi:\ \mathcal{J} \to \mathcal{J}$ η*-quasisymmetry, topologically extendable*
$\Rightarrow \exists\,\xi_n \in \hat{T} = \langle T, \iota \rangle$ η'*-quasisymmetries,* $\xi_n \to \xi$.



Other polynomial Julia sets

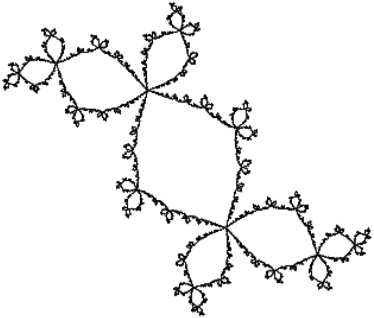


Figure: Douady rabbit.

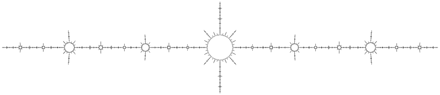


Figure: Airplane.

Gasket Julia set

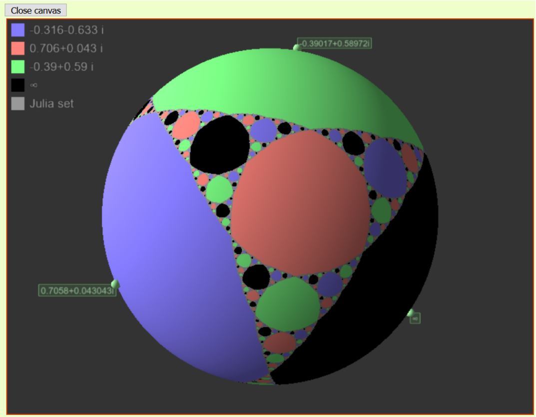


Figure: Tetrahedral Julia set, $\bar{f}(z) = \frac{2z^2}{3z^3+1}$.

Circle packings

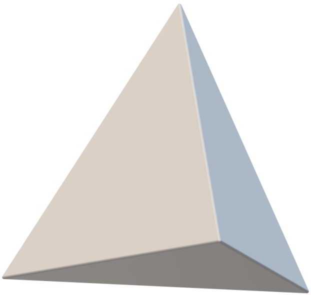


Figure: Tetrahedral triangulation.

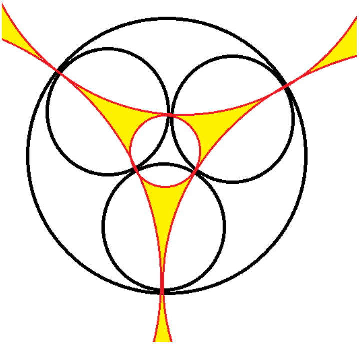


Figure: Tetrahedral circle packing, dual.

Limit sets

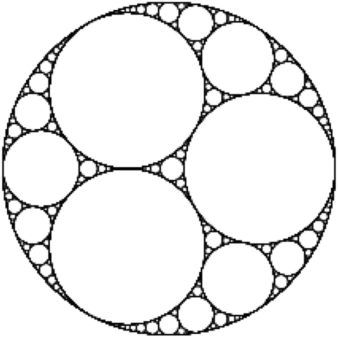


Figure: Apollonian gasket Λ.

Branched coverings

$$F \approx \begin{cases} N, & \text{white triangular interstice,} \\ \bar{z}^{d-1}, & \text{white disc.} \end{cases}$$

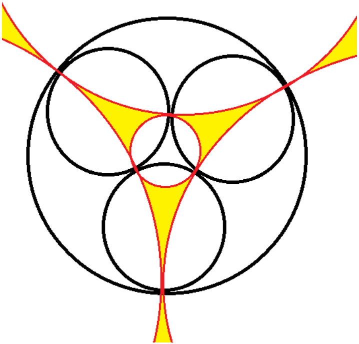


Figure: Tetrahedral branched covering.

Gasket Julia set

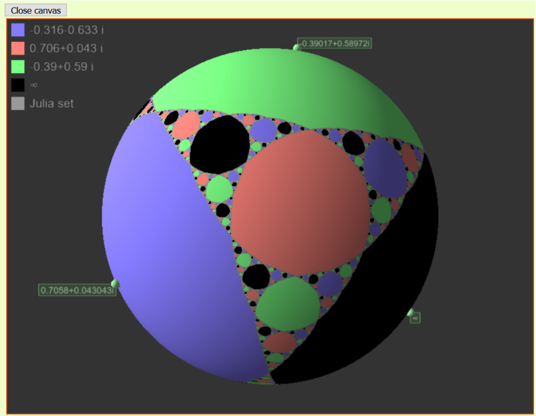


Figure: Tetrahedral Julia set, $\bar{f}(z) = \frac{2z^2}{3z^3+1}$.

Reduced triangulations

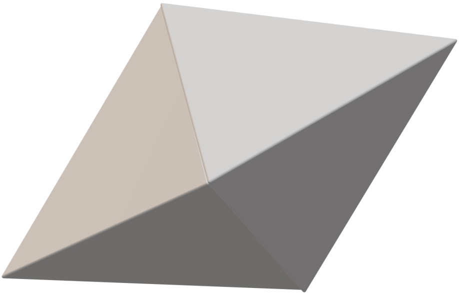


Figure: Double tetrahedron.

Reduced/non-reduced circle packings

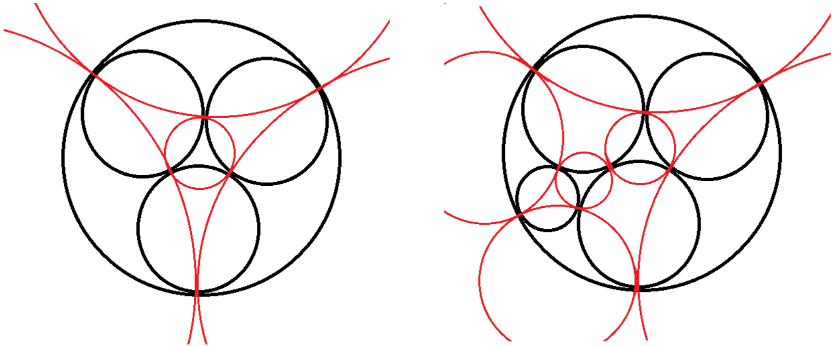


Figure: Reduced and non-reduced Apollonian triangulaions.

Reduced/non-reduced Julia sets

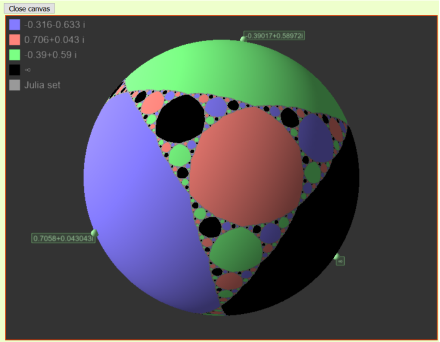


Figure: Tetrahedral Julia set.

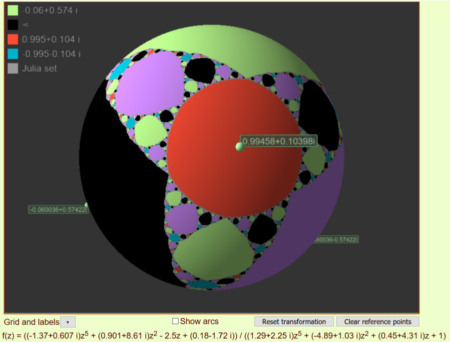
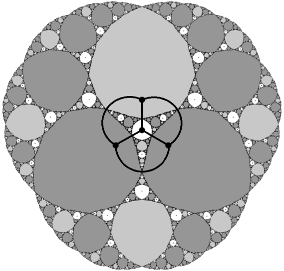


Figure: Double tetrahedron Julia set.

Quasisymmetry groups

Theorem (R. Lodge, M. Lyubich, S. M., S. Mukherjee'2019)
T reduced $\Rightarrow$

$$QS(\mathcal{J}) = \mathrm{Homeo}(\mathcal{J}) \approx \mathrm{Homeo}(\Lambda) \approx \mathrm{Aut}^T(\hat{\mathbb{C}}) \ltimes G.$$



$$g_\Delta = \begin{cases} f, & \Delta, \\ f^{-1}, & \mathcal{J} \setminus \Delta. \end{cases}$$

No qc conjugacy

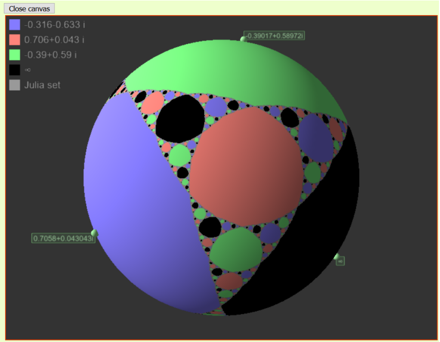


Figure: Tetrahedral Julia set.

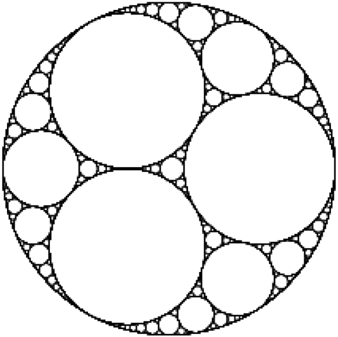


Figure: Apollonian gasket.

David maps

$U, V \subseteq \hat{\mathbb{C}}$
$H \colon U \to V$ o.p. homeo
H David if $W^{1,1}_{loc}(U)$ and $\exists\, \alpha, \epsilon_0, C > 0$,

$$\sigma\{|\mu_H| \geq 1 - \epsilon\} \leq Ce^{-\alpha/\epsilon}, \quad \forall\, 0 < \epsilon < \epsilon_0.$$

Here $\mu_H = H_{\bar{z}} / H_z$.
Equivalently,

$$\sigma\{K_H \geq K\} \leq C' e^{-\alpha' K}, \quad \forall\, K > K_0,$$

where $K_H = \frac{1+|\mu_H|}{1-|\mu_H|}$.
Equivalently,

$$\int_U \exp(pK_H)\, d\sigma < \infty, \quad \exists\, p > 0.$$

Ford circles

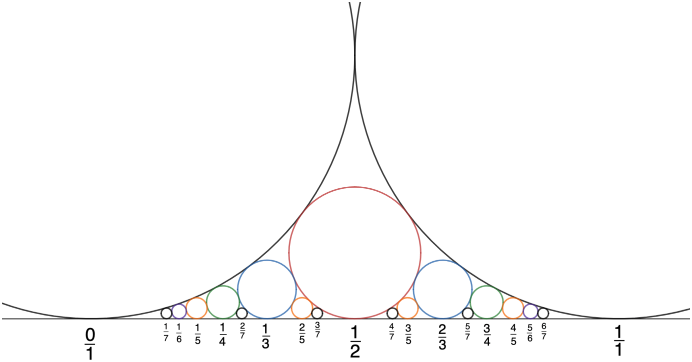


Figure: Ford circles.

$$\left| \frac{p}{q} - \frac{r}{s} \right| = \frac{1}{qs}.$$

Scalewise distortion

$$\rho_h(t) = \max_x \rho_h(x,t),$$

where

$$\rho_h(x,t) = \max\left\{\frac{h(x+t)-h(x)}{h(x)-h(x-t)}, \frac{h(x)-h(x-t)}{h(x+t)-h(x)}\right\}.$$

Angle doubling vs. Farey sequences:

$$\frac{1}{Cn} \leq \left|\frac{h(x \pm 1/2^n) - h(x)}{h(x) - h(x \mp 1/2^n)}\right| \leq Cn.$$

Theorem (J. Chen, Z. Chen, and C. He'1996; S. Zakeri'2008)

$\rho_h(t) = O\left(\log \frac{1}{t}\right), \quad t \to 0+$

$\Rightarrow$ *h has David extension.*

David conjugacy

Theorem (M. Lyubich, S. M., S. Mukherjee, D. Ntalampekos'2020)

f critically fixed anti-rational; Γ reflection group
Tischler graph dual to contact graph $\Rightarrow$ $\exists\, h$ David $h \circ f|_{\mathcal{J}} = N|_{\Lambda} \circ h$.

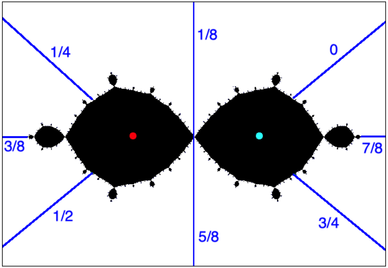


Figure: Julia set of $f(z) = \bar{z}^3 - \frac{3i}{2}\bar{z}$.

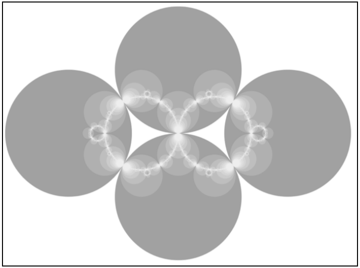


Figure: Limit set.

Mating anti-polynomials and reflection groups

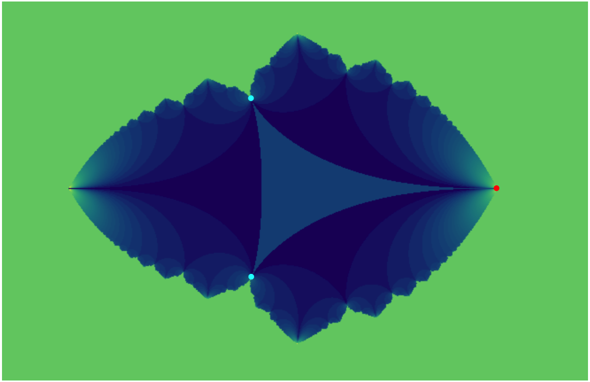


Figure: $\bar{z}^2 + \frac{1}{4}$ and triangle reflection group.
Picture courtesy: Seung-Yeop Lee

Schwarz reflection map

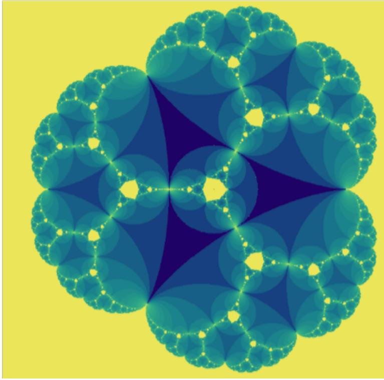


Figure: Tetrahedral J and triangle reflection in 3 components.
Picture courtesy: Seung-Yeop Lee