David homeomorphisms in analysis and dynamics

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On geometric complexity of Julia sets - IV
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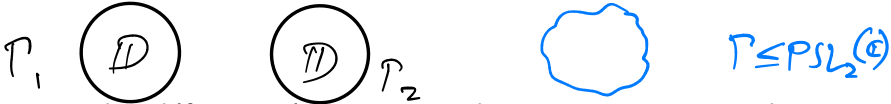
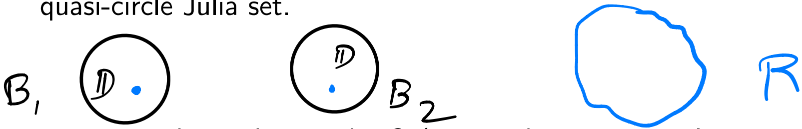
Quasi-Fuchsian and quasi-Blaschke

- Bers' simultaneous uniformization theorem: can combine two Fuchsian group actions on $\mathbb{D}$, and obtain a unique quasi-Fuchsian group.



- Using the Ahlfors-Beurling extension theorem, one can prove that two Blaschke products of the same degree, each having an attracting fixed point in $\mathbb{D}$, can be mated to obtain a unique rational map with a quasi-circle Julia set.



- How to go beyond expanding? (Haissinsky gave partial asnwers.)

- How to combine the above two operations?

David homeomorphisms

- $U, V \subset \widehat{\mathbb{C}}$, σ spherical measure.
- $H : U \to V$ o.p. homeo is *David* if $H \in W^{1,1}_{\mathrm{loc}}(U)$ and $\exists \alpha, C, \varepsilon_0 > 0$ such that

$$\sigma(\{|\mu_H| \geq 1 - \varepsilon\}) \leq Ce^{-\alpha/\varepsilon}, \quad \varepsilon \leq \varepsilon_0, \tag{1}$$

where $\mu_H = H_{\bar{z}}/H_z$.

Theorem (David Integrability Theorem)

Let $\mu\colon \widehat{\mathbb{C}} \to \mathbb{D}$ be a David coefficient; i.e., μ is a measurable function satisfying Condition (1)*.*

Then there exists a homeomorphism $H\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of class $\in W^{1,1}(\widehat{\mathbb{C}})$ that solves the Beltrami equation

$$H_{\bar{z}} = \mu H_z.$$

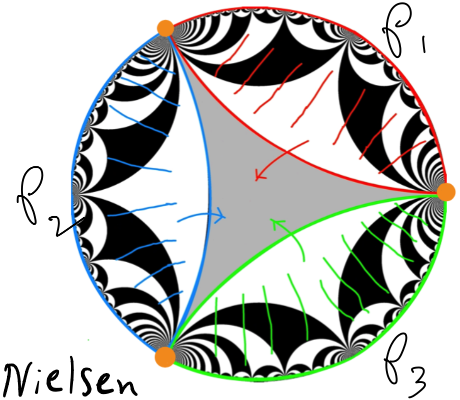
Moreover, H is unique up to postcomposition with Möbius transformations.

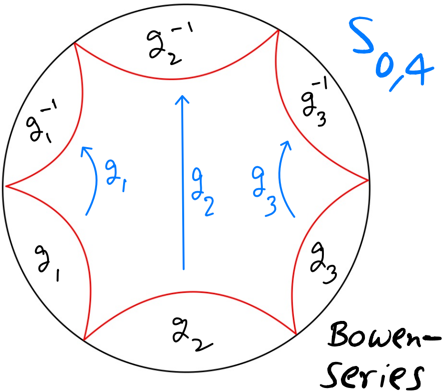
Maps orbit equivalent to groups

To address the incompatibility of group dynamics vs semigroup dynamics of maps, associate a piecewise (anti-)Möbius map $A : \mathbb{S}^1 \to \mathbb{S}^1$ to a Fuchsian/reflection group Γ that is

1) topologically z^d or $\overline{z}^d$, and

2) **orbit equivalent** to Γ, i.e., $\Gamma.x =$ Grand orbit of x under A, $\forall x \in \mathbb{S}^1$.





Special case of a David extension theorem

Theorem (Lyubich-Merenkov-Ntalampekos-M)

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a piecewise analytic, C^1, expansive covering map of degree d (with $|d| \geq 2$) such that the pieces of f satisfy a 'complex Markov property'.

Then there exists an orientation-preserving homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ that conjugates the map $z \mapsto z^d$ or $z \mapsto \overline{z}^d$ to f and continuously extends to a David homeomorphism of $\mathbb{D}$.

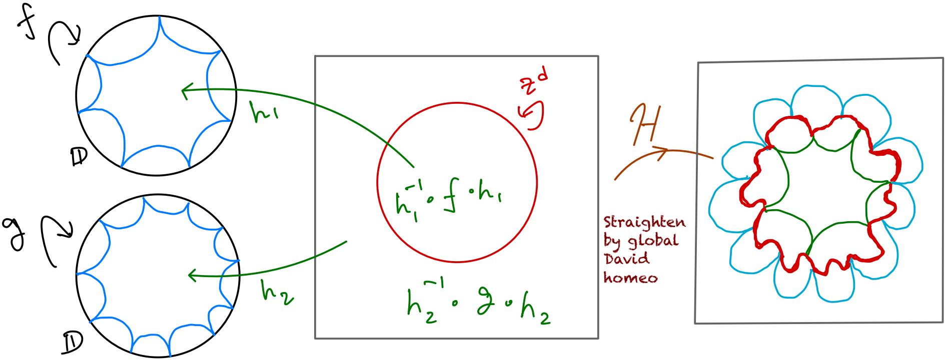
- The above result, combined with the David Integrability Theorem, gives a unified approach to
  - turn hyperbolic (anti-)rational maps to parabolic ones,
  - construct Kleinian reflection groups from critically fixed anti-rational maps,
  - combine (anti-)polynomials and Fuchsian/reflection groups to produce hybrid dynamical systems, and
  - construct Bullett-Penrose type correspondences (not today).

Mating piecewise analytic circle maps

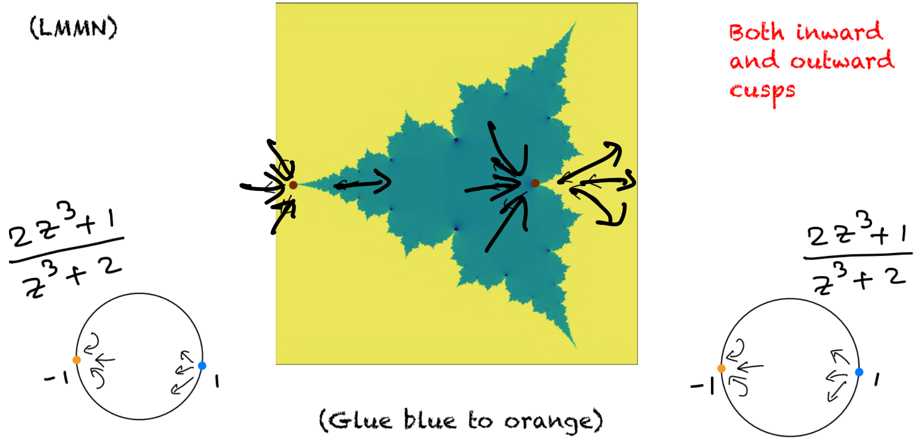
Theorem (Lyubich-Merenkov-M-Ntalampekos)

Let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be two piecewise analytic, C^1, expansive covering maps of the same degree d (with $d \geq 2$) such that the pieces of f and g satisfy a 'complex Markov property'.

Then the piecewise extensions of f and g (to subsets of $\mathbb{D}$) are conformally mateable.



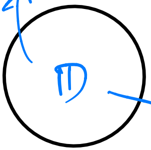
Doubly cusped conformally removable Julia set

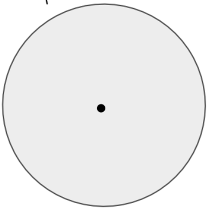


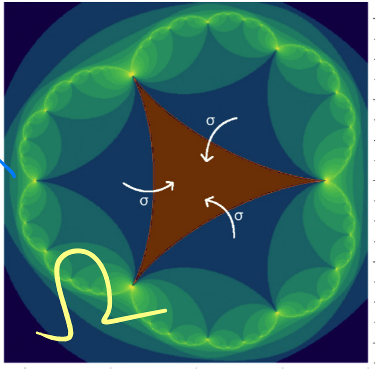
- The above mating is a cubic rational map with two parabolic fixed points.
- More generally, a connected Julia set of a geometrically finite rational map with a completely invariant Fatou component is conf. removable.

Deltoid reflection as a mating

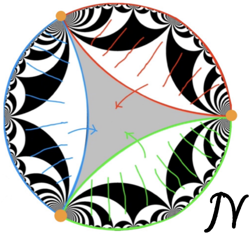
Mating must be a Schwarz reflection





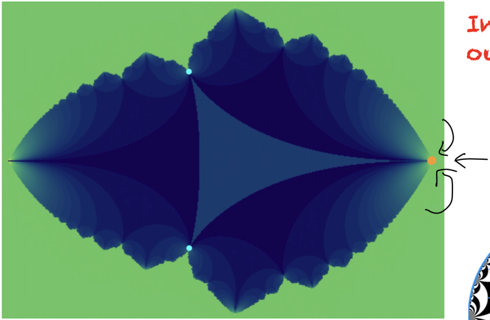


Ideal triangle group



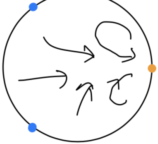
Mating ideal triangle with cauliflower

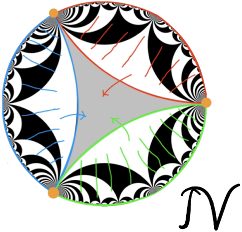
(LMMN)



Inward cusps and outward sectors

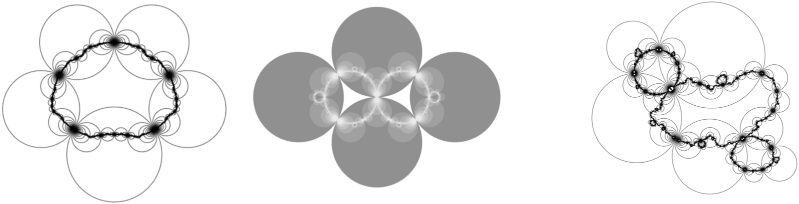
$$\frac{3\bar{z}^2+1}{\bar{z}^2+3}$$





Necklace reflection groups

- Necklace reflection groups: Closure of the Bers slice of an ideal polygon reflection group.



- Necklace groups are in bijection with critically fixed anti-polynomials.

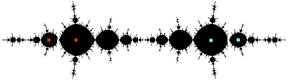
- Their limit sets are conformally removable.

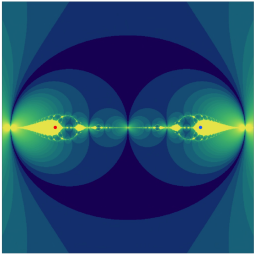
Mating anti-polynomials with necklace groups

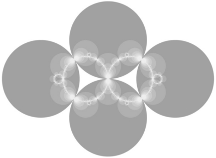
Theorem (Lyubich-Merenkov-Ntalampekos-M)

Let P be a post-critically finite, hyperbolic anti-polynomial of degree d, and Γ be a necklace group of rank $d+1$. Then, P and $\mathcal{N}_\Gamma$ are conformally mateable $\iff$ they are Moore-unobstructed (i.e., they are topologically mateable).

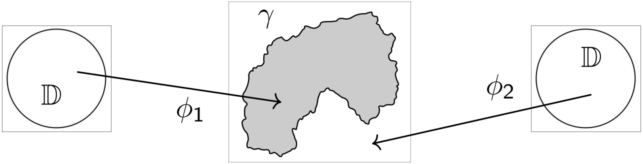
Moreover, the conformal mating is a piecewise Schwarz reflection map.







Conformal welding



- $\phi_2^{-1} \circ \phi_1 : \mathbb{S}^1 \to \mathbb{S}^1$ is called the *welding map* of the Jordan curve γ.
- It is well-known that quasisymmetric circle homeomorphisms are welding maps for unique Jordan curves.

Theorem (Lyubich-Merenkov-M-Ntalampekos)

Let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be two piecewise analytic, C^1, expansive covering maps of the same degree d (with $d \geq 2$) such that the pieces of f and g satisfy a 'complex Markov property'.

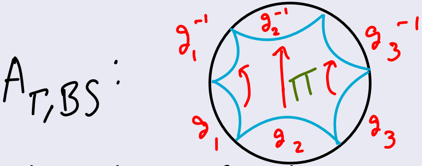
Then, f and g are topologically conjugate, and any conjugacy is a welding map. Moreover, the associated Jordan curve is conformally removable, and hence the welding solution is unique.

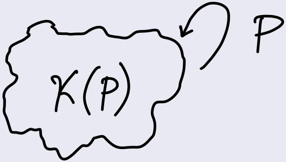
Mating polynomials with Fuchsian punctured sphere groups

Theorem (M-Mj)

Let $\Gamma \in \mathrm{Teich}(S_{0,d+1})$ and $P \in \mathcal{H}_{2d-1}$ (where $\mathcal{H}_{2d-1}$ is the principal hyperbolic component of degree $2d-1$ polynomials).

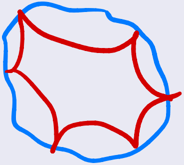
Then the Bowen-Series map $A_{\Gamma,\mathrm{BS}} : \overline{\mathbb{D}} \setminus \operatorname{int} \Pi \to \overline{\mathbb{D}}$ can be conformally mated with $P : \mathcal{K}(P) \to \mathcal{K}(P)$.





The resulting conformal mating $F : \Omega \to \widehat{\mathbb{C}}$ is given by:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{R} & \Omega \\ \big\downarrow {\scriptstyle 1/z} & & \big\downarrow {\scriptstyle F} \\ \widehat{\mathbb{C}} & \xrightarrow{R} & \widehat{\mathbb{C}} \end{array}$$



where $\mathcal{D}$ is a Jordan domain that is mapped inside out by $1/z$, and R is a rational map univalent on $\mathcal{D}$ with $\Omega = R(\mathcal{D})$.

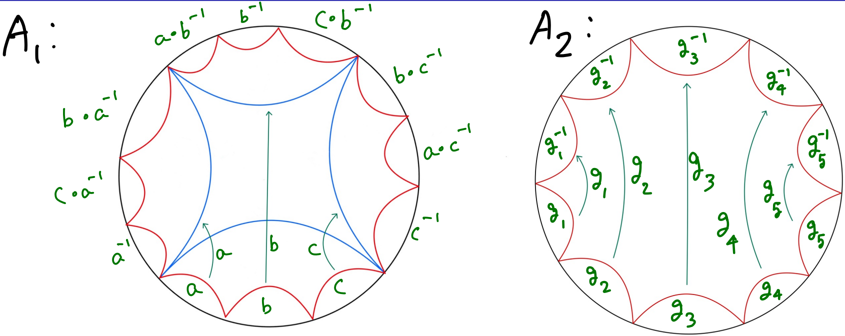
Mating polynomials with Fuchsian punctured sphere groups

- The parameter space of matings produced by the previous theorem

$$= \mathrm{Teich}(S_{0,d+1}) \times \mathcal{H}_{2d-1}.$$

- Questions:
  - Describe compactifications of various Bers slices in the above space of matings.
  - Study continuity/discontinuity of boundary extensions of conformal isomorphisms between Bers slices.
  - Where is the HD of the limit set minimized?
  - Does the HD of the limit set vary real-analytically?
  - Is the variation of the HD related to natural Riemannian metrics on the moduli space?

Mating groups in different Teichmüller spaces



- Both A_1, A_2 are piecewise Möbius, C^1, expansive degree 9 covering of $\mathbb{S}^1$ satisfying the complex Markov property.
- A_1 is orbit equivalent to $S_{0,4}$, while A_2 is orbit equivalent to $S_{0,6}$
- Conformal matings of A_1, A_2 are parametrized by $\mathrm{Teich}(S_{0,4}) \times \mathrm{Teich}(S_{0,6})$.
- Questions: What is the space of matings?

Thank you!