Definitions of quasiconformality and exceptional sets

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On geometric complexity of Julia sets - IV

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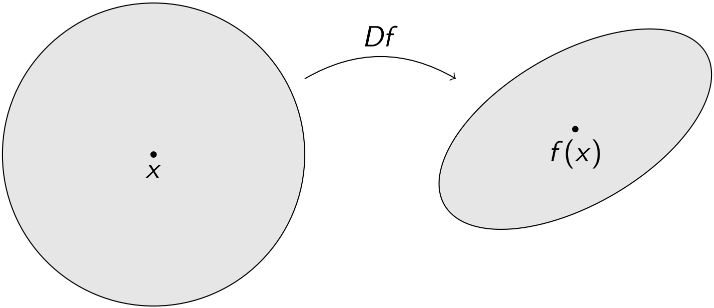
Classical definitions of quasiconformality

$\Omega \subset \mathbb{R}^n$ open set

$f\colon \Omega \to \mathbb{R}^n$ orientation-preserving topological embedding

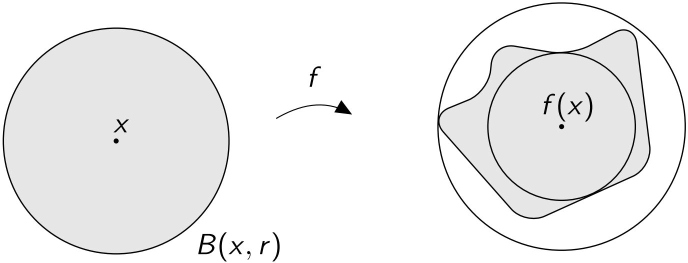
Definition (Analytic definition)

f is K-quasiconformal if $f \in W^{1,n}_{\mathrm{loc}}(\Omega)$ and $\|Df\|^n \leq KJ_f$ a.e. in Ω.



Metric definition

$$L_f(x,r) = \sup\{|f(x)-f(y)| : |x-y| \le r\}$$
$$l_f(x,r) = \inf\{|f(x)-f(y)| : |x-y| \ge r\}$$
$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)}$$



Theorem (Gehring 1960)

f is quasiconformal $\iff \limsup_{r\to 0} H_f(x,r) \le H$ *for* **every** $x \in \Omega$.

Liminf metric definition

Theorem (Heinonen–Koskela 1995)

f is quasiconformal $\iff \liminf_{r\to 0} H_f(x,r) \leq H$ *for every* $x \in \Omega$.

We only need to check quasiconformality at a **sequence** of scales!

Applications in rigidity problems in complex dynamics:

Przytycki–Rohde, Graczyk–Smirnov, Haïssinsky, Kozlovski-Shen-van Stiren, Smania

Question

When is a topological conjugacy between dynamical systems (quasi)conformal?

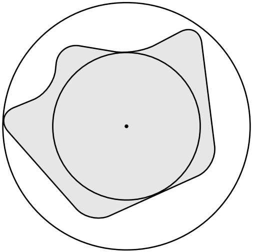
Generalized metric definition

Question

What if we require that sets of bounded eccentricity are mapped to sets of bounded eccentricity?

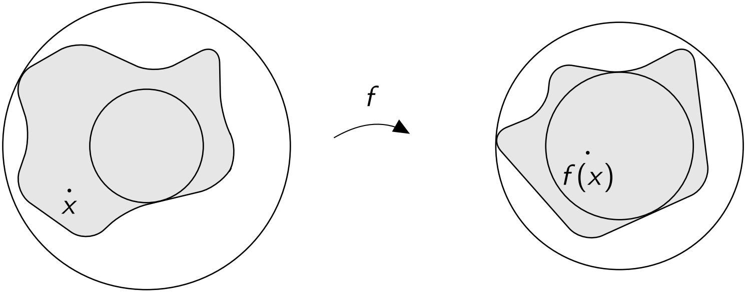
$A \subset \mathbb{R}^n$ bounded open set

$E(A) = \inf\{M \geq 1 : \text{there exists a ball } B \text{ such that } B \subset A \subset MB\}$



Generalized metric definition

$$E_f(x,r) = \inf\{M \geq 1 : \text{there exists an open set } A \text{ with } x \in A$$
$$\operatorname{diam}(A) \leq 2r,\ E(A) \leq M,\ E(f(A)) \leq M\}$$



Theorem (N. 2021)

f is quasiconformal $\iff \lim_{r\to 0} E_f(x,r) \leq H$ *for every* $x \in \Omega$.

Equivalently: there exist of sets A_n, $f(A_n)$ of uniformly bounded eccentricity with $A_n \to x$ and $f(A_n) \to f(x)$.

Exceptional/removable sets

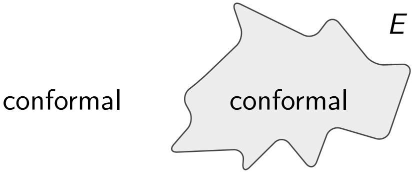
Question

Do we need to assume that the limit of $H_f(x,r)$ or $E_f(x,r)$ is uniformly bounded at all x?

We cannot remove a set of measure zero! **Too large!**

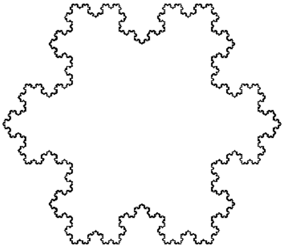
Definition

A **closed** set E is **(quasi)conformally removable** if every homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that is (quasi)conformal outside E is (quasi)conformal everywhere.



Removable sets

- Sets of σ-finite $(n-1)$-measure ($n = 2$: Besicovitch '31; $n > 2$: Gehring '60, Kallunki–Koskela '00, N. '21)
- Quasicircles
- Boundaries of John/Hölder domains (Jones '91, Jones–Smirnov '00, N. '21)
- *NED* sets (Negligible for Extremal Distance) (Ahlfors–Beurling '50)



NED sets

Γ family of paths in $\mathbb{R}^n$

$\rho\colon \mathbb{R}^n \to [0,\infty]$ Borel function

ρ is admissible for Γ if $\int_\gamma \rho\, ds \geq 1$ for all rectifiable $\gamma \in \Gamma$.

$$\operatorname{Mod}_n \Gamma = \inf_{\rho} \int \rho^n$$

Definition

A closed set $E \subset \mathbb{R}^n$ is *NED* if

$$\operatorname{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \operatorname{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n \setminus E)$$

for every pair of disjoint continua $F_1, F_2 \subset \mathbb{R}^n \setminus E$.

Definition

A set $E \subset \mathbb{R}^n$ is *NED* if

$$\operatorname{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \operatorname{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_0(E))$$

CNED sets

$\mathcal{F}_\sigma(E)$= curves in $\mathbb{R}^n$ intersecting E at countably many points

CNED= Countably Negligible for Extremal Distances

Definition

A set $E \subset \mathbb{R}^n$ is *CNED* if

$$\mathrm{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \mathrm{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_\sigma(E))$$

for every pair of disjoint continua $F_1, F_2 \subset \mathbb{R}^n$.

Theorem (N. 2021)

Suppose that $E \in CNED$ *and*

$$\lim_{r \to 0} E_f(x, r) \leq H$$

for $x \notin E$. *Then* f *is quasiconformal.*

Examples of CNED sets

All previous examples:

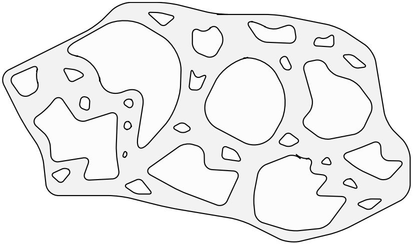
- Rectifiability: Sets of σ-finite $(n-1)$-measure
- Geometry: Quasicircles, boundaries of John/Hölder domains
- Potential theory: *NED* sets

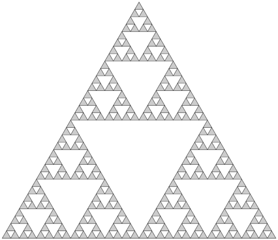
New examples:

- Non-measurable sets can be *CNED* (Sierpiński 1920)
- Unions of closed *CNED* sets

Examples of non-*CNED* sets

- Sets of positive area
- $C \times [0,1]$
- Sierpiński carpets (also non-removable N. 2019)
- Sierpiński gasket (also non-removable N. 2019)





Unions of CNED sets

Question

Is the union of two removable closed sets removable?

Yes for disjoint sets (trivial)
Yes for Cantor sets and quasicircles (Younsi 2016)

Question

Is the union of two CNED sets CNED?

Theorem (N. 2021)

Suppose that E_i is **closed** *and $E_i \in (C)NED$ for each $i \in \mathbb{N}$. Then*

$$\bigcup_{i\in\mathbb{N}} E_i \in (C)NED.$$

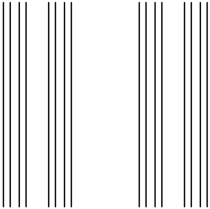
The union can be dense in $\mathbb{R}^n$!

Unions of CNED sets

Theorem (N. 2021)

There exist **Borel** *sets* $E_1, E_2 \in NED$ *such that* $E_1 \cup E_2 \notin CNED$.

In fact, $E_1 \cup E_2 = C \times [0,1]$.



$E_1 =$ countable union of NED Cantor sets $\Rightarrow NED$

$E_2 =$ Borel set whose projections to axes have measure zero $\Rightarrow NED$

Open problems

Problem

Removable sets coincide with CNED sets.

Implications:

- Union of two removable closed sets is removable.
- Removability is a local condition.
- Removable closed sets coincide with removable sets for continuous $W^{1,2}$ functions.

Thank you!