THE MAXIMUM MODULUS SET OF AN ENTIRE MAP

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Bedlewo, On geometric complexity of Julia sets IV

Let $f: \mathbb{C} \to \mathbb{C}$ be an entire map. As usual,

$$M(r, f) := \max_{|z|=r} |f(z)|, \quad \text{for } r \geq 0$$

is the maximum modulus.

Definition

The set of points where f attains its maximum modulus is its **maximum modulus set**. That is,

$$\mathcal{M}(f) := \{z \in \mathbb{C} \colon |f(z)| = M(|z|, f)\}.$$

Remark: If f is a monomial, then $\mathcal{M}(f) = \mathbb{C}$. We ignore this case.

Goal: Historical overview of results + open questions.

- Blumenthal's pioneering work
- Discontinuities of $\mathcal{M}(f)$
- Singletons in $\mathcal{M}(f)$
- Structure near the origin
- Structure near infinity

BLUMENTHAL'S PIONEERING WORK

BLUMENTHAL'S PIONEERING WORK

Otto Blumenthal seems to be the first person who studied this set:

Theorem (Blumenthal, 1907)

For f entire, $\mathcal{M}(f)$ consists of an, at most countable, **union of closed curves**, which are analytic except at their endpoints, and may or may not be unbounded.

$\star$ For fixed r, we are interested in a subset of maxima of $\vartheta \mapsto |f(re^{i\vartheta})|$.

Proof can also be found in:

Valiron, G. *Lectures on the general theory of integral functions*. Chelsea (1949).

Blumenthal studied *exceptional values* of r, where analyticity of maximum curves is "broken", and classified them into two types:

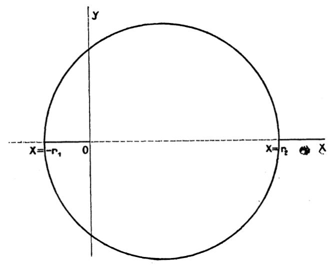
- *first kind*: values for which different maximum curves meet.
- *second kind*: values for which some maximum curve has an endpoint.

We refer to those of the second kind as *discontinuities:*

Definition

$\mathcal{M}(f)$ has a **discontinuity of modulus r** if there is a connected component Γ of $\mathcal{M}(f)$ such that $\min\{|z| : z \in \Gamma\} = r$.

Blumenthal provided an example of a quadratic polynomial p for which $\mathcal{M}(p)$ has exceptional values of the **first kind**:



Remark: Nobody else seems to have studied this phenomenon!

He didn't provide any example of f for which $\mathcal{M}(f)$ has discontinuities, but he **conjectured** that there is a cubic polynomial with this property.

DISCONTINUITIES OF $M(f)$

Hardy was the first to construct maximum modulus sets with discontinuities.

Theorem (Hardy, 1909)

For the transcendental entire function

$$f(z) := \alpha \exp\left(e^{z^2} + \sin z\right), \quad \text{with } \alpha > 0 \text{ large,}$$

$\mathcal{M}(f)$ has infinitely many discontinuities.

Namely, far away from 0, $\mathcal{M}(f)$ consists of real intervals.

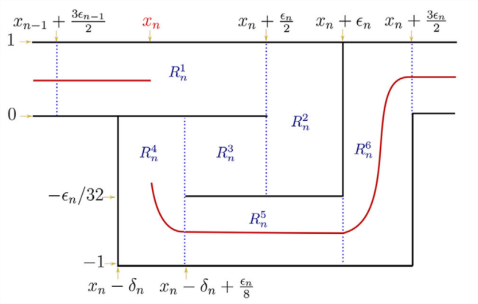
We generalize Hardy's result by specifying the moduli of the discontinuities.

Theorem (P., Sixsmith, 2020)

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers tending to infinity. Then there is a transcendental entire function $f \in \mathcal{B}$ such that $\mathcal{M}(f)$ has a discontinuity at r_n, for all $n \in \mathbb{N}$.
If there is $C > 1$ such that $r_{n+1} > Cr_n$ for all n, then f can be chosen of *finite order*.

- $f \in \mathcal{B}$ if $\overline{\{\text{ asymptotic and critical values of } f\ \}}$ is bounded; **(Eremenko-Lyubich class)**.
- f has *finite order of growth* if $\log\log|f(z)| = O(\log|z|)$.

-We take *logarithmic coordinates* and design *tracts* that force $\mathcal{M}(f)$ to "jump" at real parts close to $x_n = \log r_n$.



-We extend to an entire map using *Cauchy integrals*; [Rempe 14'].

Question: Can we specify the *location* of the discontinuities?

In 1986, Jassim and London solved Blumenthal's conjecture, proving that for

$$p(z) = z^3 + az^2 - z + b \quad \text{with } b > 0 \text{ and } ab > 1 + 2b,$$

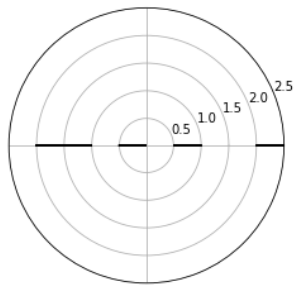
$M(p)$ must have a discontinuity.

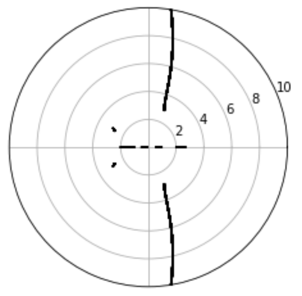
We can again specify moduli of discontinuities:

Theorem (P., Sixsmith, 2021)

Suppose $r_1, r_2, ..., r_n$ is a finite sequence of positive real numbers. Then there exists a polynomial p, of degree $2n + 1$, such that $\mathcal{M}(p)$ has discontinuities of moduli $r_1, r_2, ..., r_n$.

This time, our methods are much more elementary.





$$p(z) := 1000(z^2 + 1) + z(z^2 - 0.25)(z^2 - 1)(z^2 - 4).$$

Proof

Questions: Can we specify the *location* of the discontinuities? Can we improve the degree of the polynomials?

SINGLETONS IN M(f)

SINGLETONS IN $M(f)$

Clunie asked whether $\mathcal{M}(f)$ could have isolated points.

In his PhD thesis, Tyler, a student of Hayman, answered positively Clunie's question:

Theorem (Tyler, 2000)

The following transcendental entire function

$$f(z) := \alpha \exp\left(e^{z^2} + 2z\sin^2 z\right)$$

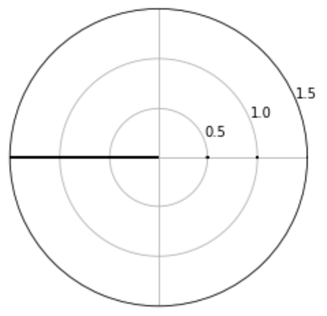
and the polynomial

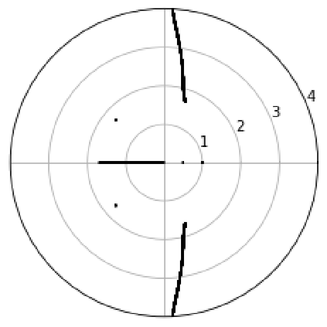
$$p(z) = \alpha(1+z^2)^2 + z(z^2-1)^2, \text{ where } \alpha > 1,$$

have isolated points in their maximum modulus sets.

Theorem (P., Sixsmith, 2021)

Suppose $r_1, r_2, \ldots, r_n$ is a finite sequence of distinct positive real numbers. Then there exists a polynomial p, of degree $4n+1$, such that $\mathcal{M}(p)$ has **singletons** at the points $r_1, r_2, \ldots, r_n$.





$$\tilde{p}(z) := 100(z^2+1) - z(z^2-0.25)^2(z^2-1)^2.$$

Proof

Theorem (P., Sixsmith, 2021)

Suppose $r_1, r_2, \ldots, r_n$ is a finite sequence of distinct positive real numbers. Then there exists a polynomial p, of degree $4n + 1$, such that $\mathcal{M}(p)$ has **singletons** at the points $r_1, r_2, \ldots, r_n$.

Questions:

- Can we specify the *location* of the singletons?
- Can we improve the degree of the polynomials?
- Can we prescribe singletons in $\mathcal{M}(f)$ for f transcendental entire?

STRUCTURE NEAR THE ORIGIN

In 1951, Hayman studied the structure of $\mathcal{M}(f)$ near the origin.

Remark: For f, g entire, if $a \neq 0$, $m \in \mathbb{Z}$ and $g(z) = az^m f(z)$, then $\mathcal{M}(g) = \mathcal{M}(f)$.

From now on we assume that f is entire and of the form

$$f(z) = 1 + az^k + h.o.t, \qquad \text{for } a \neq 0 \text{ and } k \in \mathbb{N}.$$

Theorem (Hayman, 1951)

Near the origin, $\mathcal{M}(f)$ consists of **at most** k analytic curves, only meeting at zero, any of two making an angle of $2m\pi/k$ with each other, for some $m \in \mathbb{Z}$.

Definition

For f entire as before, its **inner degree** is the maximal $\mu \in \mathbb{N}$ such that $f(z) = \tilde{f}(z^\mu)$ for some entire $\tilde{f}$.

If finitely many terms in the Taylor expansion of f satisfy *a certain algebraic condition*, then we say that f is exceptional. AC

Remark: Most entire f are **not** exceptional.

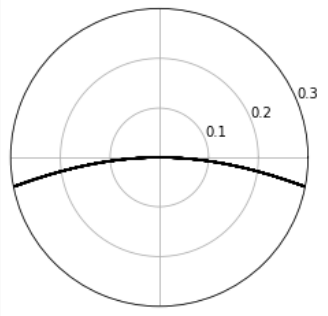
Theorem (P., Sixsmith, Evdoridou, 2021)

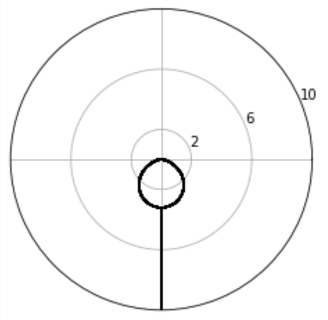
If f is entire and **not** *exceptional*, then $\mathcal{M}(f)$ consists of **exactly** μ analytic curves that meet at zero. Otherwise, it consists of a multiple of μ.

Let J_f be the number of analytic curves that comprise $\mathcal{M}(f)$ near zero.

Definition

We say that f is **magic** if $J_f > \mu_f$.





$$p(z) := 1 + z^2 + iz^3$$

Let J_f be the number of analytic curves that comprise $\mathcal{M}(f)$ near zero.

Definition

We say that f is **magic** if $J_f > \mu_f$.

Remark: If f is magic, then f is exceptional.

The converse is not true (J. Osborne, personal communication).

Questions:

- Is there a necessary and sufficient condition for a function to be magic?
- How many curves are there near the origin for a magic function?

STRUCTURE NEAR INFINITY

For polynomials, this is essentially the same question as near the origin:

- For p a polynomial of degree n, consider its ***reciprocal polynomial***

$$q(z) := z^n p(1/z).$$

- It holds that

$$z \in \mathcal{M}(p) \setminus \{0\} \iff 1/z \in \mathcal{M}(q) \setminus \{0\}.$$

- Hence, the results near the origin for $\mathcal{M}(q)$ translate into results near infinity for $\mathcal{M}(p)$.

For transcendental entire maps, this question is **open**.

For each $r > 0$ and f entire, let

$$v(r) := \#(\mathcal{M}(f) \cap \{z \in \mathbb{C} \colon |z| = r\}). \quad (1)$$

Erdős' questions (1964)

Let f be a non-monomial entire function.

(a) Can the function v be unbounded, i.e., can $\limsup_{r\to\infty} v(r) = \infty$?

(b) Can the function v tend to infinity, i.e., can $\liminf_{r\to\infty} v(r) = \infty$?

- In 1968, Hergoz and Piranian constructed f such that $v(n) = n$ for all $n \in \mathbb{N}$, answering (a) positively.
- (b) remains open to this date.

Fix f entire. For each $r, \epsilon > 0$, let $v(r, \epsilon)$ be the number of arcs in

$$\{\vartheta \in [0, 2\pi) \colon |f(re^{i\vartheta})| > M(r, f) - \epsilon\}.$$

Theorem (P., Glücksam, 2022)

There is an entire function f for which for every $\epsilon > 0$ and $N \in \mathbb{N}$, there exists $R = R(\epsilon, N)$ so that for all $r > R$,

$$v(r, \epsilon) > N.$$

- Our techniques rely on *approximation theory*, and so we cannot guarantee that $\mathcal{M}(f)$ intersects all arcs.

Proof

Theorem (Fletcher, Sixsmith, 2021)

Let $n \geq 2$, and let $T \subset \mathbb{R}^n$ be a closed set which contains a point of every modulus. Then, there is a quasiregular map h such that $\mathcal{M}(h) = T$.

THANKS FOR YOUR ATTENTION!

- Can we prescribe exceptional values of the first kind?
- Can we specify the *location* of discontinuities?
- Can we improve the degree of the polynomials on the results on discontinuities and singletons of $\mathcal{M}(p)$?
- Can we prescribe singletons of $\mathcal{M}(f)$ for f transcendental entire?
- Is there a necessary and sufficient condition for a function to be magic?
- How many curves are there near the origin for a magic function?
- (Erdős) Is it possible that $\liminf_{r\to\infty} \mathcal{M}(f) \cap \partial\mathbb{D}_r = \infty$?

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EXTRAS

DISCONTINUITIES OF POLYNOMIALS

Let $\hat{p}$ be a polynomial with only real coefficients, and choose $0 < R < R'$. For $a > 0$, set

$$p(z) := a(z^2 + 1) + \hat{p}(z).$$

If a is sufficiently large, then

$$z \in \mathcal{M}(p) \text{ and } R \le |z| \le R' \implies \operatorname{Im} z = 0.$$

If $\hat{p}$ is odd, then

$$\mathcal{M}(p) \cap \{z \in \mathbb{C} : |z| = r\} = \begin{cases} \{r\} & \text{if } \hat{p}(r) > 0, \\ \{-r\} & \text{if } \hat{p}(r) < 0, \\ \{-r, r\} & \text{if } \hat{p}(r) = 0. \end{cases}$$

We choose

$$\hat{p}(z) := z(z^2 - r_1^2)(z^2 - r_2^2) \ldots (z^2 - r_n^2).$$

Polynomial discontinuities

Let $\hat{p}$ be a polynomial with only real coefficients, and choose $0 < R < R'$. For $a > 0$, set

$$p(z) \coloneqq a(z^2 + 1) + \hat{p}(z).$$

If a is sufficiently large, then

$$z \in \mathcal{M}(p) \text{ and } R \leq |z| \leq R' \implies \operatorname{Im} z = 0.$$

If $\hat{p}$ is odd, then

$$\mathcal{M}(p) \cap \{z \in \mathbb{C} : |z| = r\} = \begin{cases} \{r\} & \text{if } \hat{p}(r) > 0, \\ \{-r\} & \text{if } \hat{p}(r) < 0, \\ \{-r, r\} & \text{if } \hat{p}(r) = 0. \end{cases}$$

We choose

$$\hat{p}(z) \coloneqq -z(z^2 - r_1^2)^2(z^2 - r_2^2)^2 \dots (z^2 - r_n^2)^2.$$

Let f be entire and of the form

$$f(z) := 1 + az^k + \sum_{\sigma=k+1}^{\infty} b_\sigma z^\sigma .$$

Let $p_k(z) := 1 + az^k$, and for each $n > k$, define

$$p_n(z) := 1 + az^k + \sum_{\sigma=k+1}^{n} b_\sigma z^\sigma .$$

There is some least $N \geq k$ such that $\mu_{p_N} = \mu_f$.

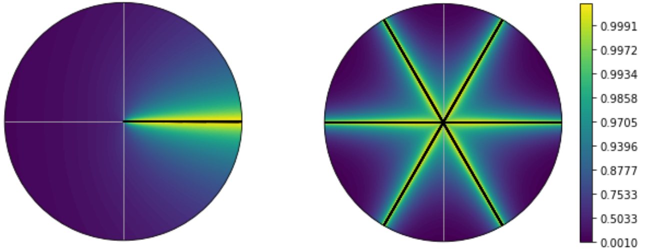
Definition

We say that f is *exceptional* if there exist $m \in \{1, \ldots, 2k-3\}$, $m' \in \mathbb{Z}$, and $\sigma \in \{k+1, \ldots, N\}$, such that $b_\sigma \neq 0$ and also

$$m\pi = \frac{k}{\sigma}(m'\pi - \arg b_\sigma) + \arg a.$$

SKETCH OF PROOF

- For each $n \in \mathbb{N}$, let $e_n(z) := \exp(z^n)$. Then, for each $r > 0$, $\nu(r) = n$ and between every two maximum modulus points, there is an arc with $|e_n| < 1$.



- Our function f acts like e_{2^n} in pieces of sectors S_n, up to an error $\epsilon_n \to 0$, and $\max|f|$ is smaller elsewhere. Erdős