A counterexample to Eremenko's conjecture

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University of Liverpool

Bedlęwo, August 19, 2022

A fine paper of 1989

[E] Alexandre Eremenko, ***On the iteration of entire functions***, in: Dynamical systems and ergodic theory, Banach Center Publications, 1989.

$f:\mathbb{C}\to\mathbb{C}$ transcendental entire function.

$I(f) := \{z\in\mathbb{C}: f^n(z)\to\infty\}$ "Escaping Set"

Theorem (Eremenko, 1989)

Let f be a transcendental entire function.

- $I(f)\neq\emptyset$;
- *Every connected component of* $\overline{I(f)}$ *is* ***unbounded***.

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Why the escaping set?

- f polynomial: ***basin of infinity*** – open set foliated by ***external rays***.
- $\rightsquigarrow$ puzzle techniques.
- f transcendental: f is a $F_{\sigma\delta}$ but never F_σ (R., 2022).
- ***But*** $I(f)$ often contains ***structures*** such as ***curves to infinity***.
- $\rightsquigarrow$ understand the dynamics by understanding the structure of $I(f)$.

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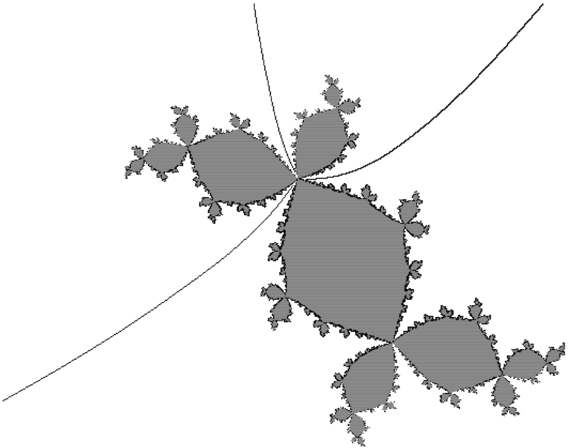

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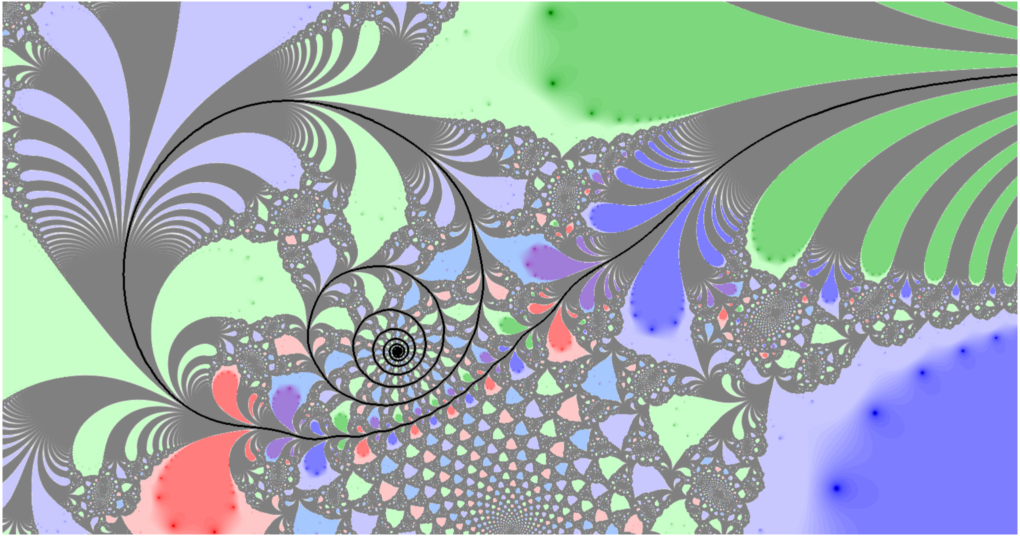

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Eremenko's conjecture

$$I(f) := \{z \in \mathbb{C} \colon f^n(z) \to \infty\}$$

"It is plausible that the set $I(f)$ has ***no bounded connected components***."

Eremenko's conjecture

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It seems that each escaping point
can to infinity be joined
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Conjecture (Eremenko's conjecture)

Let f be a transcendental entire function. Then every connected component of $I(f)$ is unbounded.

Eremenko's conjecture

Theme and variations

- ***Variation I:*** $I(f)$ has ***at least one*** unbounded connected component.
  True! (Rippon–Stallard 2005).
- ***Variation II:*** $I(f) \cup \{\infty\}$ is ***connected***.
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Progress on Eremenko's conjecture

Theme and variations

- ***Variation III*** ("Strong version of Eremenko's conjecture"): $I(f) \cup \{\infty\}$ is ***path-connected***.
  False! (Rottenfußer–Rückert–R–Schleicher, 2011).
- ***Variation IV*** ("uniform version"): If $z \in I(f)$, then there is an unbounded connected set A such that $f^n|_A \to \infty$ ***uniformly***.
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Positive results on Eremenko's conjecture

Positive results

Eremenko's conjecture ***holds*** if:

- The ***postsingular set*** of f is bounded.
- The ***singular set*** of f is bounded and f has ***finite order***.
- $F(f) = \mathbb{C} \setminus \partial I(f)$ has a ***multiply-connected*** component.
- Functions of order $< 1/2$ and ***regular growth***.
- ***Real*** functions of order $< 1/2$ with only negative roots.
- Functions of ***very*** small growth.

[R07] L. Rempe, *On a question of Eremenko concerning escaping sets of entire functions*, Bull. Lond. Math. Soc. (2007)

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Challenges of finding a counterexample

- What could a counterexample look like?
- How to obtain sufficient control to realise this structure?

A counterexample to Eremenko's conjecture

Theorem (Martí-Pete–R–Waterman 2022)

There exists a transcendental entire function f such that $I(f)$ has a connected component consisting of ***a single point****.*

[MRW] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering lakes of Wada, and maverick points*, Preprint (2022)

Wandering domains

$f : \mathbb{C} \to \mathbb{C}$ transcendental entire

- ***Fatou set*** $F(f)$: locus of ***normality*** of (f^n).
- ***Julia set*** $J(f) = \mathbb{C} \setminus F(f) = \partial I(f)$: locus of non-normality.

A ***wandering domain*** is a connected component U of $F(f)$ such that

$$f^n(U) \cap f^m(U) = \emptyset, \quad n \neq m.$$

In transcendental dynamics, ***wandering domains exist***!

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Approximation theory

Theorem ((A special case of) Arakelian's theorem)

*Let $A \subset \mathbb{C}$ be closed such that $(\mathbb{C} \setminus A) \cup \{\infty\}$ is **connected, and locally connected at** ∞.*

Suppose $g\colon A \to \mathbb{C}$ is continuous on A and holomorphic on its interior. Then, for every $\varepsilon > 0$, there is an entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $|f - g| < \varepsilon$ on A.

Use of approximation theory to construct wandering domains: Eremenko–Lyubich, 1989.

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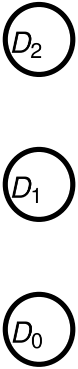
Existence of WD via Arakelian's theorem



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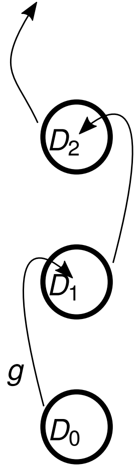

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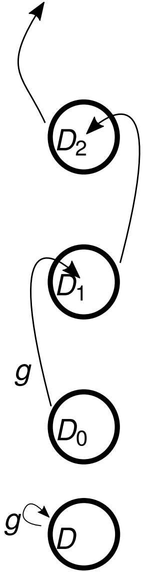

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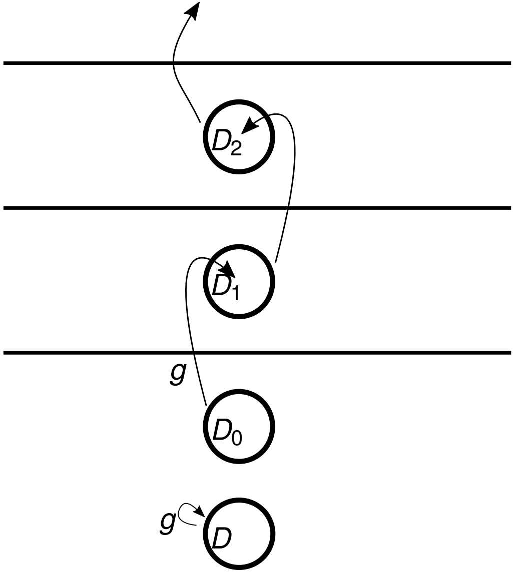

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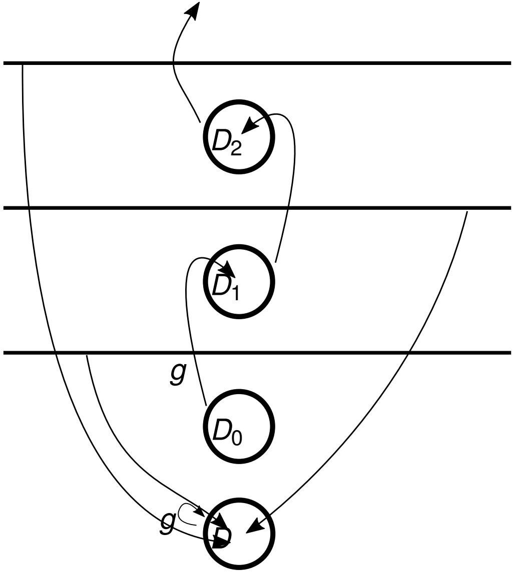

Types of wandering domains

The wandering domain we just constructed is in the ***escaping set***.

- A wandering domain U with $U \subset I(f)$ is called ***escaping***.
- A wandering domain U is called ***oscillating*** if orbits in U are ***unbounded*** but $U \not\subset I(f)$.

Theorem (Eremenko–Lyubich 1989)

There exists an entire function with an oscillating wandering domain.

- Eremenko and Lyubich build a partially-defined function g with the desired properties, and then apply a suitable approximation theorem.

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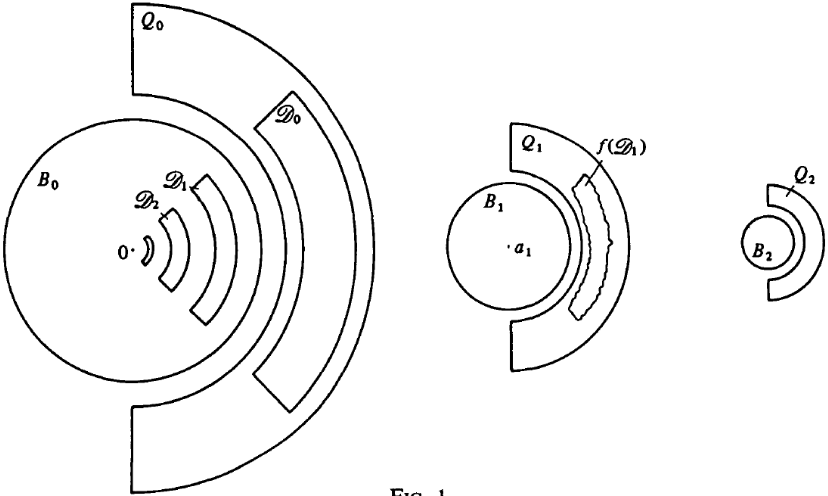

FIG. 1

Oscillating Wandering Domains

Key idea: Instead of obtaining f by ***one*** approximation, build a ***sequence*** f_n, where f_{n+1} approximates a map g_n that agrees with f_n on a (large) set.

- May use the properties of f_n in defining g_{n+1}.

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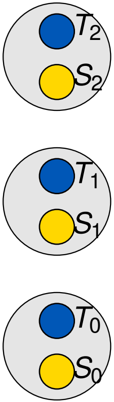

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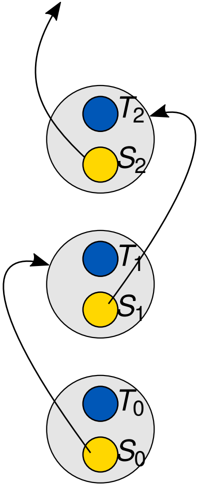

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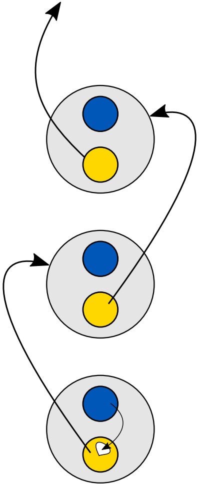

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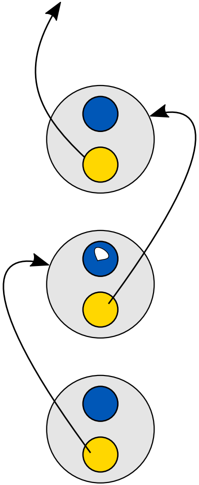

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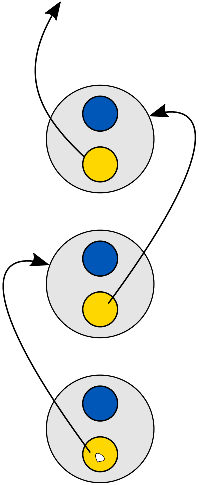

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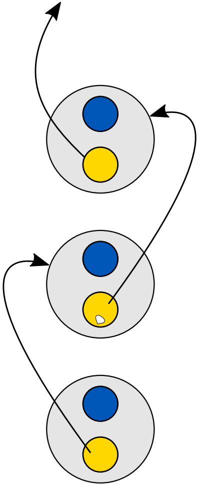

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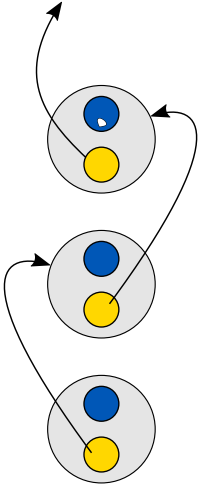

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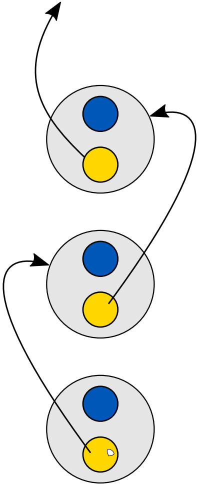

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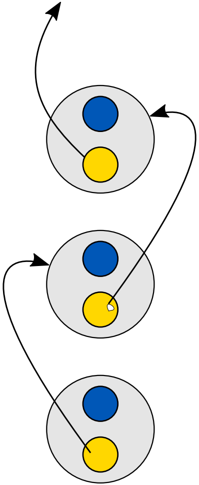

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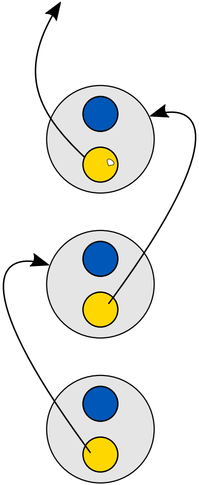

Behaviour of boundary points

Can $I(f)$ contain a ***bounded*** wandering domain U with $\partial U \cap I(f) = \emptyset$?

Theorem (Rippon–Stallard 2011)

*If $U \subset I(f)$ is a wandering domain, then $I(f) \cap \partial U$ has **full harmonic measure**.*

Question (Rippon)

Suppose U is a bounded ***escaping*** wandering domain of f. Is $\partial U \subset I(f)$?

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Suppose U is a bounded ***oscillating*** wandering domain of f. Is $\partial U \cap I(f) = \emptyset$?

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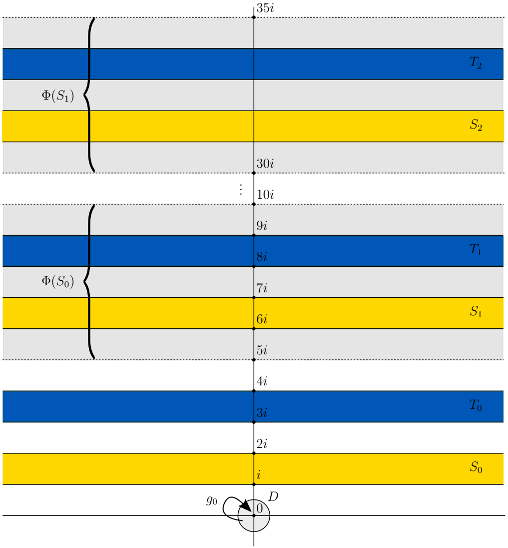
Scaffolding



Boc-Thaler's result

- Boc-Thaler (2020): many simply-connected domains can be realised ***exactly*** as wandering domains of entire functions.
- The proof uses a similar approach to the above (***next talk***!).
- We can use a similar approach to ensure that our WD is a ***half-strip***.

[B21] B. Boc Thaler, *On the geometry of simply connected wandering domains*, B. London Math. Soc. (2021)

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The example

Theorem (Martí-Pete–R–Waterman)

Define

$$U := \{a + ib \colon a > 0, |b| < 1\}.$$

There exists an entire function f with the following properties.

- *U is an oscillating wandering domain of f.*
- $0 \in I(f)$.
- *$\overline{U}$ is a connected component of the set of points with unbounded orbits.*

By a result of Rippon and Stallard, $I(f) \cap \partial U$ has ***zero harmonic measure***. Hence $\{0\}$ is a connected component of $I(f)$.

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Further results

Our method also yields:

- An escaping wandering domain whose ***boundary contains non-escaping ("maverick") points***.
- ***Lakes of Wada*** in complex dynamics.
- A counterexample to a ***question of Boc Thaler*** concerning simply-connected wandering domains.
- ***Wandering continua*** of transcendental entire functions.

See the next talk by David Martí-Pete!

[MRW] D. Martí-Pete, L. Rempe and J. Waterman, *Eremenko's conjecture, wandering lakes of Wada, and maverick Points*, Preprint (2022)

Dziękuję!