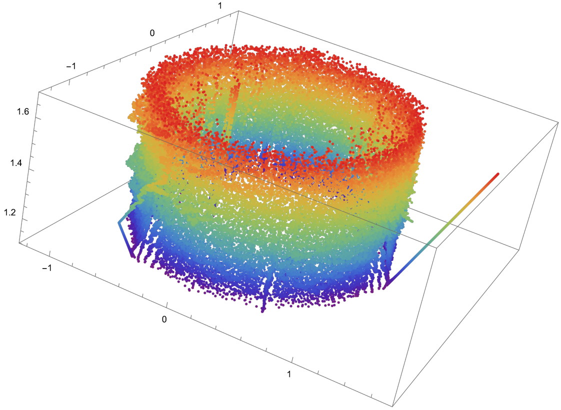

Core entropy along the Mandelbrot set and Thurston's *Master Teapot*

Giulio Tiozzo - University of Toronto

Summary

1. Topological entropy

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2. The core entropy

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5. Kneading theory for veins

Joint work with Kathryn Lindsey and Chenxi Wu.

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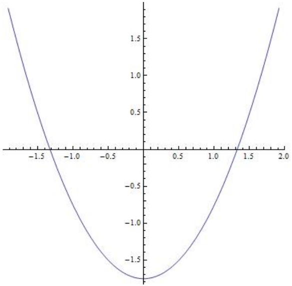

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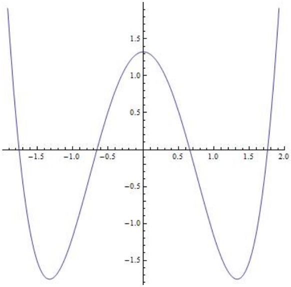

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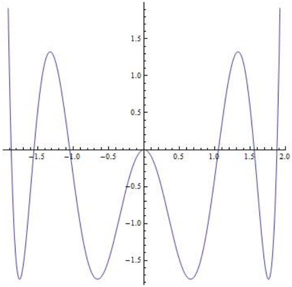

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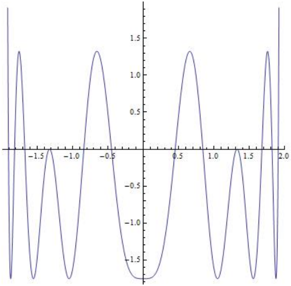

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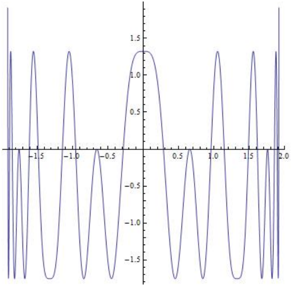

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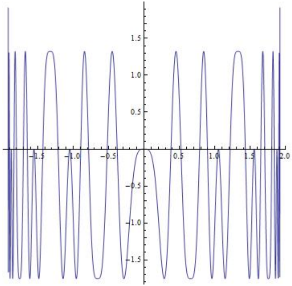

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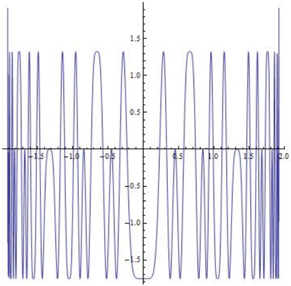

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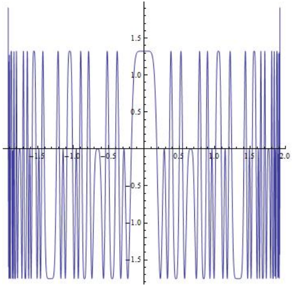

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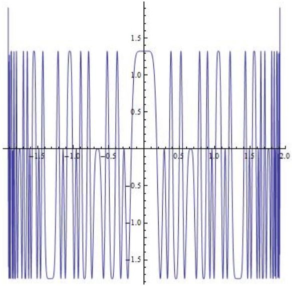

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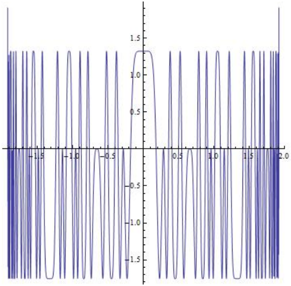

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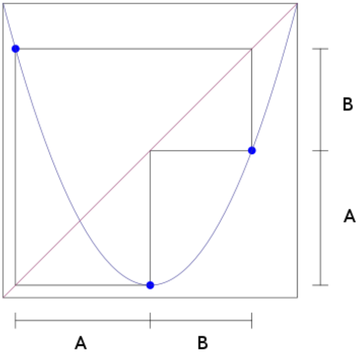

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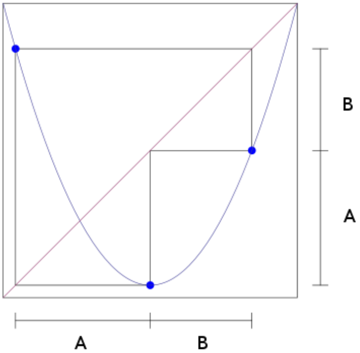

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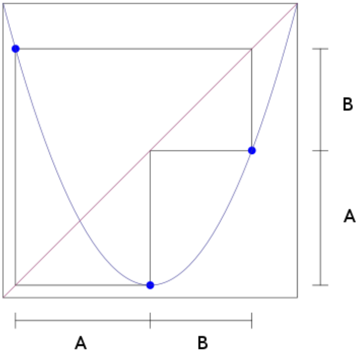

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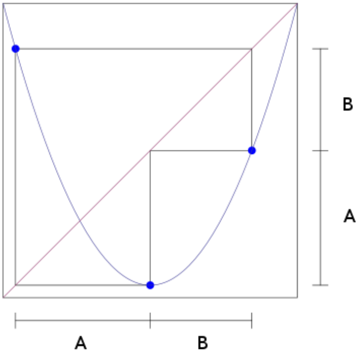

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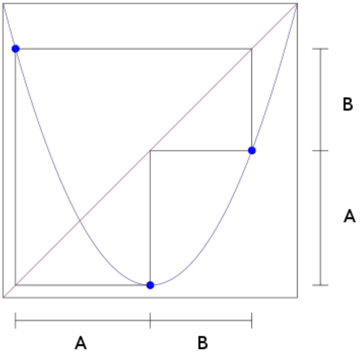

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- What if you fix the degree of the polynomial?

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Consider the real quadratic family

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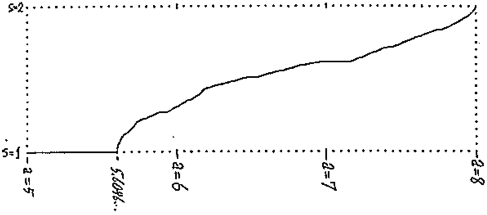

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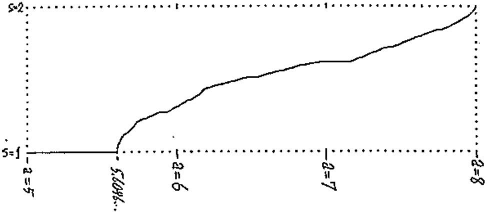

[Picture is for $f_a(x) = ax(1-x)$.]

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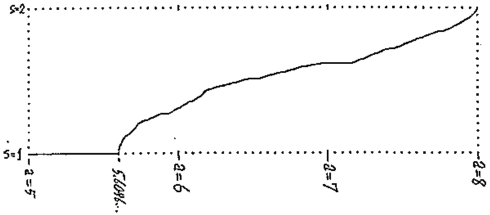

Question : Can we extend this theory to complex polynomials?

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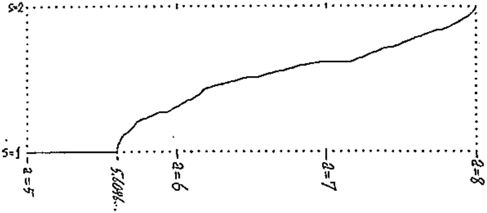
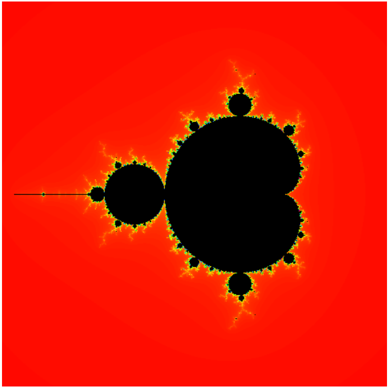


Remark. If we consider $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f_c, \hat{\mathbb{C}}) = \log 2$. (Lyubich 1980)

Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the connectedness locus of the quadratic family

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^n(0) \nrightarrow \infty\}$$



External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

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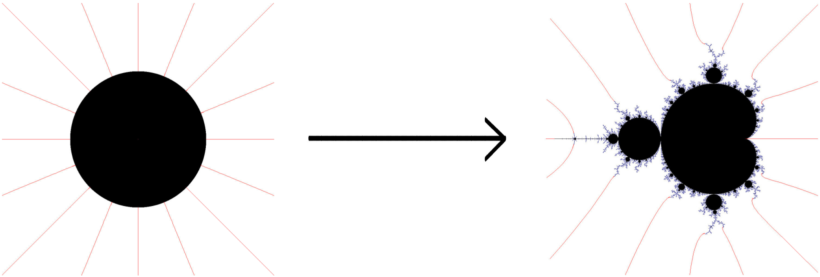

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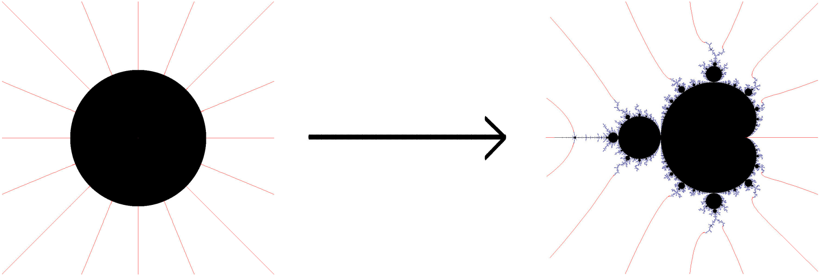

The images of radial arcs in the disk are called external rays.

$$R(\theta) := \Phi_{\mathcal{M}}(\{\rho e^{2\pi i\theta} : \rho > 1\})$$

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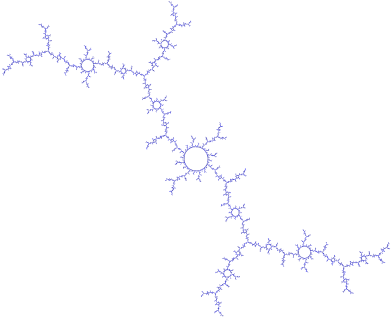

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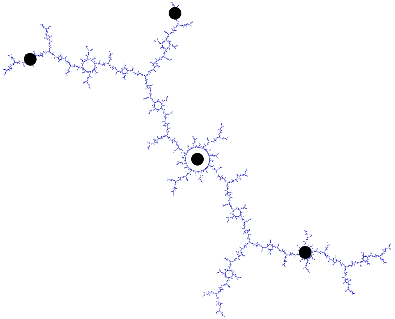

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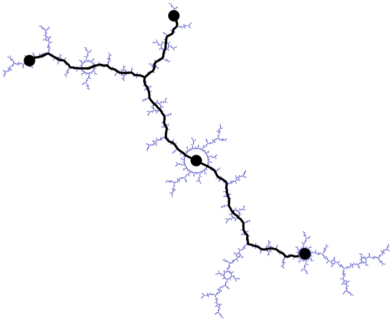

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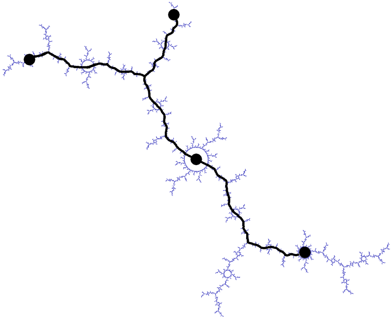

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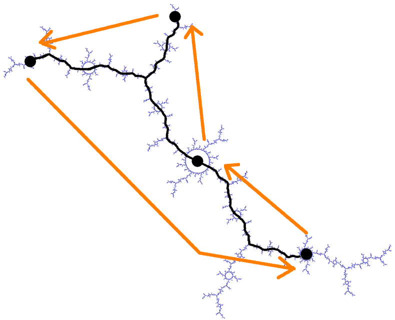

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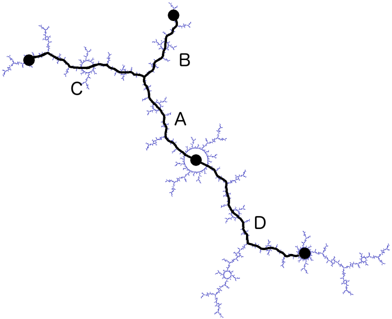

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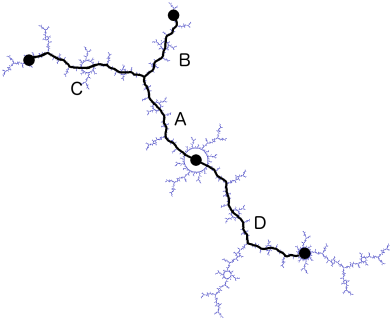

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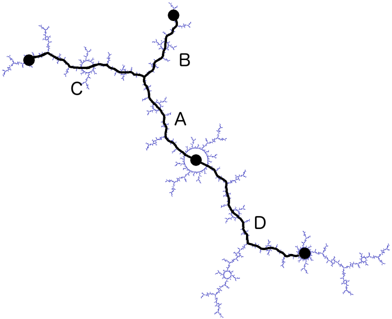

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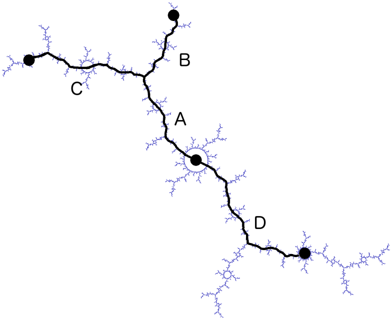

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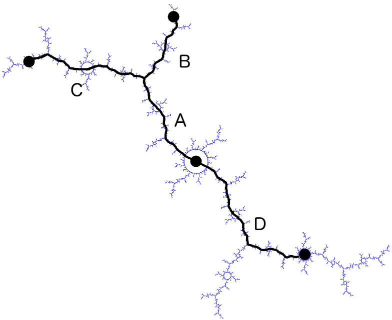

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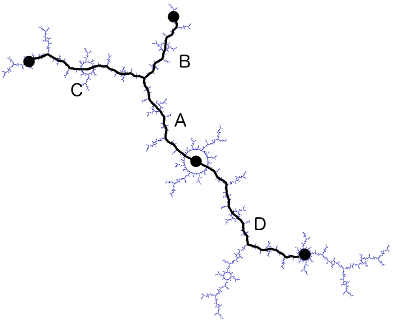

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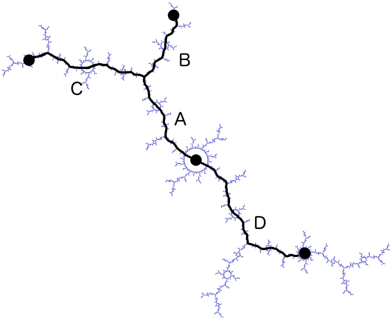

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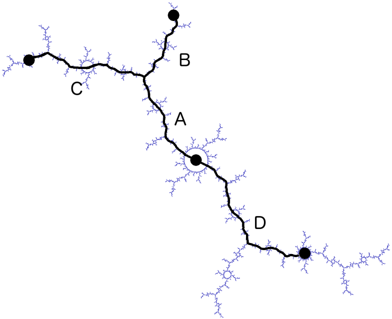

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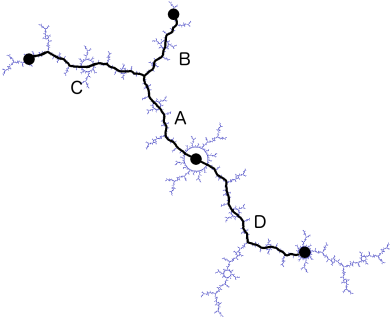

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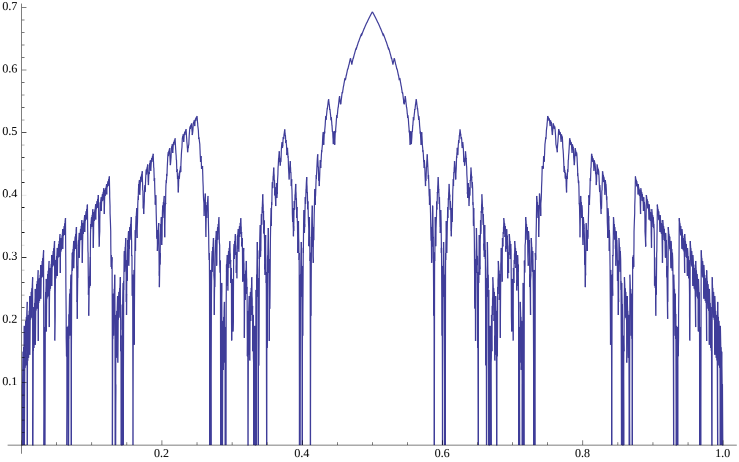
Question: How does $h(\theta)$ vary with the parameter θ?

Core entropy as a function of external angle (W. Thurston)



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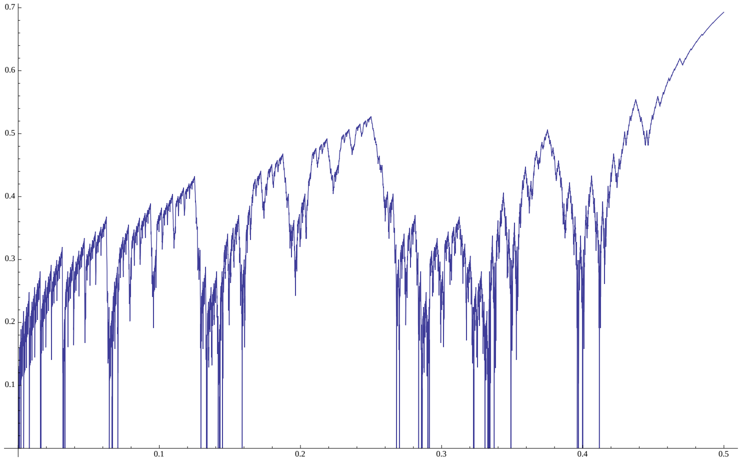

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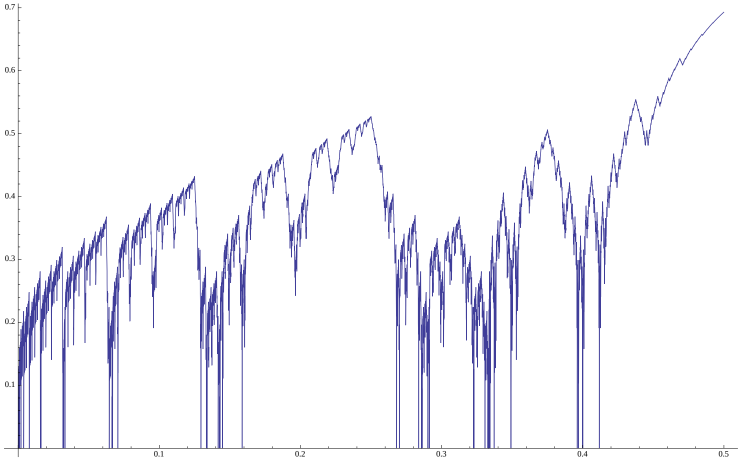
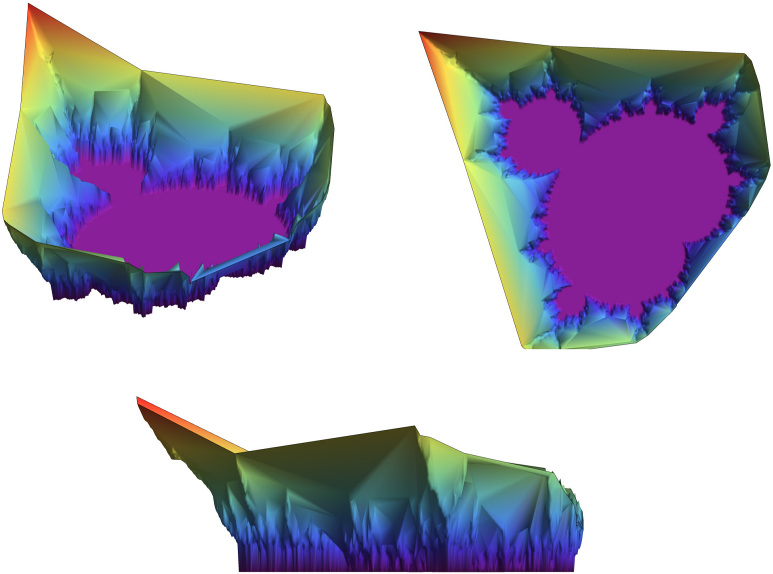


Question Can you see the Mandelbrot set in this picture?

Core entropy as a function of c



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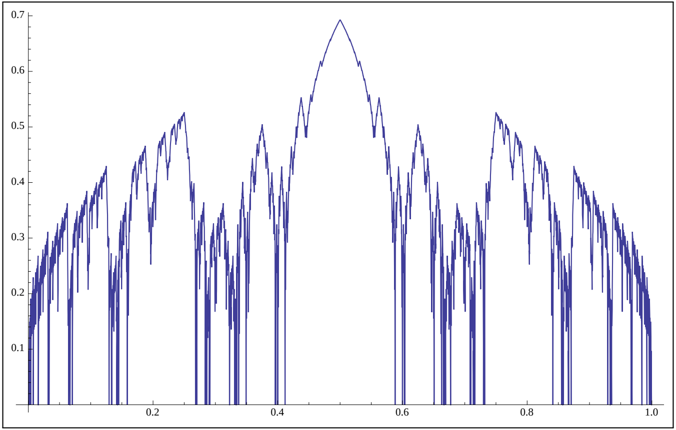
Theorem (Li Tao; Penrose; Tan Lei; Zeng Jinsong)

If $\theta_1 <_{\mathcal{M}} \theta_2$, then

$$h(\theta_1) \leq h(\theta_2)$$

The core entropy as a function of external angle

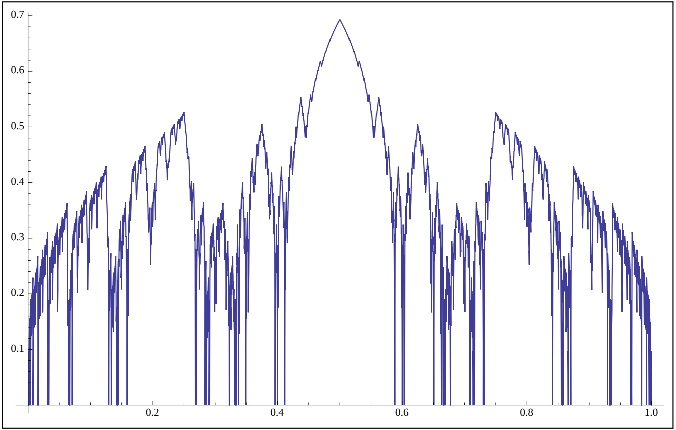
Question (Thurston, Hubbard):
Is $h(\theta)$ a continuous function of θ?



Continuity of core entropy

Theorem (T.; Dudko-Schleicher)

The core entropy function $h(\theta)$ extends to a continuous function from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.



Regularity properties of the core entropy

In fact:

Theorem (T. '15)

The core entropy is locally Hölder continuous at θ *if* $h(\theta) > 0$, *and not locally Hölder at* θ *where* $h(\theta) = 0$.

Theorem (T. '17)

Let $h(\theta)$ *be the entropy of the* <u>*real*</u> *quadratic polynomial with external ray* θ*. Then the local Hölder exponent* $\alpha(h, \theta)$ *of h at* θ *satisfies*

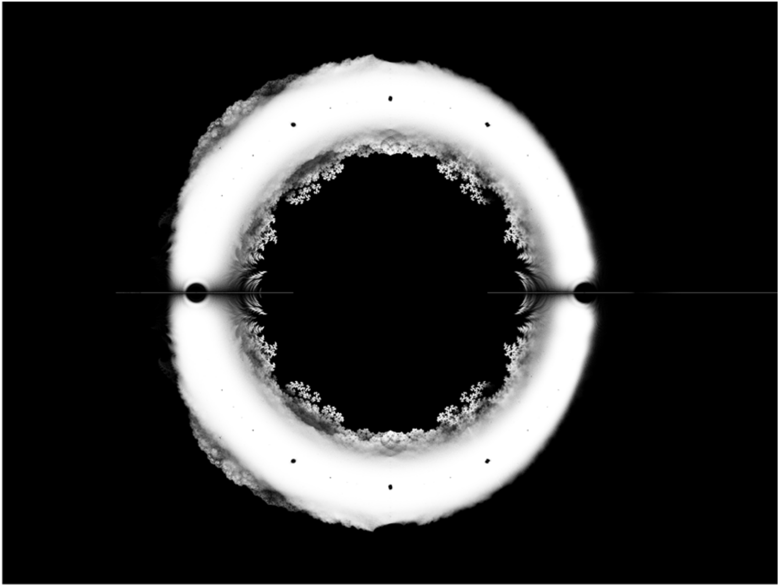
$$\alpha(h, \theta) := \frac{h(\theta)}{\log 2}$$

(Conjectured Isola-Politi, 1990)

Further questions

Question. What about the other eigenvalues?

The entropy spectrum (W. Thurston ’12)



Thurston's entropy spectrum

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$$\mathcal{M}_0 := \{c \in \mathbb{R} \ : \exists n \text{ s.t. } f_c^n(0) = 0\}$$

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where $\mathrm{Gal}(\lambda)$ is the set of Galois conjugates (i.e., roots of the same minimal polynomial) of λ.

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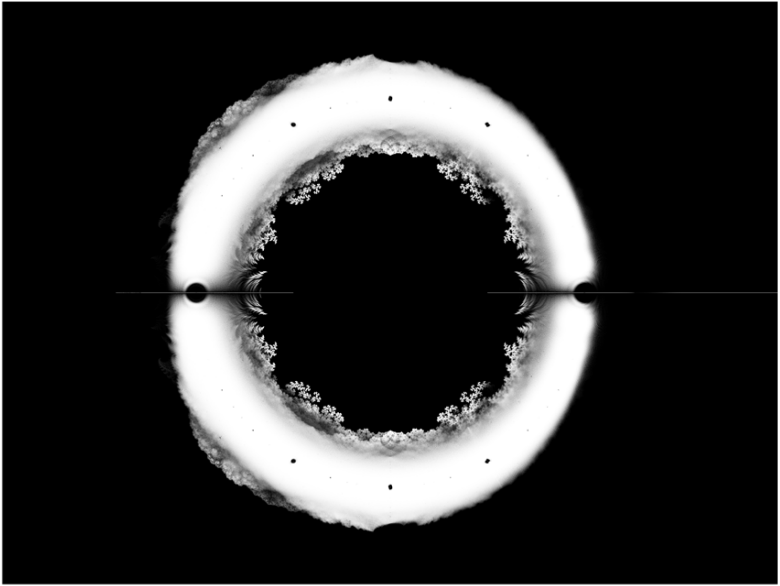

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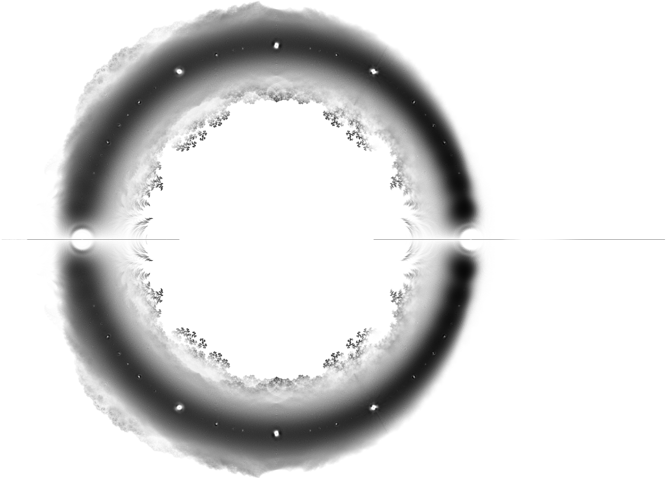

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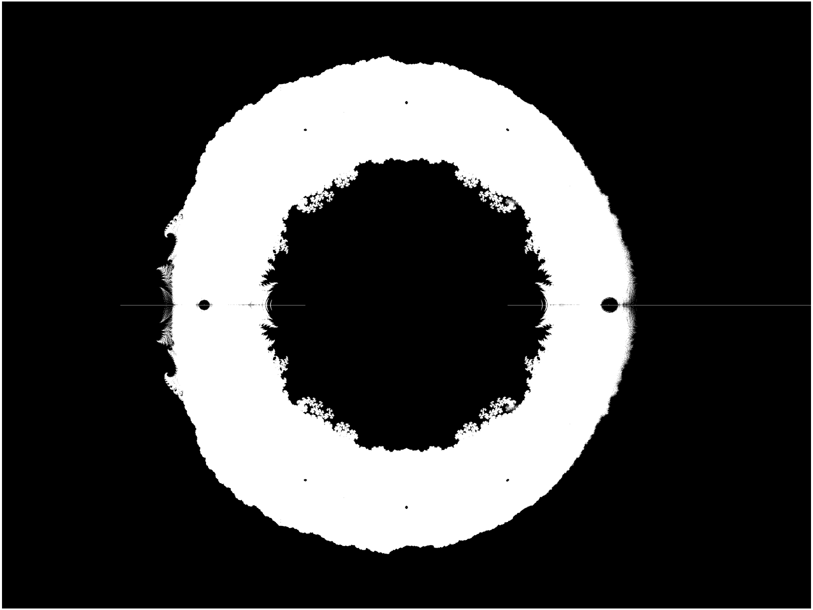
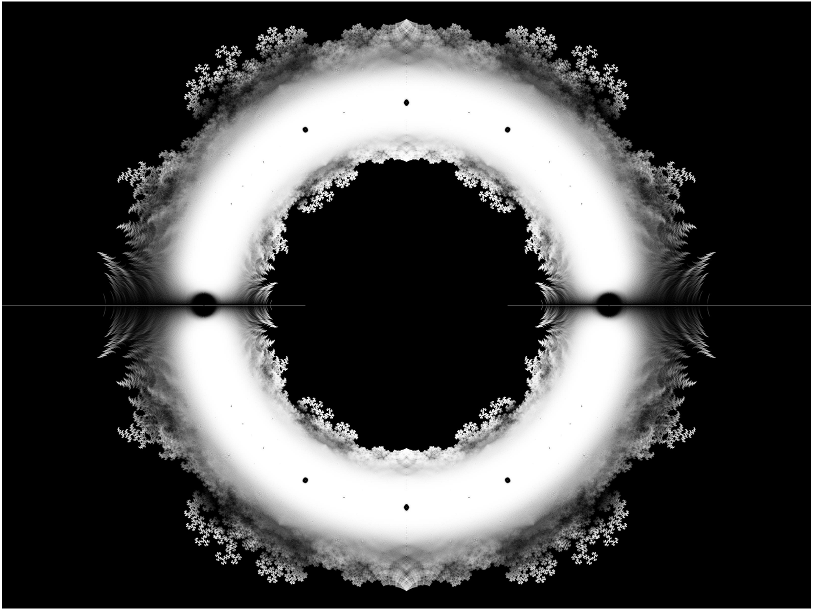
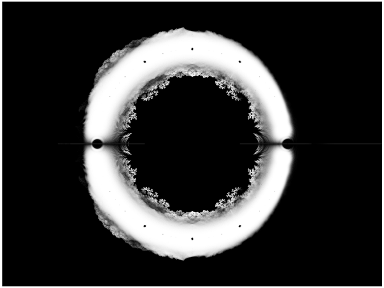


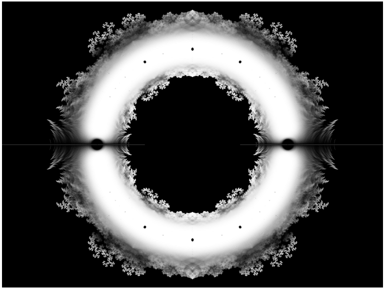
Zeros of polynomials with coefficients ± 1 (Bousch)



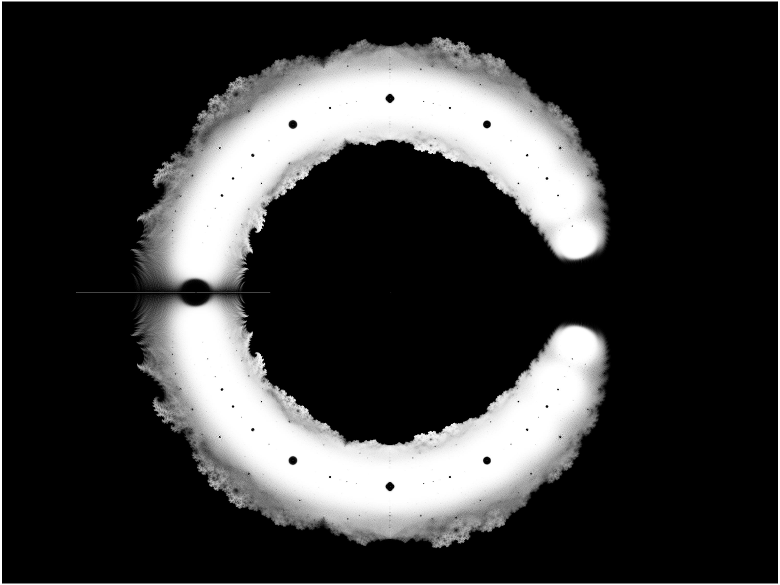
Comparison

$$\Sigma_{\pm} := \overline{\left\{ z \in \mathbb{C} \ : \ \exists (\epsilon_k) \in \{\pm\}^n \ : \ \sum_{k=1}^{n} \epsilon_k z^k = 0 \right\}}$$





Zeros of polynomials with coefficients $0, 1$ (Odlyzko-Poonen)



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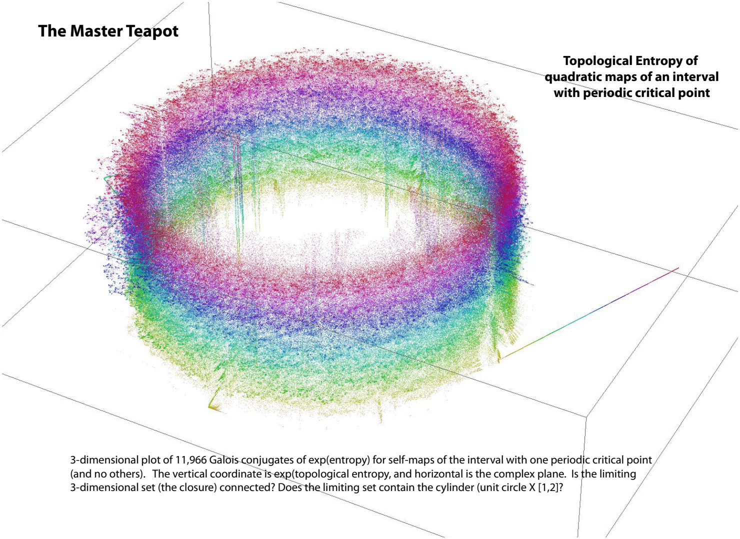
$$\Sigma \cap \mathbb{D} = \Sigma_{\pm} \cap \mathbb{D}.$$

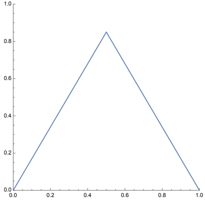
The Master Teapot for real maps



3-dimensional plot of 11,966 Galois conjugates of exp(entropy) for self-maps of the interval with one periodic critical point (and no others). The vertical coordinate is exp(topological entropy, and horizontal is the complex plane. Is the limiting 3-dimensional set (the closure) connected? Does the limiting set contain the cylinder (unit circle X [1,2]?

The Master Teapot for tent maps

For each $\lambda \in [1, 2]$, consider the tent map $T_\lambda : [0, 1] \to [0, 1]$



$$T_\lambda(x) := \begin{cases} \lambda x & \text{if } x \leq 1/2 \\ \lambda(1 - x) & \text{if } x > 1/2 \end{cases}$$

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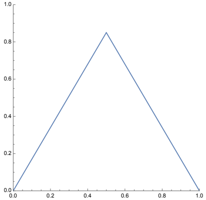

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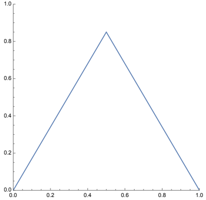

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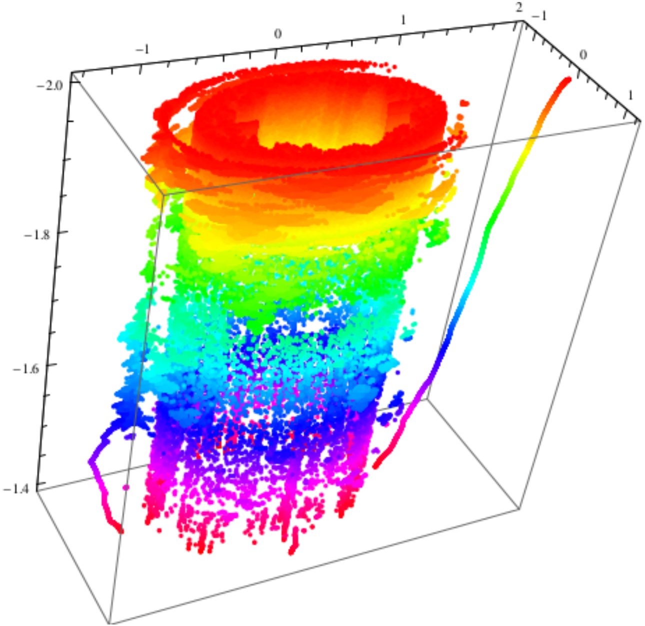
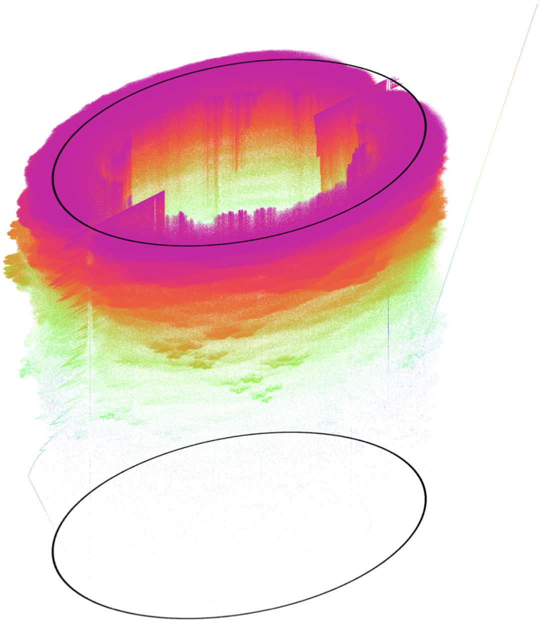
The Master Teapot for real polynomials



A three-dimensional object



Geometry of the teapot

Video (by D. Davis): `https://vimeo.com/259921275`

3D view:

`http://www.math.toronto.edu/tiozzo/teapot.html`

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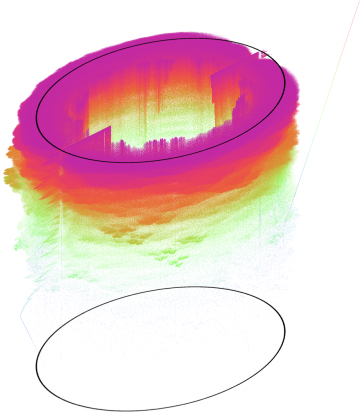

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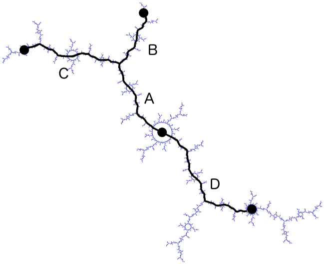

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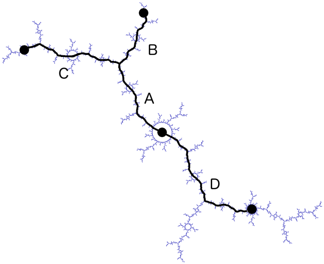

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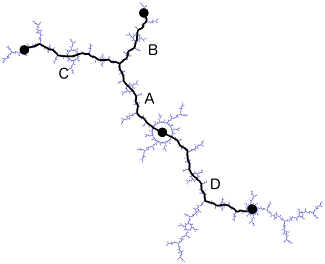

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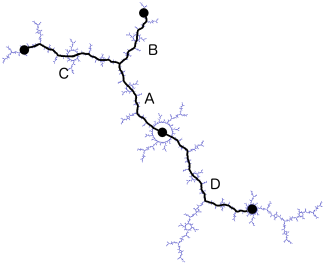

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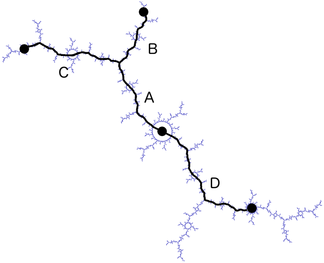

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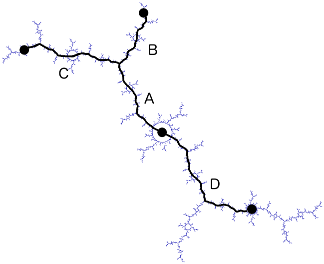

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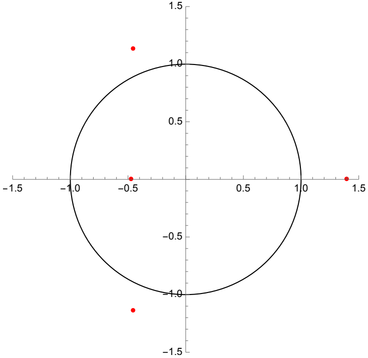

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Denote as $Com^+(\mathbb{C})$ the collection of compact subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$, with the Hausdorff topology.
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But: this is <u>not</u> true for the part inside $\mathbb{D}$!

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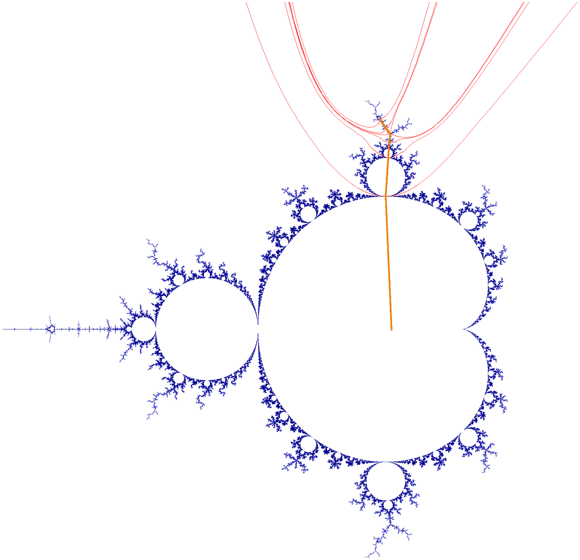

Principal veins

For each p, q with $gcd(p, q) = 1$, there is a parameter $c_{p/q}$ with:

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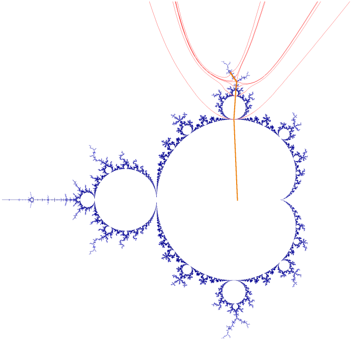

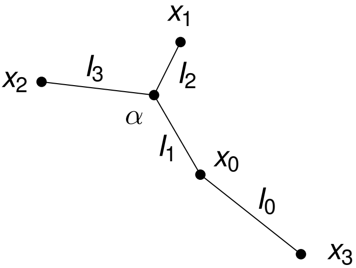

Principal veins

Definition

The $\frac{p}{q}$-principal vein is the vein joining 0 with $c_{p/q}$.

Let $\Theta^{per}_{p/q}$ the periodic external angles whose rays land on the p/q-vein.

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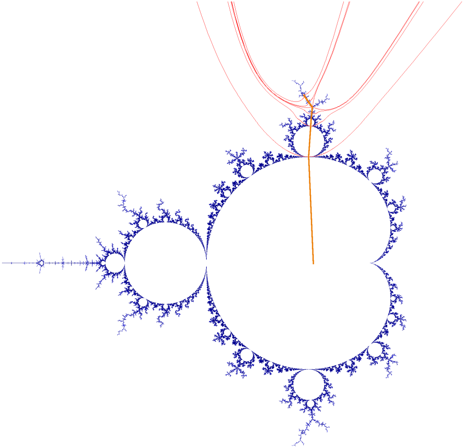
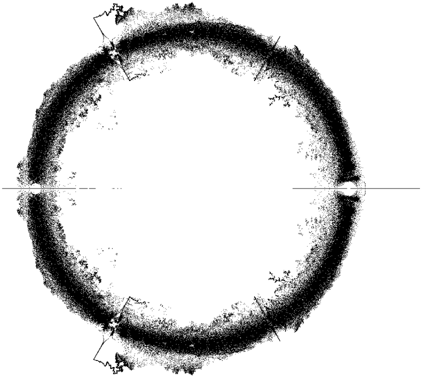


The Thurston set for a principal vein

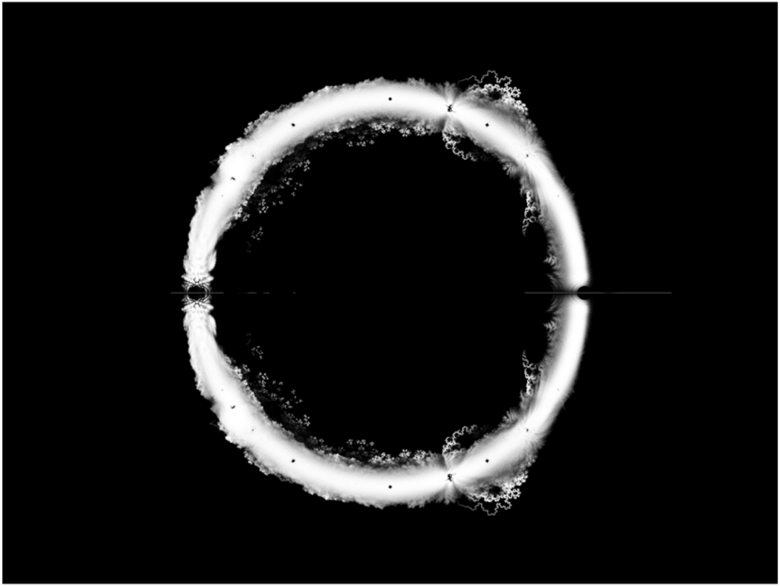
We define the Thurston set for the principal p/q-vein as

$$\Sigma_{p/q} := \overline{\left\{ z \in \mathbb{C} \mid \det(M_\theta - zI) = 0 \text{ for some } \theta \in \Theta^{per}_{p/q} \right\}}.$$

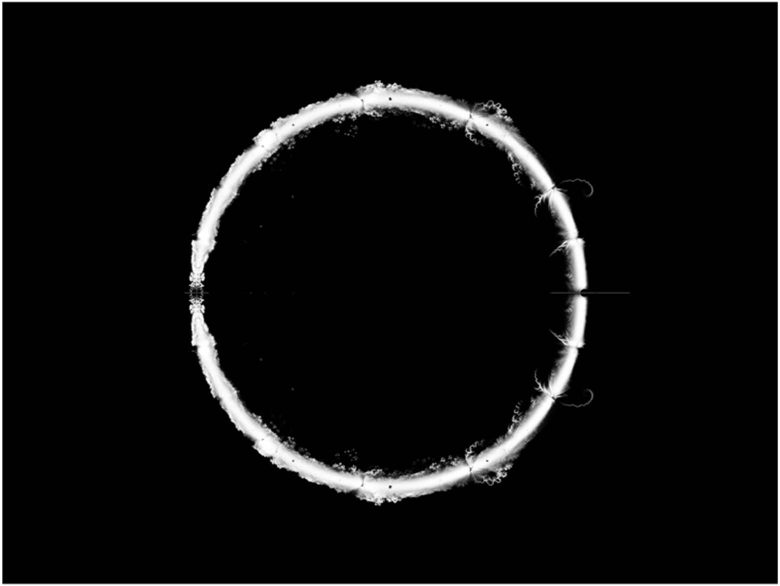


Galois conjugates of entropies of complex maps: 1/3 vein

Galois conjugates of entropies of complex maps: 1/5 vein



Galois conjugates of entropies of complex maps: 1/11 vein



Connectivity of the $\frac{p}{q}$-Thurston set

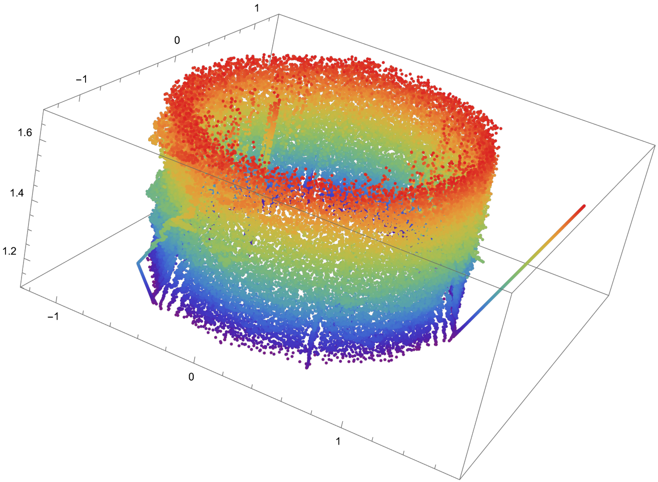
Corollary (Lindsey-T.-Wu '21)

For any (p, q) coprime, the Thurston set

$$\Sigma_{p/q} \cap \{z \in \mathbb{C} \ : \ |z| \geq 1\}$$

is path connected and locally connected.

Inside view: the Teapot



The Master Teapot for principal veins

For each λ, define

$$\mathcal{Z}(\lambda) := \{z \in \mathbb{C} \mid \det(M_\theta - zI) = 0 \ \forall \theta \in \Theta_{p/q} \text{ s.t. } \lambda = e^{h(\theta)}\}$$

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Definition

We define the $\frac{p}{q}$-Master Teapot to be the set

$$\Upsilon_{p/q} := \overline{\left\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = e^{h(\theta)} \text{ for some } \theta \in \Theta^{per}_{p/q},\ z \in \mathcal{Z}(\lambda)\right\}}$$

The Persistence Theorem - complex veins

Theorem (Persistence - Lindsey-T.-Wu '21)

For $z \in \mathbb{D}$, $(z, \lambda) \in \Upsilon_{p/q}$ *implies* $\{z\} \times [\lambda, \lambda_q] \subset \Upsilon_{p/q}$.

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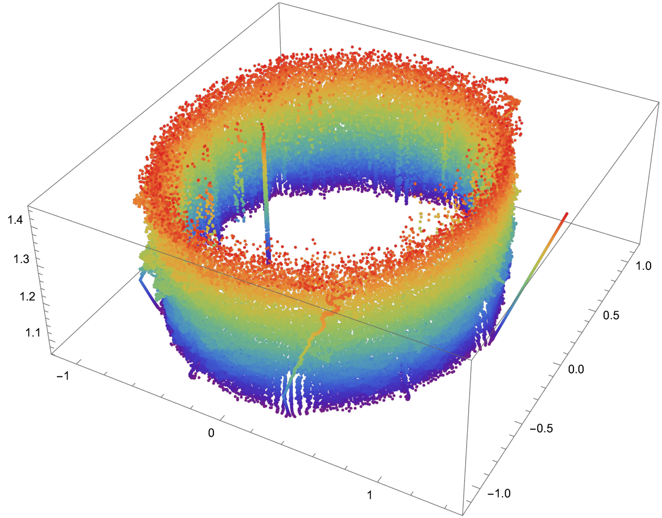

Computing the entropy

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Idea 2: (Thurston): look at set of pairs of postcritical points, which correspond to arcs between postcritical points.
Denote $c_i := f^i(0)$ the i^{th} iterate of the critical point, and let

$$P := \{(c_i, c_j) \;\; i, j \geq 0\}$$

the set of pairs of postcritical points

Computing the entropy: non-separated pair

A pair (i, j) is non-separated if c_i and c_j lie on the same side of the critical point.

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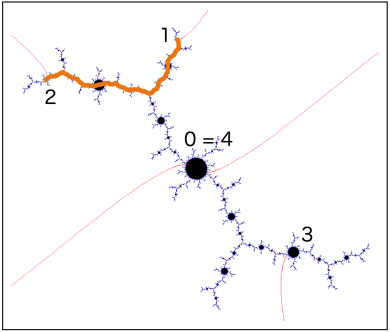

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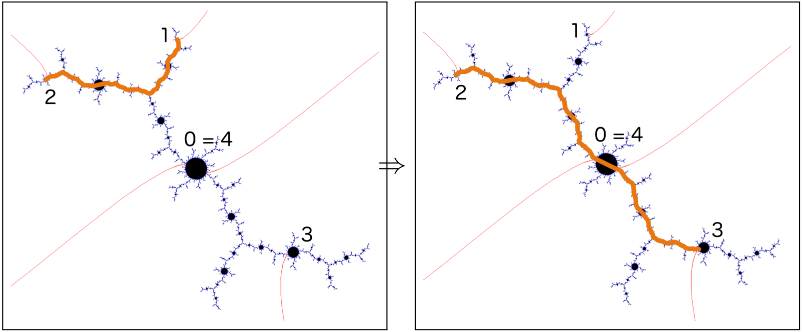

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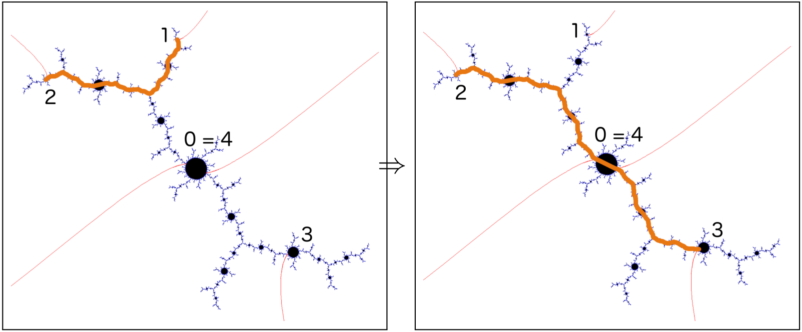

$(1, 2) \quad \Rightarrow \quad (2, 3)$

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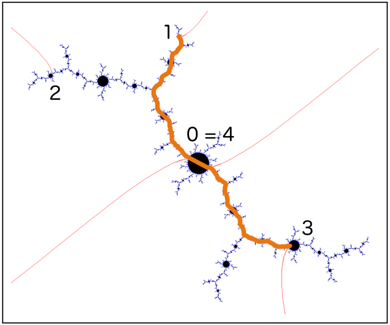

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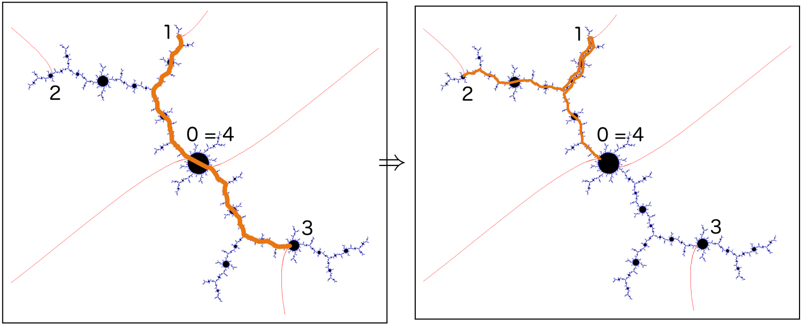

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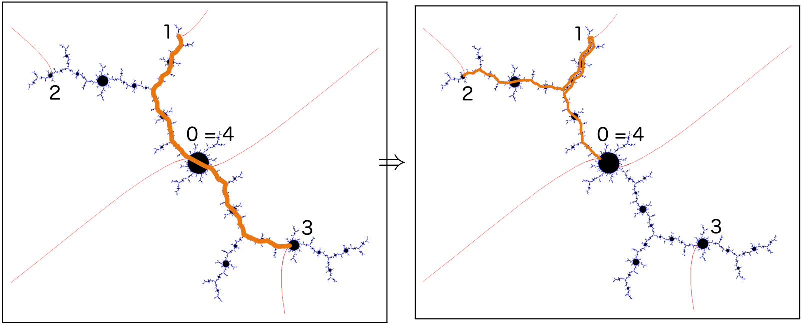

$$(1, 3) \quad \Rightarrow \quad (1, 2) + (1, 4)$$

The algorithm

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Theorem (Thurston; Tan Lei)

The entropy of f_θ is given by

$$h(\theta) = \log \lambda$$

where λ is the leading eigenvalue of A.

See also Gao Yan, Wolf Jung.

Coincidence of entropy algorithms

Theorem (Lindsey-T.-Wu '21)

Let f be a postcritically finite quadratic polynomial.

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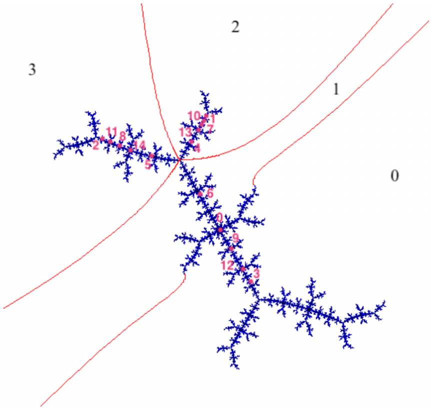
If f is critically periodic and belongs to a principal vein, a third polynomial that has the same roots off the unit circle is

(3) *the principal vein kneading polynomial $D(t)$.*

Kneading theory for principal veins

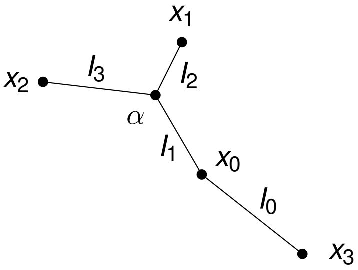


We define the itinerary $\mathrm{It}(x) \in \{0, 1, 2\}^{\mathbb{N}}$ as the itinerary for the first return map on $I_0 \cup I_1 \cup I_2$.

Kneading theory for principal veins

Let us define the “piecewise linear model"

$$F_{0,q,\lambda}(x) := \lambda x + \lambda + 1$$
$$F_{1,q,\lambda}(x) := -\lambda x + \lambda + 1$$
$$F_{2,q,\lambda}(x) := -\lambda^{q-1} x + \lambda^{q-1} + 1$$



Let $\epsilon_j \in \{+1, -1\}$ and $q_j \in \mathbb{N}^+$, and polynomial B_j be such that

$$F_{j,q,1/t}(x) := \frac{\epsilon_j}{t^{q_j}} x + \frac{B_j(t)}{t^{q_j}}$$

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Let

$$\mathrm{It}(c_0) = (w_0)^\infty, \qquad \mathrm{It}(c_1) = (w_1)^\infty$$

be the itineraries of two critically periodic parameters $c_0 <_{\mathcal{M}} c_1$ in $\mathcal{V}_{p/q}$. Then, given any $N > 0$, there is a critically periodic parameter c_2 such that the itinerary of c_2 is

$$\mathrm{It}(c_2) = \left((w_1)^N u (w_0)^N\right)^\infty$$

and c_2 is the smallest parameter on the vein with the given core entropy.

Corollary

Let $(z, \lambda_0) \in \Upsilon$, let $\lambda_1 > \lambda_0$. For N large, $c_0 <_{\mathcal{M}} c_2 <_{\mathcal{M}} c_1$, hence $\lambda_0 < \lambda_2 < \lambda_1$ but since $\mathrm{It}(c_2)$ and $\mathrm{It}(c_0)$ have the same "end" we have $d(z, \mathcal{Z}(c_0)) < \epsilon$, hence $(z, \lambda_1) \in \Upsilon$
$\Rightarrow$ persistence.

Thank you!