Rational maps with smooth degenerate Herman rings

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ON GEOMETRIC COMPLEXITY OF JULIA SETS - IV

Będlewo (online)

Aug 16, 2022

Origin

Fatou, 1920[1]:

Il nous resterait à étudier les courbes analytiques invariantes par une transformation rationnelle et dont l'étude est intimement liée à celle des fonctions étudiées dans ce Chapitre. Nous espérons y revenir bientôt.

It would remain for us to study the **invariant analytic curves** of a rational transformation and whose study is intimately linked to that of the functions studied in this chapter. We hope to return there soon.

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Motivations: decomposing dynamics.

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Progresses

Theorem (Azarina, 1989)

Suppose $f(\gamma) \subset \gamma$, where f is ***entire*** *and γ is a Jordan* ***analytic curve****. Then either*

- *γ is a circle and f is conformally conjugate to $z \mapsto z^n$ with $n \in \mathbb{Z}$; or*
- *γ is a level curve of the linearizing function of some Siegel disk.*

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Invariant analytic curves under **rational maps**:

Circles, level curves of linearizing functions of Siegel disks or Hermann rings.

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Invariant analytic curves under **rational maps**:

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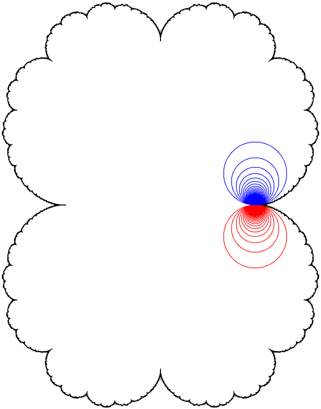
Theorem (Eremenko, 2012)

*There are two types (algebraic and transcendental) of Jordan analytic curves which are not circles and invariant under the **Lattés maps** (Julia set $= \widehat{\mathbb{C}}$), and moreover, the restriction on each of these curves is **not** a homeomorphism.*

Invariant smooth curves

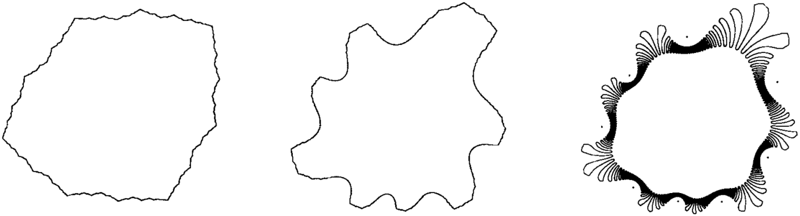
Level curves (C^1-smooth) of attracting Fatou coordinates in **parabolic basins**:



Invariant smooth curves

Boundaries of **Siegel disks**:

- C^∞-smooth (Pérez-Marco 1997; Avila-Buff-Chéritat 2004, 2020; Geyer 2008)
- C^n but not C^{n+1}; C^0 but not Hölder (Buff-Chéritat 2007)

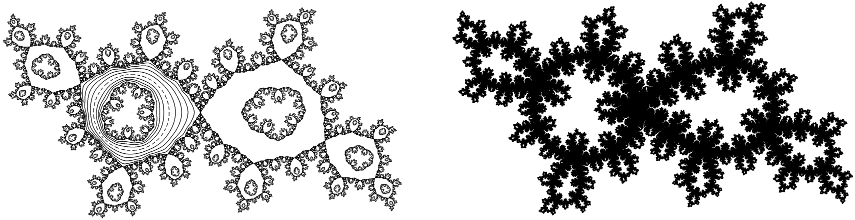


Construction by perturbation (Figures from [ABC04])

Boundaries of **Herman rings**:

- C^∞-smooth (Buff, unpublished; Avila 2003)

Herman rings and degeneration



Invariant circles of Blaschke products $z \mapsto e^{2\pi i t} z^2 \frac{z-a}{1-az}$, where $a = 4, 3$

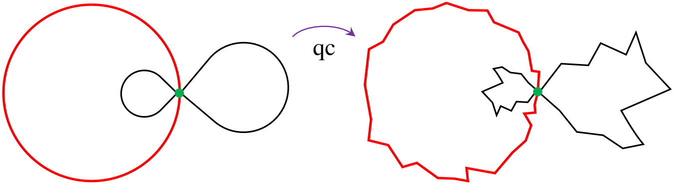
Definition (Degenerate Herman ring)

A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ is called a **degenerate Herman ring** of a meromorphic function f if γ is **not a spherical circle** and satisfies the following properties:

- γ is contained in the Julia set of f;
- γ is not a boundary component of any Siegel disk or Herman ring of f;
- $f(\gamma) = \gamma$ and $f : \gamma \to \gamma$ is conjugate to an irrational rotation.

Eremenko's question

A degenerate Herman ring by quasiconformal deformation (somewhat trivial):



$$\text{From} \quad z \mapsto e^{2\pi i t} z^2 \frac{z-3}{1-3z} \quad \text{to} \quad z \mapsto \frac{\lambda z + a z^2 (z-3)}{1 + (\frac{\lambda}{a} - 3) z} (0 < |\lambda| < 1).$$

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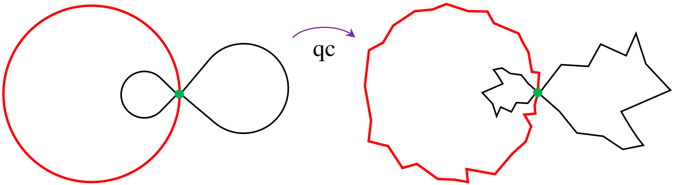

From $\quad z \mapsto e^{2\pi \mathrm{i} t} z^2 \dfrac{z-3}{1-3z} \quad$ to $\quad z \mapsto \dfrac{\lambda z + a z^2 (z-3)}{1 + (\frac{\lambda}{a} - 3) z} (0 < |\lambda| < 1).$

2 years ago, in the open problems session of the conference "On Geometric Complexity of Julia Sets II", Eremenko raised the following:

Question (Eremenko)

Do there exist (C^∞-)**smooth** degenerate Herman rings?

Main result

Theorem (Y., 2022)

There exist cubic rational maps having a smooth degenerate Herman ring.

Main ideas in the proof:

(1) Construction of smooth Siegel disks by Avila-Buff-Chéritat;

(2) Classical Siegel-to-Herman qc surgery by Shishikura;

(3) Control of the loss of Lebesgue measure of quadratic filled-in Julia sets by Buff-Chéritat;

(4) Rigidity of the cubic maps having bounded type Herman rings.

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Remark

Avila's construction of smooth ("degenerate") Herman rings does not need (2)(3)(4) since all perturbations can be done in the 1-parameter family $f_t(z) = e^{2\pi i t} z^2 \frac{z-a}{1-az}$, where $a > 3$ is given.

Siegel disk and conformal radius

Let $\alpha = [0; a_1, a_2, \cdots, a_n, \cdots] \in \mathbb{R} \setminus \mathbb{Q}$. Then $\frac{p_n}{q_n} := [0; a_1, a_2, \cdots, a_n] \to \alpha$ as $n \to \infty$.

If α is of Brjuno, i.e., $\sum_{n \geqslant 1} \frac{\log q_{n+1}}{q_n} < +\infty$, then

$$P_\alpha(z) := e^{2\pi i \alpha} z + z^2$$

has a **Siegel disk** Δ_α centered at the origin.

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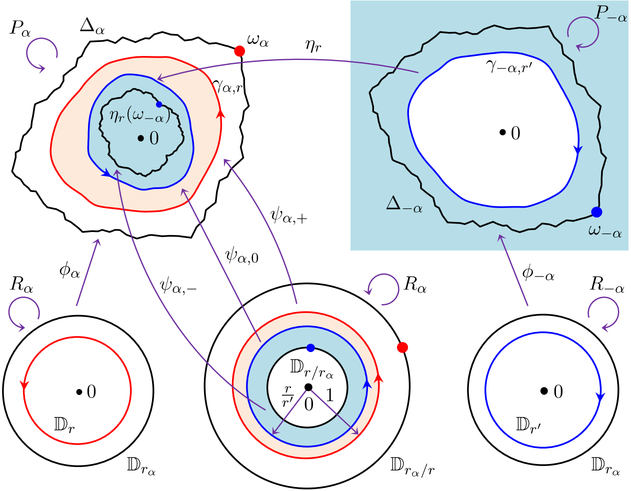
$\exists 1$ $r_\alpha > 0$ and conformal map $\phi_\alpha : \mathbb{D}_{r_\alpha} \to \Delta_\alpha$ with $\phi_\alpha(0) = 0$ and $\phi'_\alpha(0) = 1$, s.t.

$$\begin{array}{ccc} \mathbb{D}_{r_\alpha} & \xrightarrow{R_\alpha} & \mathbb{D}_{r_\alpha} \\ \downarrow \phi_\alpha & & \downarrow \phi_\alpha \\ \Delta_\alpha & \xrightarrow{P_\alpha} & \Delta_\alpha. \end{array}$$

The number $r_\alpha > 0$ is called the **conformal radius** of Δ_α.

Siegel-to-Herman qc surgery



Pasting $P_{\pm\alpha}(z) = e^{\pm 2\pi i\alpha} z + z^2$ together ($0 < r < r' < r_\alpha$) to obtain F_α (quasi-regular)

Lemma

$\forall$ Brjuno α, $\exists$ qc $\Phi_\alpha : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying $\Phi_\alpha(0) = 0$, $\Phi_\alpha(\infty) = \infty$ and $\Phi_\alpha(\omega_\alpha) = 1$ s.t.

$$Q_\alpha(z) := \Phi_\alpha \circ F_\alpha \circ \Phi_\alpha^{-1}(z) = uz^2 \frac{z-a}{1 - \frac{2a-3}{a-2} z}$$

has **4 critical points** $\left\{0, 1, \infty, c = \frac{a(a-2)}{2a-3}\right\}$ and a fixed **Herman ring** A_α with rotation number α and **modulus** $\frac{1}{\pi} \log \frac{r_\alpha}{r}$, where $u \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{C} \setminus \left\{0, 1, \frac{3}{2}, 2, 3\right\}$.

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Lemma

$\forall$ Brjuno α, we have

$$\text{area}(J(Q_\alpha)) = 0 \quad \Longleftrightarrow \quad \text{area}(J(P_\alpha)) = 0.$$

If α is of bounded type (or more generally, $\alpha \in \mathbf{PZ}$), then each component of ∂A_α is a Jordan curve passing through exactly one critical point and $\text{area}(J(Q_\alpha)) = 0$.

Petersen-Zakeri type: $\mathbf{PZ} := \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : \log a_n = \mathcal{O}(\sqrt{n}) \text{ as } n \to \infty\}$.

Rigidity of the surgery

Lemma

Suppose

$$\alpha \in \mathbf{PZ}, \quad 0 < r < r_\alpha \quad \text{and} \quad \theta \in [0, 2\pi). \qquad (*)$$

Then $\exists 1\ a = a(\alpha, r, \theta) \in \mathbb{C} \setminus \{0, 1, \frac{3}{2}, 2, 3\}$ and $u = u(\alpha, r, \theta) \in \mathbb{C} \setminus \{0\}$ s.t.

$$Q_{a,u}(z) := uz^2 \frac{z-a}{1 - \frac{2a-3}{a-2} z}$$

has a fixed **Herman ring** $A := A_{\alpha,r,\theta}$ in $\mathbb{C}$ satisfying:

- $Q_{a,u} : A \to A$ has rotation number α;
- $\mathrm{mod}(A) = \frac{1}{\pi} \log \frac{r_\alpha}{r}$; and
- $\partial_+ A$ and $\partial_- A$ are Jordan curves passing through the critical points 1 and $c = \frac{a(a-2)}{2a-3}$ respectively, and the **conformal angle** between 1 and c is θ.

We use $Q_{a,u}$ and $Q_{\alpha,r,\theta}$ alternately under the assumption (*).

Siegel disks and perturbations

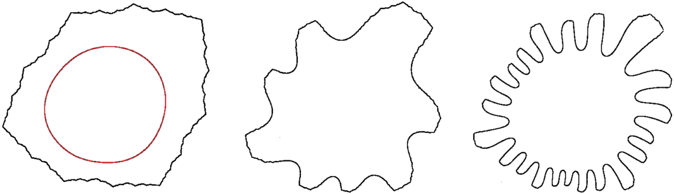
Lemma (Avila-Buff-Chéritat, 2004)

For any Brjuno number $\alpha := [0; a_1, a_2, \cdots, a_n, \cdots]$, any bounded type number $\beta := [0; t_1, t_2, \cdots, t_n, \cdots]$ and any radius r_0 with $0 < r_0 < r_\alpha$, let

$$\alpha_n := [0; a_1, a_2, \cdots, a_n, A_n, t_1, t_2, t_3, \cdots],$$

where $A_n := \lfloor (r_\alpha / r_0)^{q_n} \rfloor$. Then

1. $\alpha_n \to \alpha$ and $r_{\alpha_n} \to r_0$ as $n \to \infty$;
2. For any $\varepsilon > 0$, if n is large, then P_{α_n} has a repelling cycle which is ε-close to $\phi_\alpha(\mathbb{T}_{r_0})$.



Surgery's continuity

For any

$$\alpha \in \mathbf{PZ}, \quad 0 < r < r_\alpha \quad \text{and} \quad \theta \in [0, 2\pi),$$

$\exists 1$ conformal map $\chi_{\alpha,r,\theta} : \mathbb{A}_{r/r_\alpha, r_\alpha/r} \to A_{\alpha,r,\theta}$ with $\chi'_{\alpha,r,\theta}(1) > 0$, s.t.

$$\begin{array}{ccc} \mathbb{A}_{r/r_\alpha, r_\alpha/r} & \xrightarrow{R_\alpha} & \mathbb{A}_{r/r_\alpha, r_\alpha/r} \\ \Big\downarrow \chi_{\alpha,r,\theta} & & \Big\downarrow \chi_{\alpha,r,\theta} \\ A_{\alpha,r,\theta} & \xrightarrow{Q_{\alpha,r,\theta}} & A_{\alpha,r,\theta}. \end{array}$$

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High type numbers:

$$\mathrm{HT}_N := \left\{ \alpha = [0; a_1, a_2, \cdots, a_n, \cdots] \in \mathbb{R} \setminus \mathbb{Q} \mid a_n \geqslant N \text{ for all } n \geqslant 1 \right\}.$$

Let $N \geq 1$ be large enough s.t. the **near-parabolic renormalization** operator can be acted infinitely many times on P_α whenever $\alpha \in \mathrm{HT}_N$ (Inou-Shishikura, 2008).

Surgery's continuity

Main Lemma

For given $\alpha \in \mathbf{PZ} \cap \mathrm{HT}_N$, $0 < r < r_0 < r_\alpha$ and $\theta \in [0, 2\pi)$, we set

$$\alpha_n := [0; a_1, a_2, \cdots, a_n, A_n, N, N, N, \cdots] \quad \text{with } A_n = \lfloor (r_\alpha / r_0)^{q_n} \rfloor.$$

Then

1. $\alpha_n \to \alpha$, $r_{\alpha_n} \to r_0$ and $Q_{\alpha_n, r, \theta} \to Q_{\alpha, r, \theta}$ uniformly on $\widehat{\mathbb{C}}$ as $n \to \infty$;
2. $\chi_{\alpha_n, r, \theta} \to \chi_{\alpha, r, \theta}$ locally uniformly in $\mathbb{A}_{r/\rho, \rho/r}$ as large $n \to \infty$, where $\rho \in (r, r_0)$;
3. For any $\varepsilon > 0$, if n is large, then $Q_{\alpha_n, r, \theta}$ has two repelling cycles which are ε-close to $\chi_{\alpha, r, \theta}(\mathbb{T}_{r_0/r})$ and $\chi_{\alpha, r, \theta}(\mathbb{T}_{r/r_0})$ respectively.

Key point in the proof: $Q_{\alpha_n, r, \theta} = \Phi_{\alpha_n} \circ F_{\alpha_n} \circ \Phi_{\alpha_n}^{-1}$, where

$$F_{\alpha_n} \to F_\alpha \quad \text{and} \quad \Phi_{\alpha_n} \to \Phi_\alpha \quad \text{uniformly on } \widehat{\mathbb{C}} \text{ as } n \to \infty.$$

The convergence of Φ_{α_n} is based on the control the loss of area of quadratic filled-in Julia sets (Buff-Chéritat, 2012), and this is the only place where we need to restrict the irrational numbers to be of high type.

The proof

Fréchet space: Complete metrizable locally convex vector space. For example:

$$C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{C}) := \big\{f:\mathbb{R}/\mathbb{Z}\to\mathbb{C} \text{ is a } C^\infty\text{-function}\big\}.$$

Theorem (Y., 2022)

Assume that $\alpha\in\mathrm{HT}_N$ *is of bounded type,* $0<r<r_\alpha$, $\theta\in(0,2\pi)$ *and* $\varepsilon>0$. *Let*

$$g(t)=\chi_{\alpha,r,\theta}(e^{2\pi\mathrm{i}t}):\mathbb{R}/\mathbb{Z}\to\mathbb{C}, \quad \text{where } Q_{\alpha,r,\theta}=Q_{a,u}.$$

Then, $\exists\ \alpha'\in\mathrm{HT}_N$, *a cubic rational map* $Q_{a',u'}$ *and a* C^∞*-embedding* $h:\mathbb{R}/\mathbb{Z}\to\mathbb{C}$ *s.t.*

- $|\alpha-\alpha'|<\varepsilon$, $|a-a'|<\varepsilon$ *and* $|u-u'|<\varepsilon$;
- $h(\mathbb{R}/\mathbb{Z})$ *is a* ***smooth degenerate Herman ring*** *of* $Q_{a',u'}$ *with rotation number* α' *which is accumulated by repelling cycles from both sides;*
- g *and* h *are* ε*-close in the Fréchet space* $C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{C})$.

Area of Julia sets

Fatou, 1919[2]:

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substitution inverse, et telle que la substitution donnée la transforme en une autre qui lui soit complètement intérieure, de manière qu'elle fasse partie du domaine d'attraction du point double, si en outre, sur les courbes antécédentes de C, on a à partir d'un certain rang

$$|R'(z)| > k > 1,$$

le domaine total du point double a pour frontière un ensemble parfait partout discontinu; cet ensemble est de mesure linéaire nulle si k est supérieur au degré d de $R(z)$, de mesure superficielle nulle si $k > \sqrt{d}$.

[2] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), 161–271.

Area of Julia sets

Lebesgue measure of nowhere dense Julia sets of rational maps:

Zero area: Very fruitful results (The place is too small to write all of them down).

Positive area:

- (Buff-Chéritat, 2012): Siegel, Cremer, ∞-satellite renorm. with unbounded combinatorics
- (Avila-Lyubich, 2022): ∞-primitive renorm. with bounded combinatorics
- (Dudko-Lyubich, 2018): ∞-satellite renorm. with bounded combinatorics

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Theorem (Y., 2022)

There exist cubic ***rational maps*** *having a nowhere dense Julia set of* ***positive area*** *for which these maps have no irrationally indifferent periodic points, no Herman rings, and are not renormalizable.*

Intuitively, such rational maps can be seen to be obtained by "pasting" two quadratic Siegel polynomials along their Siegel disk boundaries.

Alternative path to Eremenko's question

W. R. Lim (2022) gives another method of constructing **non-trivial** degenerate Herman rings (i.e., not by a qc deformation of Blaschke products), based on the study of a prior bounds of bounded type Herman rings.

The degenerate Herman rings constructed there are **quasi-circles** passing through critical points (hence are not smooth).

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Question (Eremenko, 2012)

Does there exist a Jordan **analytic** invariant curve of a rational map, different from a circle, which is mapped onto itself homeomorphically and intersects the Julia set?

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Question

Do there exist **analytic** degenerate Herman rings?

Further questions

Questions

(1) Does there exist a smooth degenerate Herman ring whose rotation number is **not of Brjuno type**?

(2) Does there exist a degenerate Herman ring which is **not a quasi-circle**?

(3) (Eremenko) Do there exist smooth degenerate Herman rings for **transcendental meromorphic** functions?

(4) (folk) Do there exist at most finitely renormalizable **polynomials** without irrationally indifferent periodic points whose Julia sets have positive area?

Thank you for your attention !