Gdańsk-Kraków-Łódź-Warszawa Workshop in Singularity Theory

A special session dedicated to the memory of
Stanisław Łojasiewicz

December 12-16, 2022

Booklet of abstracts

Institute of Mathematics
Polish Academy of Sciences
Warsaw, Poland

IMPAN, 2022

About the conference

Conference website:

https://www.impan.pl/en/activities/banach-center/conferences/22-lojasiewicz

Hosting institution:

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, 00-656 Warsaw, Poland

Scientific Committee:

Jérôme Bolte (TSE & University of Toulouse)
Bronisław Jakubczyk (IMPAN Warsaw)
Stanisław Janeczko (Warsaw University of Technology)
Tadeusz Mostowski (University of Warsaw)
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Supported by:

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Foreword

The aim of the workshop is to commemorate the 20th anniversary of the death of Professor Stanisław Łojasiewicz – who played a crucial role in the creation of the Gdańsk-Kraków-Łódź-Warszawa Seminar in Singularity Theory, which is currently held online each month at IMPAN – by bringing together world-class experts in real and complex analytic geometry and in singularity theory – a vast area of research to which S. Łojasiewicz has made a fundamental contribution – as well as other researchers eager to connect with techniques in these domains. In particular, the diffusion of these techniques and their potential in current research problems, especially among young researchers, is one of the objectives of the meeting.

Topics that will be discussed during the meeting include the geometry of semi- and subanalytic sets, o-minimal structures, Łojasiewicz inequalities and their applications to non-standard and functional analysis, Lipschitz geometry, stratification theory, equisingularity and invariants of singularities.

We wish you all a nice and fruitful stay in Warsaw!

The organizers

Contents

Abstracts of mini-courses

Symplectic Monodromy "at radius 0" and equimultiplicity of families of hypersurfaces with constant Milnor number

12, 15 Nov 14:00
13, 16 Nov 9:15

Javier Fernández de Bobadilla
IKERBASQUE & BCAM Bilbao

In recent work we proved the Zariski multiplicity conjecture for families of isolated hypersurface singularities. For it we show how to construct a symplectic monodromy "at radius 0" with very special dynamical properties, which is symplectically isotopic by radius variation to the usual symplectic monodromy at positive small radius. For this we use a hybrid construction employing log-geometry (Kato-Nakayama spaces) and tropical geometry. In particular our construction provides a smooth atlas in the topological space used by A'Campo for his study of monodromy zeta function and Lefschetz numbers, and provides an alternate construction of special symplectic monodromy representatives due to McLean which is better suited for the study of families and degenerations. Then we use a slight generalization of a spectral sequence in Floer Homology (due to McLean) to recover multiplicity, and properties of invariance of Floer homology along symplectic isotopies to prove its constancy in Milnor number constant families.

During the course I will provide details on certain aspects of the constructions described above.

This is a joint work with Tomasz Pełka.

Gradient inequalities, generalizations and quantitative aspects

12, 15 Nov 9:15
13, 16 Nov 14:00

Krzysztof Kurdyka
Savoie Mont Blanc University

In 4 lectures I will describe several generalizations, variants and applications of the celebrated Łojasiewicz gradient inequality

Theorem. (Łojasiewicz 1962). *Let $f : U \to \mathbb{R}$ be a real analytic function, where $U \subset \mathbb{R}^n$ is open and $K \subset U$ a compact set. Then for any $x^* \in K$ there exists constants $C > 0$, $\varepsilon >$ and $\rho < 1$ such that*

$$\|\nabla f(x)\| \geq C|f(x) - f(x^*)|^\rho, \tag{1}$$

for $x \in K$ such that $|f(x) - f(x^*)| \leq \varepsilon$.

Tentative plan

Lecture 1: Talweg and the length of gradient trajectories

For a real valued smooth function f on a Riemannian M manifold, by the *gradient extremal set* of f we mean all the critical points of the square of the norm of the gradient restricted to the fibres of f. If f is a generic (Morse) function, then this set is a union of smooth curves which intersect 'transversally' at critical points. The part of the gradient extremal set corresponding to the local minima is called a *Talweg* or *ridge and valley lines*. It can be used to estimate the length of gradient trajectories and possibly should organize the dynamics of the gradient flow. Talwegs play important role in numerous applications, for instance geology, theoretical chemistry, image compressing. The Talwegs of a Morse function on a compact manifold may reveal the topology of the manifold.

Lecture 2: Effective estimates for the length of gradient trajectories in the polynomial case, also for discrete trajectories

I will show how to compute effectively Talwegs for polynomials on open sets or smooth algebraic sets. Then, using the Cauchy-Crofton formula, one gets effective estimates for the length of gradient trajectories in the polynomial case as a function only of the degree and the diameter of the domain. In the case of compact algebraic sets one gets an estimate for their geodesic diameter. I will also discuss the convergence of proximal discrete gradient trajectories.

Lecture 3: From Talweg to the Łojasiewicz gradient inequality and KL-inequality

I will show how the notion of Talweg is used to prove the celebrated gradient inequality which states (in the original version) that if f is analytic, then in a neighborhood of x_0

$$||\nabla f|| \geq c|f - f(x_0)|^{\rho}, \tag{2}$$

for some $\rho < 1$ and $c > 0$. I will discuss estimates for the exponent ρ where f is a polynomial. The goal of this lecture will be to present the main ingredients of the proof of this inequality or, more precisely, its generalization in o-minimal geometry known as the *KL-inequality*.

Lecture 4

In the last lecture I would like to develop one of the following topics:

1. Generalizations of Łojasiewicz's gradient inequality (or rather the KL inequality) for the maps in the o-minimal and analytic case. It gives bounds for volume of foliations transverse to the fibers of an o-minimal map.

2. Trajectories of the horizontal gradient. If we consider a smooth function f on a manifold M equipped with a non-holonomic distribution Δ (for instance a contact or Engel manifold) and a metric g on Δ (called sub-riemannian metric), then we can define $\nabla_g f$ which is the dual to the differential of f restricted to Δ. In this case Łojasiewicz's gradient inequality (or the KL inequality) is no longer valid since the set of critical values may contain an open set. Trajectories of $\nabla_g f$ may not have a limit (they can accumulate on a 'cycle'). However, for a generic potential f they have a limit.

Abstracts of talks

Partial desingularization

16 Nov
10:30

Edward Bierstone
University of Toronto

Can we understand the nature of the singularities that have to be admitted after a blow-up sequence that preserves the normal crossings locus of an algebraic (or complex-analytic) variety X? For example, every surface can be transformed by blowings-up preserving normal crossings to a surface with at most additional Whitney umbrella singularities. We will discuss general conjectures, and solutions for X of dimension up to four. The techniques involve circulant matrices, elementary Galois theory and Newton-Puiseux expansion in several variables.

Work in collaboration with André Belotto and Ramon Ronzon Lavie.

Lipschitz Geometry of Real Surfaces. General Overview.

15 Nov
16:15

Lev Birbrair
Federal University of Ceará

I am going to give an overview of the present situation in the Lipschitz Geometry of Real Surfaces. The subject includes some new results in Local and Global Lipschitz Geometry, Outer, Inner and Ambient Lipschitz Geometry.

Poincaré-Hopf Theorem *à la Lipschitz* – inspired by Marie-Hélène Schwartz and Tadeusz Mostowski

14 Nov
9:15

Jean-Paul Brasselet
Aix-Marseille University

It is well known that the Poincaré-Hopf Theorem does not hold anymore in the case of singular varieties. Using what she called *radial extension of stratified vector fields*, Marie-Hélène Schwartz showed that one recovers the Theorem in the framework of Whitney stratified varieties. The Mostowski's conditions for defining Lipschitz stratifications allow to provide a precise construction in this setting.

This is a joint work with Tadeusz Mostowski and Nguyen Thi Bich Thuy.

Structure of the tree at infinity of a polynomial in two variables

16 Nov 11:25

Pierrette Cassou-Noguès
University Bordeaux 1

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial map. Let $\mathbb{C}^2 \subset X$ be a compactification of $\mathbb{C}^2$ where X is a smooth rational compact surface and such that there exists a map $\Phi\colon X \to \mathbb{P}^1$ which extends f. Put $\mathcal{D} = X \setminus \mathbb{C}^2$; $\mathcal{D}$ is a curve whose irreducible components are smooth rational compact curves and all its singularities are ordinary double points. The dual graph is a tree. We are interested in this tree. It can be very complicated. The aim is to understand a structure in this tree.

In this talk, we will forget about polynomials and only work with abstract trees with some properties (verified by trees arising from polynomials) and define their genus. We will consider some subtrees called *teeth*, and replace them by simpler subtrees without changing the genus. We then arrive at what we call reduced trees. Now in the reduced trees, we consider some parts that we call *combs*, with a particular structure. The main result is that we can bound the number of combs in a reduced tree in terms of its genus. A consequence is that a tree of genus 0 has only one comb, and we can describe the general shape of a tree of genus 0. Trees of genus 1 are also easy to deal with.

We hope to have time to speak about the Alexander polynomial of a tree.

This is a joint work with Daniel Daigle.

Regularity in analysis and geometry and the influence of Łojasiewicz

12 Nov 17:10

Tobias Colding
MIT

Many problems in geometry and analysis can be thought of as questions about gradient flows on infinite dimensional spaces. This occurs in the analysis of the formation of singularities in many geometric problems. This connection turns many regularity questions into questions about infinite dimensional dynamical systems. Infinite dimensional generalizations of Łojasiewicz's first and second inequality and Łojasiewicz's theorem play key roles in this.

The geometry of 2-dimensional indefinite improper affine spheres

13 Nov
14:55

Wojciech Domitrz
Warsaw University of Technology

Improper affine spheres are hypersurfaces whose affine Blaschke normal vectors are all parallel. They are given as the graphs of solutions of the classical Monge-Ampère equation. We describe generic singularities and the geometry of 2-dimensional indefinite improper affine spheres. Finally we prove Gauss-Bonnet formulas for 2-dimensional indefinite affine spheres with generic singularities. This is a joint work with M. Zwierzyński.

Lipschitz geometry of abnormal definable surface germs

16 Nov
17:10

Andrei Gabrielov
Purdue University

There are two natural metrics, inner and outer, on a surface germ definable in a polynomially bounded o-minimal structure (e.g., semialgebraic or subanalytic). A germ is normally embedded if these two metrics are equivalent. Although inner Lipschitz classification of surface germs goes back to Birbrair '99, outer Lipschitz classification remains an open problem even for Hölder triangles, the building blocks of inner classification. If a Hölder triangle T is not normally embedded, it contains an *abnormal arc* γ: there are two normally embedded subtriangles T' and T'' of T such that $T' \cap T'' = \gamma$ and $T' \cup T''$ is not normally embedded. A Hölder triangle is *abnormal* if its generic arcs are abnormal and non-generic arcs are normal. Abnormal Hölder triangles (aka *snakes*) play important role in outer Lipschitz geometry of surface germs. In this talk, we review their geometric and combinatorial properties.

This is a joint work with Emanoel Sousa.

On the Tjurina and Milnor numbers of a foliation

16 Nov
14:55

Evelia R. García Barroso
University of La Laguna

We present the relationship between the Tjurina and Milnor numbers of a singular foliation $\mathcal{F}$, in the complex plane, with respect to a balanced divisor of separatrices of $\mathcal{F}$. This is a joint work with Arturo Fernández Pérez and Nancy Saravia Molina.

Equi-singularity of real families and Curvatures at infinity

12 Nov
16:15

Vincent Grandjean
Federal University of Ceará

In this joint work with Nicolas Dutertre (Université d'Angers), we relate certain equisingularity criterion of real families definable in a polynomially bounded o-minimal structures with the continuity, in the parameters, of densities at infinity obtained from the Lipschitz-Killing curvatures of the elements of the family.

Łojasiewicz inequalities in a certain class of smooth functions

15 Nov
10:30

Hà Huy Vui
TIMAS - Thang Long University

We consider the Łojasiewicz inequalities for smooth functions satisfying the Kamomoto and Nose condition. We indicate some cases where the Łojasiewicz exponents can be computed explicitly. Some relationship between the Łojasiewicz exponents and the so called line type of smooth convex functions is established.

This is a joint work with Ha Minh Lam, Institute of Mathematics, VAST, Vietnam.

The tropical bifurcation set of polynomial maps on a plane

13 Nov
10:30

Boulos El-Hilany
TU Braunschweig

The bifurcation set of a polynomial map $X \to Y$ between two smooth affine varieties is the smallest set of points in Y outside of which the map is a locally trivial smooth fibration.

Standard methods for characterizing the bifurcation set rely on elimination techniques that can often be inefficient. This talk concerns polynomial maps on the two-dimensional torus defined over the field of generalized Puiseux series with complex coefficients. I will present a combinatorial procedure for computing the tropical curve of the bifurcation set of general polynomial maps with given support. I will also apply this new description to compute the Newton polytope of the bifurcation set for a complex polynomial map on the plane.

Unfolding theory, Stable maps and Mather-Yau/Gaffney-Hauser results in arbitrary characteristic

15 Nov
14:55

Dmitry Kerner
Ben Gurion University of the Negev

In 40's Whitney studied maps of $\mathcal{C}^\infty$ manifolds. When a map is not an immersion/submersion, one tries to deform it locally, in hope to make it 'generic'. This approach has led to the rich theory of stable maps, developed by Mather, Thom and many others. The main 'engine' was vector field integration. This chained the whole theory to the $\mathcal{C}^\infty$, or $\mathbb{R}/\mathbb{C}$-analytic setting. I will present the purely algebraic approach, studying maps of germs of Noetherian schemes, in any characteristic. The relevant groups of equivalence admit 'good' tangent spaces. Submodules of the tangent spaces lead to submodules of the group orbits. Then goes the theory of unfoldings (triviality and versality). Then I will discuss the new results on stable maps and theorems of Mather-Yau/Gaffney-Hauser.

Three hypotheses on the Łojasiewicz exponent

12 Nov
14:55

Tadeusz Krasiński
University of Łódź

This is a joint work with Szymon Brzostowski.

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function possessing an isolated critical point at the origin. In this talk, we will discuss three conjectures concerning the Łojasiewicz exponent $ł(f)$ of the function f: attaining on special curves, topological invariance and effective formulae in non-degenerate cases. Discussing the evidence gathered in favour of the conjectures, we will in particular focus on recent results.

Approximation and homotopy in real algebraic geometry

12 Nov
10:30

Wojciech Kucharz
Jagiellonian University

I will talk about a joint paper with Jacek Bochnak containing an appendix written by János Kollár. Let X be a real algebraic variety and let Y be a homogeneous space for some linear real algebraic group. We prove that a continuous map $f\colon X \to Y$ can be approximated by regular maps in the compact-open topology if and only if it is homotopic to a regular map. Taking $Y = \mathbb{S}^p$, the unit p-dimensional sphere, we obtain solutions to several problems that have been open since the 1980s. This has several consequences for approximation of maps between unit spheres. For example, we prove that for every positive integer n every continuous map from $\mathbb{S}^n$ into $\mathbb{S}^n$ can be approximated by regular maps. So far such a result has only been known for five special values of n, namely, $n = 1, 2, 3, 4$ or 7.

The Łojasiewicz exponent of the gradient of a plane complex curve with respect to its polar curve

14 Nov 11:25

Andrzej Lenarcik

Kielce University of Technology

Let $f = 0$ be a germ of an isolated singularity at $0 \in \mathbb{C}^2$ defined by $f \in \mathbb{C}\{X, Y\}$. The Łojasiewicz exponent $\mathcal{L}_0(f)$ is the best $\theta \geq 0$ in the local inequality $(*)$ $|\mathrm{grad} f(z)| \geq c|z|^\theta$. Every nonsingular $\lambda \in \mathbb{C}\{X, Y\}$ (local parameter) defines the polar curve $\Gamma_{f,\lambda}$: $(\partial\lambda/\partial X)(\partial f/\partial Y) - (\partial\lambda/\partial Y)(\partial f/\partial X) = 0$. Restricting $(*)$ to $z \in \Gamma_{f,\lambda}$ we consider the relative Łojasiewicz exponent $\mathcal{L}_0(f|\Gamma_{f,\lambda})$. After the articles of Bogusławska, Kuo-Parusiński and Płoski, we know that $\mathcal{L}_0(f) = \mathcal{L}_0(f|\Gamma_{f,\lambda})$ when f and λ are transverse. Therefore, we focus mostly on the case with f and λ tangent. The relative Łojasiewicz exponent does not change after analytical coordinate change. We ask whether or not this exponent is an equisingularity invariant of the pair f, λ. We found a specific equisingularity classes with negative answer. These classes may be described by using Newton polygons. After a coordinate change with $\lambda = X$ (in this case $\Gamma_{f,\lambda} = \{\partial f/\partial Y = 0\}$), the Newton polygon of f has one segment that joins $(p, 0)$ and $(0, q)$ with $\mathrm{ord} f = p < q$. Moreover f is nondegenerate on this segment. In such class we state that $\mathcal{L}_0(f|\partial f/\partial Y = 0)$ is greater than or equal to $p - 1$ and less than or equal to $q - 1$. We found examples with different exponents in the same class. For other classes the relative Łojasiewicz exponent is the equisingularity invariant. We describe its value by using the Eggers tree of f and by observing where λ leaves the tree.

Tame geometry in Hensel minimal structures: Łojasiewicz inequalities and applications

13 Nov 11:25

Krzysztof Jan Nowak

Jagiellonian University

In my talk, I will present tame topology in Hensel minimal structures from my recent papers, with special attention focused on the Łojasiewicz inequalities. The theory of those structures, introduced in a recent article by Cluckers-Halupczok-Rideau, seems to be a suitable non- Archimedean counterpart of o-minimality from real algebraic geometry. I additionally require that every definable subset in the imaginary sort RV be already definable in the pure valued field language. This condition is satisfied by many of the classical tame structures on Henselian fields (including Henselian fields with analytic structure, V-minimal fields and polynomially bounded o-minimal structures with a convex subring), and ensures that the residue field is orthogonal to the value group.

At the center of my approach is my closedness theorem to the effect that the projections with projective fiber are definably closed maps. This, along with quantifier elimination for ordered Abelian groups, allows me to reduce the problems behind the Łojasiewicz inequalities to those of semilinear geometry in the value group sort.

Among the applications of the Łojasiewicz inequalities is, for instance, that every closed definable subset is the zero set of a global continuous definable function. This,

in turn, is an ingredient of the proof of the embedding theorem for regular definable spaces, which goes back to Robson and van den Dries in the case of o-minimal geometry. Also discussed will be extension of continuous definable functions.

Motivic, logarithmic, and topological Milnor fibrations

14 Nov
10:30

Adam Parusiński
University of Nice

We compare the topological Milnor fibration and the motivic Milnor fibre of a regular complex function with only normal crossing singularities by introducing their common extension: the complete Milnor fibration. We give two equivalent constructions: the first one extending the classical Kato–Nakayama log-space, and the second one, a version of the real oriented deformation to the normal cone. In particular, we recover A'Campo's model of the topological Milnor fibration, by quotienting the motivic Milnor fibration with suitable powers of $R > 0$, and show that it determines the classical motivic Milnor fibre.

This is a joint work with J.-B. Campesato and G. Fichou.

On the main scientific achievements of Stanisław Łojasiewicz.

15 Nov
17:10

Wiesław Pawłucki
Jagiellonian University

During the talk we will present the main scientific achievements of Stanisław Łojasiewicz (1926-2002).

The extra-nice dimensions

13 Nov
16:15

Maria Aparecida Soares Ruas
ICMC-USP

We define the extra-nice dimensions and prove that the subset of locally stable 1-parameter families in $\mathcal{C}^\infty(N \times [0,1], P)$ is dense if and only if the pair of dimensions $(\dim N, \dim P)$ is in the extra-nice dimensions. This result is parallel to Mather's characterization of the nice dimensions as the pairs (n, p) for which stable maps are dense. The extra-nice dimensions are characterized by equisingularity properties of generic hyperplane sections of discriminants of stable germs in one dimension higher. They are also related to the simplicity of $\mathscr{A}_e$-codimension 2 germs. The results raise interesting questions about the geometry of generic sections of discriminants of stable maps in the nice dimensions.

This is a joint work with Raul Oset-Sinha (U.València) and Roberta Wik Atique (ICMC-USP).

Some applications of the Łojasiewicz inequality in optimization

12 Nov
11:25

Stanisław Spodzieja
University of Łódź

We show that if a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is nonnegative on a closed basic semialgebraic set $X = \{x \in \mathbb{R}^n \colon g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$, where $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$, then f can be approximated uniformly on compact sets by polynomials of the form $\sigma_0 + \varphi(g_1)g_1 + \ldots + \varphi(g_r)g_r$, where $\sigma_0 \in \mathbb{R}[x_1, \ldots, x_n]$ and $\varphi \in \mathbb{R}[t]$ are sums of squares of polynomials. In particular, if X is compact, and $h(x) := R^2 - |x|^2$ is positive on X, then $f = \sigma_0 + \sigma_1 h + \varphi(g_1)g_1 + \ldots + \varphi(g_r)g_r$ for some sums of squares $\sigma_0, \sigma_1 \in \mathbb{R}[x_1, \ldots, x_n]$ and $\varphi \in \mathbb{R}[t]$, where $|x|^2 = x_1^2 + \ldots + x_n^2$.

We give stronger versions of known approximation and representation theorems with sums of squares of polynomials. Then, using the Łojasiewicz inequality, we give quantitative versions of these results and explain some applications to semidefinite optimization methods.

This is a joint work with Krzysztof Kurdyka.

A view of point for the 2-dimensional Jacobian conjecture

16 Nov
16:15

Nguyen Thi Bich Thuy
University of São Paulo

We construct a class of non-proper polynomial maps $F \colon \mathbb{K}^2 \to \mathbb{K}^2$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and study the "nowhere vanishing Jacobian" condition of this class. We obtain on the one hand a large class of polynomial maps satisfying the 2-dimensional Jacobian conjecture for both real and complex cases and, on the other hand a method on the way to construct a counter-example, if such counter-example exists. We prove also that the 2-dimensional complex Jacobian conjecture is true up to the degree 104. We present also the results of joint works with Anna Valette and Guillaume Valette, and Maria Aparecida Soares Ruas, about an approach to polynomial maps using intersection homology. We end our talk by presenting the singular variety associated to a Pinchuk map.

Parallel hiking trails in the mountains on the torus

15 Nov
11:25

Michał Zwierzyński
Warsaw University of Technology

Let us consider a continuous function $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = f(1) \bmod 1$. Let us assume that f has finitely many local extrema and the values at these extrema are pairwise different non-integers modulo 1. Then, let $\hat{f}(x) = f(x) \bmod 1$ for $x \in [0,1]$. The graph of $\hat{f}$ is called a *mountain*. Note that $\hat{f}\colon [0,1] \to [0,1]$ and if we glue the endpoints of the interval $[0,1]$, the graph of $\hat{f}$ is a subset of the torus.

Now, let us imagine two points (the *hikers*) on a mountain m which is the graph of $\hat{f}$. The hikers can move on m but all the time they must be on the same level on m. Now, we will add more restrictions for moving in the mountains. The hikers always move forward (left or right) except when one of them encounters a peak or valley (an extremum of $\hat{f}$) – then the other hiker turns back. Thanks to this strange procedure, the hikers will be on the same level all the time.

Finally, we are asking: how many different hiking trails are for a fixed mountain? If both hikers starts in the same peak of valley – will they ever meet again (if so, where?)? During the talk, we will get to know partial answers to the questions posed.

The problem posed above only seemingly has nothing to do with singularity theory and differential geometry. During the lecture, we will associate the properties of hiking trails with the global geometry of singular sets such as the Wigner caustic, the Centre Symmetry Set, the secant caustic, the constant measure set.
