Partial desingularization

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Introduction

Motivating question. Given an algebraic (or complex-analytic) variety X, can we find a sequence of blowings-up $\sigma : X' \to X$ such that σ preserves the normal crossings locus of X, and X' has only normal crossings singularities?

Normal crossings at a point a: X can be defined by a monomial equation

$$x_{j_1} \cdots x_{j_k} = 0$$

in local coordinates $(x_1, \dots, x_n)$ at a.

Answer depends on the meaning of local coordinates; e.g.,

regular coordinates (or regular system of parameters)

simple normal crossings snc

(locally, irreducible components smooth, transverse)

local analytic (or étale or formal) coordinates (after finite field extension)

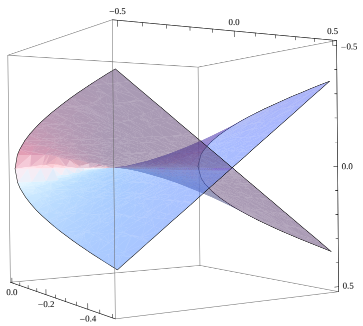
normal crossings nc

Examples

$$y^2 = x^2 + x^3 \qquad \text{nc, not snc}$$
$$y^2 + x^2 = 0 \qquad \text{nc, snc iff } \sqrt{-1} \in \mathbb{K}$$

Answer is yes for snc [B-Milman 1997], Szabo, Kollár, . . .
no for nc, in general.

Example. Whitney umbrella $z^2 - wx^2 = 0$.



nc(2) along nonzero w-axis
pinch point pp at 0

There is no proper birational morphism that eliminates pp without modifying nc points.

Theorem [BM 2012]. Suppose $\dim X = 2$. Then there is a smooth blowing-up sequence $\sigma : X' \to X$ preserving X^{nc}, such that X' has only nc and pp singularities.

Remark. Whitney's umbrella has smooth normalization:
Set $w = v^2$. Then $z^2 - wx^2$ splits as $(z - vx)(z + vx)$.

Conjecture A. For any algebraic (or complex-analytic) variety X, there is a finite composite of admissible smooth blowings-up $\sigma : X' \to X$, preserving X^{nc}, such that X' has smooth normalization.

Admissible means, in particular, centres normal crossings with the exceptional divisor.

More concretely:

Conjecture B. ... such that X' has only minimal singularities from an explicit finite list (with smooth normalization).

Theorem. The conjectures are true for $\dim X \leq 4$.

For example, proof for $\dim X = 2$

We consider a hypersurface in 3 variables.

Then nc(3) points are isolated, and the nc(2)-locus has codimension 2.

We can blow up in the complement of nc(3) without modifying nc(2), until the maximum value of the desingularization invariant inv is inv(nc(2)). Then the locus inv = inv(nc(2)) is a smooth curve C.

At any point of C, we can choose local coordinates in which

$$X: \; z^2 + w^k x^2 = 0$$

(w an exceptional divisor; X is nc(2) on $\{z = x = 0, \; w \neq 0\}$.)

By finitely many blowings-up with centre $\{z = w = 0\}$, we transform X to either

$$\begin{aligned} z^2 + x^2 &= 0 \qquad \text{nc(2)} \\ \text{or} \quad z^2 + wx^2 &= 0 \qquad \text{pp} \end{aligned}$$

Note that $z^2 + w^k x^2$ splits as a polynomial in $w^{1/2}, x, z$.

The proofs of our main results are combinations of methods from resolution of singularities and splitting techniques. The latter will be the main focus of the second part of the talk.

Circulant singularities

Circulant matrix

$$C_k(X_0, X_1, \ldots, X_{k-1}) := \begin{pmatrix} X_0 & X_1 & \cdots & X_{k-1} \\ X_{k-1} & X_0 & \cdots & X_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & \cdots & X_0 \end{pmatrix}$$

Eigenvectors

$$V_\ell = (1, \varepsilon^\ell, \varepsilon^{2\ell}, \ldots, \varepsilon^{(k-1)\ell}), \quad \ell = 0, \ldots, k-1,$$

where $\varepsilon = e^{2\pi i/k}$;

corresponding eigenvalues

$$Y_\ell = X_0 + \varepsilon^\ell X_1 + \cdots + \varepsilon^{(k-1)\ell} X_{k-1}$$

Set $\Delta_k := \det C_k$.

Circulant singularity $\mathrm{cp}(k)$ is defined by

$$\Delta_k(x_0,\ w^{1/k}x_1,\ \ldots,\ w^{(k-1)/k}x_{k-1})$$
$$= \prod_{\ell=0}^{k-1}\left(x_0 + \varepsilon^{\ell}w^{1/k}x_1 + \cdots + \varepsilon^{(k-1)\ell}w^{(k-1)/k}x_{k-1}\right)\ ;$$

an irreducible polynomial in $w, x_0, \ldots, x_{k-1}$.

$\mathrm{cp}(k)$ is a limit of k-fold normal crossings $\mathrm{nc}(k)$ singularities in $k+1$ variables.

$\mathrm{cp}(k)$ has smooth normalization: splits on substituting $w = v^k$.

For example,

$\mathrm{cp}(1) = \text{smooth}$

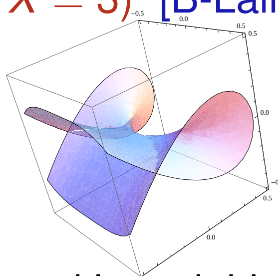
$\mathrm{cp}(2) = \mathrm{pp}:\quad \Delta_2(z,\ w^{1/2}x) = z^2 - wx^2$

$\mathrm{cp}(3):\quad \Delta_3(z,\ w^{1/3}y,\ w^{2/3}x) = z^3 + wy^3 + w^2x^3 - 3wxyz$

Minimal singularities: products of circulant singularities and their singular neighbours.

Example. 4 variables ($\dim X = 3$) [B-Lairez-Milman, 2012].

Neighbours of $\mathrm{cp}(3)$:



$\mathrm{nc}(2)$, $\mathrm{nc}(3)$,

$\Delta_3(z,\ w^{1/3}y,\ w^{2/3})$

Minimal singularities: $\mathrm{cp}(3)$ and its neighbours, together with

$\mathrm{nc}(4)$, $\mathrm{cp}(2)$, $\text{smooth} \times \mathrm{cp}(2)$: $y(z^2 - wx^2) = 0$.

5 variables (dim $X = 4$)

Neighbours of cp(4), $\Delta_4(x_0, w^{1/4}x_1, w^{2/4}x_2, w^{3/4}x_3) = 0$:

nc(k), $k = 2, 3, 4$, together with

(1) $\Delta_4\left(z, w^{1/4}x_1, w^{2/4}x_2, w^{3/4}\right)$

(2) $\Delta_4\left(z, w^{1/4}x_1, w^{2/4}, w^{3/4}x_3\right)$

(2′) $\Delta_4\left(z, w^{1/4}x_1, w^{2/4}, w^{3/4}x_2x_3\right)$, x_2 exceptional

(3) $\Delta_4\left(z, w^{1/4}x_1, w^{2/4}, w^{3/4}\right)$

Minimal singularities include the following limits of nc(4):

cp(4), smooth $\times$ cp(3), cp(2) $\times$ cp(2),

smooth $\times$ smooth $\times$ cp(2) = nc(2) $\times$ cp(2) .

Approach to the main problems

We can assume that, X is a hypersurface, because nc are hypersurface singularities, and the desingularization algorithm blows up points of highest embedding dimension.

More precise version of Conjecture B

Given k, there is a finite composite of admissible smooth blowings-up $\sigma : X' \to X$, preserving nc of order up to k, such that X' has only minimal singularities.

Inductive strategy

- Blow up following the desingularization algorithm until $\text{inv} \leq \text{inv}(\text{nc}(k))$.

- Modify non-nc points of

$$S_k := \{\text{inv} = \text{inv}(\text{nc}(k))\}$$

  to get minimal singularities in $\Sigma_k = S_k \cup D_k$,
  where $D_k \subset E$ is a distinguished nc divisor,
  and only nc in $U \backslash \Sigma_k$, for some neighbourhood U of Σ_k.

- Apply the inductive hypothesis in the complement of Σ_k; the centres of blowing up extend to global admissible centres.

Modification of a stratum S_k at a non-nc point

At a limit of $\text{nc}(k)$, we can reduce to the case that

$$X : \quad f(w_1, \ldots, w_r, u_1, \ldots, u_q, x_1, \ldots, x_{k-1}, z) = 0\,,$$

where

(1) $f(w,u,x,z) = z^k + a_1(w,u,x)z^{k-1} + \cdots + a_k(w,u,x)\,;$

(2) The $\text{nc}(k)$-locus is $\{x = z = 0\,,\ w_1 \cdots w_r \neq 0\}$, where w represents the exceptional divisor; in particular, f is generically $\text{nc}(k)$ on $\{x = z = 0\}\,;$

(3) $\text{inv} = \text{inv}(\text{nc}(k))$ on $\{x = z = 0\}\,.$

We are interested in the splitting of f as

$$f(w,u,x,z) = \prod_{j=1}^{k}(z - b_j(w,u,x)) \,.$$

Notation.

$\mathbb{C}(w)$ (or $\mathbb{C}(\!(w)\!)$) field of fractions of $\mathbb{C}[w]$ (or $\mathbb{C}[\![w]\!]$).

$\overline{\mathbb{C}(w)}$, $\overline{\mathbb{C}(\!(w)\!)}$ algebraic closures.

$\overline{\mathbb{C}(w)}$: subfield of $\overline{\mathbb{C}(\!(w)\!)}$ of elements algebraic over $\mathbb{C}[w]$.

In a single variable w, $\mathbb{C}(\!(w)\!)$ = formal Laurent series with finitely many negative exponents, and

$$\overline{\mathbb{C}(\!(w)\!)} = \bigcup_{p\in\mathbb{N}} \mathbb{C}(\!(w^{1/p})\!) \qquad \text{(Newton-Puiseux theorem)}$$

Basic example

$$f(w,x,z) = z^2 + (w^3 + x)x^2 \,.$$

f is nc(2) on $\{z = x = 0,\ w \neq 0\}$.

f does not split over $\mathbb{C}[\![w,x]\!]$, but

$$f(v^2,x,z) = z^2 + v^6\left(1 + \frac{x}{v^6}\right)x^2 \,,$$

so $f(w,x,z)$ splits over $\mathbb{C}(w^{1/2})[\![x]\!]$.

Note. $f(v^2,x,z)$ is not nc as an element of $\mathbb{C}[\![v,x,z]\!]$, but is nc in $\mathbb{C}(v)[\![x,z]\!]$.

Blow up 0: The nc locus lifts to the w-chart, given by the substitution (w,wx,wz), and the strict transform f' of f is

$$f'(w,x,z) := w^{-2}f(w,wx,wz) = z^2 + w(w^2 + x)x^2 \,.$$

After 2 more blowings-up of 0, we get

$$f'(w,x,z) = z^2 + w^3(1+x)x^2\,,$$

so $f'(w,x,z)$ splits over $\mathbb{C}[\![w^{1/2},x]\!]$.

After an additional cleaning blow-up, centre $z = w = 0$, we get a pinch point.

Recall that, in general. we can begin with the assumption that

$$f(w,u,x,z) = z^k + a_1(w,u,x)z^{k-1} + \cdots + a_k(w,u,x)\,,$$

where f is nc(k) on $\{x = z = 0,\ w_1 \cdots w_r \neq 0\}$, and w represents the exceptional divisor,

and we are interested in the splitting of f as

$$f(w,u,x,z) = \prod_{j=1}^{k}(z - b_j(w,u,x))\,.$$

Then:

(1) From the generic splitting condition, it follows that there is a unique splitting in $\overline{\mathbb{C}(w)}[\![u,x]\!][z]$, and each

$$b_j(w,u,x) \in \overline{\mathbb{C}(w)}[\![u,x]\!]$$

belongs to the ideal generated by $x_1, \dots, x_{k-1}$.

(2) It follows that f splits over $L[\![u,x]\!]$, where L is a finite normal extension of $\mathbb{C}(w)$ in $\overline{\mathbb{C}(w)}$, using the fundamental theorem of Galois theory.

For example, if there is a single w variable, then f splits over $\mathbb{C}(w^{1/p})[\![u,x]\!]$, for some p, by the Newton-Puiseux theorem,

and we can take $p = k$, if f is irreducible.

Theorem 1

(Case $q = 0$, $r = 1$: limits of $\mathrm{nc}(k)$ in $k+1$ variables.)

Assume that

$$f(w, x_1, \dots, x_{k-1}, z) = z^k + a_1(w,x)z^{k-1} + \cdots + a_k(w,x)$$

is $\mathrm{nc}(k)$ on $\{z = x = 0,\ w \neq 0\}$. Then, after a finite number of blowings-up of 0, f splits over $\mathbb{C}[\![w^{1/p}, x]\!]$, for some p.

Proof uses (1) above and a multivariate Newton-Puiseux theorem of Soto-Vicente (2011).

Splitting question

Assume that

$$f(w_1, \dots, w_r,\ u_1, \dots, u_q,\ x_1, \dots, x_{k-1},\ z)$$

is $\mathrm{nc}(k)$ on $\{x = z = 0,\ w_1 \cdots w_r \neq 0\}$ (with the additional hypothesis $\mathrm{inv}(0) = \mathrm{inv}(\mathrm{nc}(k))$, if needed).

Is it true that, after finitely many blowings-up with centres

$$\{z = x = w_{j_1} = \cdots = w_{j_s} = 0\},$$

f splits over $\mathbb{C}[\![w_1^{1/p}, \dots, w_r^{1/p},\ u, x]\!]$, for some p ?

Theorem 2. True for $k \leq 3$ (with the additional hypothesis on inv).

Theorem 3. Circulant normal form

Assume that f satisfies the hypotheses of Thm 1 or 2, and that

$$\mathrm{inv}(0) = \mathrm{inv}(\mathrm{nc}(k))\,.$$

Then there is a finite sequence of admissible blowings-up preserving the $\mathrm{nc}(k)$ locus, after which the only singularities that can occur as limits of $\mathrm{nc}(k)$ points, are products of circulant singularities;

i.e., singularities of the form

$$\prod_{i=1}^{p} \Delta_{k_i}\left(y_{i0},\ w^{1/k_i} y_{i1},\ \dots,\ w^{(k_i-1)/k_i} y_{i,k_i-1}\right) = 0\,,$$

in suitable étale coordinates $\left(w, (y_{i\ell})_{\ell=0,\dots,k_i-1,\, i=1,\dots p}\right)$, where $k_1 + \cdots k_p = k$.

Summary remark

Following the inductive strategy, Theorems 1–3 can be used together with desingularization techniques to prove Conjectures A and B for $\dim X \leq 4$.

We believe a proof in the general case might be achieved by generalizing the methods used, with the exception of the splitting question, which seems to be the main challenge.

Thank you for your attention!