The tropical bifurcation set of polynomial maps on a plane

Boulos El Hilany

GKŁW Workshop in Singularity Theory 2022
dedicated to the memory of Stanisław Łojasiewicz

December 19, 2022

The bifurcation set for polynomial automorphisms

Definition

The **bifurcation set**, $B(f)$, of a dominant polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of all $y \in \mathbb{C}^n$ satisfying $|f^{-1}(y)| \neq$ **topological degree of** f.

The bifurcation set for polynomial automorphisms

Definition

The **bifurcation set**, $B(f)$, of a dominant polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of all $y \in \mathbb{C}^n$ satisfying $|f^{-1}(y)| \neq$ **topological degree of** f.

The bifurcation set is useful for understanding the topology of maps , e.g.

- the Jacobian conjecture, and
- the classification of topological types of maps with a given degree.

The bifurcation set for polynomial automorphisms

Definition

The **bifurcation set**, $B(f)$, of a dominant polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of all $y \in \mathbb{C}^n$ satisfying $|f^{-1}(y)| \neq$ **topological degree of** f.

The bifurcation set is useful for understanding the topology of maps , e.g.

- the Jacobian conjecture, and
- the classification of topological types of maps with a given degree.

Maps $F : X \to Y$ model phenomena from real-life problems.

e.g. $B(F)$ represents the region of all inputs $y \in Y$ critical to the problem.

The bifurcation set for polynomial automorphisms

Definition

The **bifurcation set**, $B(f)$, of a dominant polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of all $y \in \mathbb{C}^n$ satisfying $|f^{-1}(y)| \neq$ **topological degree of** f.

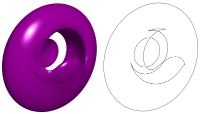
The bifurcation set is useful for understanding the topology of maps , e.g.

- the Jacobian conjecture, and
- the classification of topological types of maps with a given degree.

Maps $F : X \to Y$ model phenomena from real-life problems.

e.g. $B(F)$ represents the region of all inputs $y \in Y$ critical to the problem.

Algebraic Vision



The bifurcation set for polynomial automorphisms

Definition

The **bifurcation set**, $B(f)$, of a dominant polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of all $y \in \mathbb{C}^n$ satisfying $|f^{-1}(y)| \neq$ **topological degree of** f.

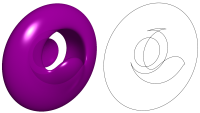
The bifurcation set is useful for understanding the topology of maps , e.g.

- the Jacobian conjecture, and
- the classification of topological types of maps with a given degree.

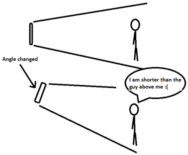
Maps $F : X \to Y$ model phenomena from real-life problems.

e.g. $B(F)$ represents the region of all inputs $y \in Y$ critical to the problem.

Algebraic Vision



Position recovery



The bifurcation set for polynomial automorphisms

Definition

The **bifurcation set**, $B(f)$, of a dominant polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of all $y \in \mathbb{C}^n$ satisfying $|f^{-1}(y)| \neq$ **topological degree of** f.

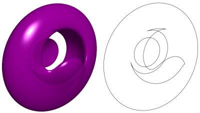
The bifurcation set is useful for understanding the topology of maps , e.g.

- the Jacobian conjecture, and
- the classification of topological types of maps with a given degree.

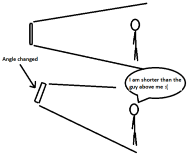
Maps $F : X \to Y$ model phenomena from real-life problems.

e.g. $B(F)$ represents the region of all inputs $y \in Y$ critical to the problem.

Algebraic Vision



Position recovery



Robotics

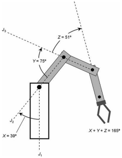


Image Credits: (1) Gallet, Lubbes, Schicho, Vršek. ArXiv:1810.05559 (2) stackoverflow.com (3) loopandbreak.com

Known results and open problems

$f := (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ – dominant polynomial map, then $B(f) = D(f) \cup S(f)$;

- $D(f)$ – **discriminant**, i.e. $f(\{$**critical points**$\})$, and
- $S(f)$ – **non-properness set** of f, i.e.

$$\left\{ w \in \mathbb{C}^n \;\middle|\; \exists \{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n,\ \|z_k\| \to \infty \quad \text{and} \quad f(z_k) \to w \right\}$$

Known results and open problems

$f := (f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ – dominant polynomial map, then $B(f) = D(f) \cup S(f)$;

- $D(f)$ – **discriminant**, i.e. $f(\{$**critical points**$\})$, and
- $S(f)$ – **non-properness set** of f, i.e.

$$\left\{ w \in \mathbb{C}^n \;\middle|\; \exists \{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n,\ \|z_k\| \to \infty \quad \text{and} \quad f(z_k) \to w \right\}$$

Computing the equations D_f and S_f for $B(f)$:

- To obtain D_f, we compute

$$\langle f_1 - w_1, \dots,\ f_n - w_n,\ \det(\mathrm{Jac}_z\, f) \rangle \cap \mathbb{C}[w_1, \dots, w_n]$$

Known results and open problems

$f := (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ – dominant polynomial map, then $B(f) = D(f) \cup S(f)$;

- $D(f)$ – **discriminant**, i.e. $f(\{$**critical points**$\})$, and
- $S(f)$ – **non-properness set** of f, i.e.

$$\left\{ w \in \mathbb{C}^n \;\middle|\; \exists \{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n,\ \|z_k\| \to \infty \quad \text{and} \quad f(z_k) \to w \right\}$$

Computing the equations D_f and S_f for $B(f)$:

- To obtain D_f, we compute

$$\langle f_1 - w_1, \ldots,\ f_n - w_n,\ \det(\mathrm{Jac}_z\, f)\rangle \cap \mathbb{C}[w_1, \ldots, w_n]$$

- [Jelonek – 93] To obtain S_f, we compute

$$\langle f_1 - w_1, \ldots,\ f_n - w_n)\rangle \cap \mathbb{C}[z_i, w_1, \ldots, w_n],\ \text{for } i = 1, \ldots, n.$$

Known results and open problems

$f := (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ – dominant polynomial map, then $B(f) = D(f) \cup S(f)$;

- $D(f)$ – **discriminant**, i.e. $f(\{$**critical points**$\})$, and
- $S(f)$ – **non-properness set** of f, i.e.

$$\left\{ w \in \mathbb{C}^n \;\middle|\; \exists \{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n,\ \|z_k\| \to \infty \quad \text{and} \quad f(z_k) \to w \right\}$$

Computing the equations D_f and S_f for $B(f)$:

- To obtain D_f, we compute

$$\langle f_1 - w_1, \ldots,\ f_n - w_n,\ \det(\mathrm{Jac}_z\, f)\rangle \cap \mathbb{C}[w_1, \ldots, w_n]$$

- [Jelonek – 93] To obtain S_f, we compute

$$\langle f_1 - w_1, \ldots,\ f_n - w_n)\rangle \cap \mathbb{C}[z_i, w_1, \ldots, w_n],\ \text{for } i = 1, \ldots, n.$$

- Bit-complexity for computing D_f and S_f is at least $\mathcal{O}\left(2^n n^{3n-2} d^{n^2+2n-2}\right)$

Known results and open problems

$f := (f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ – dominant polynomial map, then $B(f) = D(f) \cup S(f)$;

- $D(f)$ – **discriminant**, i.e. $f(\{\textbf{critical points}\})$, and
- $S(f)$ – **non-properness set** of f, i.e.

$$\left\{ w \in \mathbb{C}^n \;\middle|\; \exists \{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n,\ \|z_k\| \to \infty \quad \text{and} \quad f(z_k) \to w \right\}$$

Computing the equations D_f and S_f for $B(f)$:

- To obtain D_f, we compute

$$\langle f_1 - w_1, \dots, \ f_n - w_n, \ \det(\mathrm{Jac}_z\, f)\rangle \cap \mathbb{C}[w_1, \dots, w_n]$$

- [Jelonek – 93] To obtain S_f, we compute

$$\langle f_1 - w_1, \dots, \ f_n - w_n)\rangle \cap \mathbb{C}[z_i, w_1, \dots, w_n], \ \text{for } i = 1, \dots, n.$$

- Bit-complexity for computing D_f and S_f is at least $\mathcal{O}\left(2^n n^{3n-2} d^{n^2+2n-2}\right)$

Problem

How to determine the invariants (e.g. prime decomposition, singularities, homology, etc.) of $B(f)$ without computing its ideal?

Polyhedral invariants (an example)

P – bivariate polynomial $\sum_{a\in\mathbb{N}^n} c_a z^a$; $z^a := {z_1}^{a_1}\cdots {z_n}^{a_n}$.
supp P – **support** of P, i.e. $\{a \in \mathbb{N}^n |\ c_a \neq 0\}$
NP(P) – **Newton polytope** of P, i.e. convex hull in $\mathbb{R}^n$ of supp P.

Polyhedral invariants (an example)

P – bivariate polynomial $\sum_{a\in\mathbb{N}^n} c_a z^a$; $z^a := {z_1}^{a_1}\cdots {z_n}^{a_n}$.

$\operatorname{supp} P$ – **support** of P,   i.e. $\{a\in\mathbb{N}^n \mid c_a \neq 0\}$

$\mathrm{NP}(P)$ – **Newton polytope** of P,   i.e. convex hull in $\mathbb{R}^n$ of $\operatorname{supp} P$.

Claim: Under some genericity assumptions, some invariants of polynomial maps depend only on the supports of the polynomials.

Polyhedral invariants (an example)

P – bivariate polynomial $\sum_{a\in\mathbb{N}^n} c_a z^a$; $z^a := {z_1}^{a_1}\cdots {z_n}^{a_n}$.
supp P – **support** of P, i.e. $\{a \in \mathbb{N}^n |\ c_a \neq 0\}$
NP(P) – **Newton polytope** of P, i.e. convex hull in $\mathbb{R}^n$ of supp P.

Claim: Under some genericity assumptions, some invariants of polynomial maps depend only on the supports of the polynomials.

Example

For generic $\ell \in \mathbb{C}^4$, the below map $F_\ell : \mathbb{C}^2_{u,v} \to \mathbb{C}^2_{r,s}$ satisfies

$$(u,\ v) \mapsto (\ell_1 \cdot uv^2,\ \ell_2 \cdot uv + \ell_3 \cdot u^3 + \ell_4 \cdot v^2)$$

Polyhedral invariants (an example)

P – bivariate polynomial $\sum_{a\in\mathbb{N}^n} c_a z^a$; $z^a := {z_1}^{a_1}\cdots {z_n}^{a_n}$.
supp P – **support** of P, i.e. $\{a\in\mathbb{N}^n \mid c_a\neq 0\}$
NP(P) – **Newton polytope** of P, i.e. convex hull in $\mathbb{R}^n$ of supp P.

Claim: Under some genericity assumptions, some invariants of polynomial maps depend only on the supports of the polynomials.

Example

For generic $\ell\in\mathbb{C}^4$, the below map $F_\ell : \mathbb{C}^2_{u,v}\to\mathbb{C}^2_{r,s}$ satisfies

$$(u,\ v)\mapsto(\ell_1\cdot uv^2,\ \ell_2\cdot uv+\ell_3\cdot u^3+\ell_4\cdot v^2)$$

- top. $\deg(F_\ell)=8$, $D(F_\ell)$ is a rational curve with exactly one cusp and two nodes
- The Newton polytopes of C_{F_ℓ} and of D_{F_ℓ} are independent of ℓ.

Polyhedral invariants (an example)

P – bivariate polynomial $\sum_{a\in\mathbb{N}^n} c_a z^a$; $z^a := {z_1}^{a_1}\cdots {z_n}^{a_n}$.
supp P – **support** of P, i.e. $\{a\in\mathbb{N}^n \mid c_a\neq 0\}$
NP(P) – **Newton polytope** of P, i.e. convex hull in $\mathbb{R}^n$ of supp P.

Claim: Under some genericity assumptions, some invariants of polynomial maps depend only on the supports of the polynomials.

Example

For generic $\ell\in\mathbb{C}^4$, the below map $F_\ell : \mathbb{C}^2_{u,v}\to\mathbb{C}^2_{r,s}$ satisfies

$$(u,\ v)\mapsto(\ell_1\cdot uv^2,\ \ell_2\cdot uv+\ell_3\cdot u^3+\ell_4\cdot v^2)$$

- top. $\deg(F_\ell)=8$, $D(F_\ell)$ is a rational curve with exactly one cusp and two nodes
- The Newton polytopes of C_{F_ℓ} and of D_{F_ℓ} are independent of ℓ.

Indeed, for some $\varphi_1,\ldots\varphi_8\in\mathbb{Z}[\ell_1,\ldots,\ell_4]$, we get:

$$C_{F_\ell} = -6\ell_4\cdot u^3v-\ell_2\cdot uv^2+2\ell_3\cdot v^3,$$

$$D_{F_\ell} = \varphi_1\cdot r^3s^3+\varphi_2\cdot r^5+\varphi_3\cdot s^8+\varphi_4\cdot r^2s^5+\varphi_5\cdot r^4s^2+\varphi_6\cdot r^3s^4+\varphi_7\cdot r^5s+\varphi_8.$$

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

$f \in \mathbb{C}^A$ – generic, then A determines:

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

$f \in \mathbb{C}^A$ – generic, then A determines:

- Newton polytopes of C_f, D_f, and of S_f,

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

$f \in \mathbb{C}^A$ – generic, then A determines:

- Newton polytopes of C_f, D_f, and of S_f,
- the topological type of f (Corollary of [Sabbah 84])

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

$f \in \mathbb{C}^A$ – generic, then A determines:

- Newton polytopes of C_f, D_f, and of S_f,
- the topological type of f (Corollary of [Sabbah 84])

  Since $\mathbb{C}^A \hookrightarrow \Omega(d_1, d_2)$, where

$$\Omega(d_1, d_2) := \left\{ f : \mathbb{C}^2 \to \mathbb{C}^2 \mid \deg f_1 = d_1, \ \deg f_2 = d_2 \right\}.$$

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

$f \in \mathbb{C}^A$ – generic, then A determines:

- Newton polytopes of C_f, D_f, and of S_f,
- the topological type of f (Corollary of [Sabbah 84])

  Since $\mathbb{C}^A \hookrightarrow \Omega(d_1, d_2)$, where

$$\Omega(d_1, d_2) := \left\{ f : \mathbb{C}^2 \to \mathbb{C}^2 \mid \deg f_1 = d_1,\ \deg f_2 = d_2 \right\}.$$

Theorem (Farnik, Jelonek, Ruas – 2019)

$f \in \Omega(d_1, d_2)$ – *generic. Then,* $\mathtt{Sing}(D(f))$ *has only simple cusps and nodes,*

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

$f \in \mathbb{C}^A$ – generic, then A determines:

- Newton polytopes of C_f, D_f, and of S_f,
- the topological type of f (Corollary of [Sabbah 84])

  Since $\mathbb{C}^A \hookrightarrow \Omega(d_1, d_2)$, where

$$\Omega(d_1, d_2) := \left\{ f : \mathbb{C}^2 \to \mathbb{C}^2 \mid \deg f_1 = d_1,\ \deg f_2 = d_2 \right\}.$$

Theorem (Farnik, Jelonek, Ruas – 2019)

$f \in \Omega(d_1, d_2)$ – *generic. Then,* $\mathrm{Sing}(D(f))$ *has only simple cusps and nodes,and*

$$|\,\mathrm{Sing}(D(f))| = |\mathrm{NP}(\mathrm{D}_f)|^\circ - |\mathrm{NP}(\mathrm{C}_f)|^\circ,$$

$|\Delta|^\circ$ – *number of integer points in the relative interior of a polytope* $\Delta \subset \mathbb{R}^2$

Polyhedral invariants of polynomial maps

- A – pair of finite subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$.
- $\mathbb{C}^A$ – space of all polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, $A_1 = \operatorname{supp} f_1$ and $A_2 = \operatorname{supp} f_2$.

$f \in \mathbb{C}^A$ – generic, then A determines:

- Newton polytopes of C_f, D_f, and of S_f,
- the topological type of f (Corollary of [Sabbah 84])

  Since $\mathbb{C}^A \hookrightarrow \Omega(d_1, d_2)$, where

$$\Omega(d_1, d_2) := \left\{ f : \mathbb{C}^2 \to \mathbb{C}^2 \mid \deg f_1 = d_1, \ \deg f_2 = d_2 \right\}.$$

Theorem (Farnik, Jelonek, Ruas – 2019)

$f \in \Omega(d_1, d_2)$ *– generic. Then,* $\operatorname{Sing}(D(f))$ *has only simple cusps and nodes,and*

$$|\operatorname{Sing}(D(f))| = |\mathrm{NP}(\mathrm{D}_f)|^\circ - |\mathrm{NP}(\mathrm{C}_f)|^\circ,$$

$|\Delta|^\circ$ *– number of integer points in the relative interior of a polytope* $\Delta \subset \mathbb{R}^2$

Conjecture

Theorem above holds for a generic* $f \in \mathbb{C}^A$ *if A is "nice" enough.*

A direction for computing the Newton polytope

- $\mathbb{K}$ – field of **generalized complex Puiseux series** $\alpha(t) := c_0 t^{k_0} + c_1 t^{k_1} + \cdots$, with $k_0 < k_1 < \cdots \in \mathbb{R}$, and $c_0, c_1, \ldots \in \mathbb{C}$.

A direction for computing the Newton polytope

- $\mathbb{K}$ – field of **generalized complex Puiseux series** $\alpha(t) := c_0 t^{k_0} + c_1 t^{k_1} + \cdots$, with $k_0 < k_1 < \cdots \in \mathbb{R}$, and $c_0, c_1, \ldots \in \mathbb{C}$.
- val – function $\mathbb{K} \to \mathbb{R} \cup \{-\infty\}$, $0 \neq \alpha(t) \mapsto -k_0$.

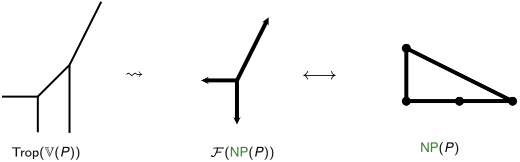
A direction for computing the Newton polytope

- $\mathbb{K}$ – field of **generalized complex Puiseux series** $\alpha(t) := c_0 t^{k_0} + c_1 t^{k_1} + \cdots$, with $k_0 < k_1 < \cdots \in \mathbb{R}$, and $c_0, c_1, \ldots \in \mathbb{C}$.
- val – function $\mathbb{K} \to \mathbb{R} \cup \{-\infty\}$, $0 \neq \alpha(t) \mapsto -k_0$.
- Trop : $\mathbb{K}^2 \to (\mathbb{R} \cup \{-\infty\})^2$, $z \mapsto (\mathrm{val}(z_1),\ \mathrm{val}(z_2))$

A direction for computing the Newton polytope

- $\mathbb{K}$ – field of **generalized complex Puiseux series** $\alpha(t) := c_0 t^{k_0} + c_1 t^{k_1} + \cdots$, with $k_0 < k_1 < \cdots \in \mathbb{R}$, and $c_0, c_1, \ldots \in \mathbb{C}$.
- val – function $\mathbb{K} \to \mathbb{R} \cup \{-\infty\}$, $0 \neq \alpha(t) \mapsto -k_0$.
- Trop : $\mathbb{K}^2 \to (\mathbb{R} \cup \{-\infty\})^2$, $z \mapsto (\text{val}(z_1),\ \text{val}(z_2))$
- $\text{Trop}(\mathbb{V}(P))$ – the **tropicalization** of $\mathbb{V}(P)$:
- is a rational piecewise-affine complex in $\mathbb{R}^2$ [Biery, Groves – 1984], and
- determines the dual fan $\mathcal{F}(\text{NP}(P))$ [Kapranov– 2000]

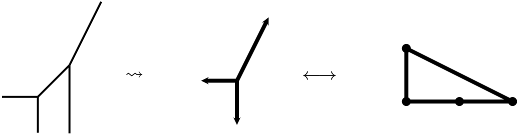
$$P := 1 + 12t^{12} - 5t^2 \cdot z_1 + (7 \cdot t^{-2} + t^{-1}) \cdot z_2 + 2t^7 \cdot {z_1}^2$$



A direction for computing the Newton polytope

- $\mathbb{K}$ – field of **generalized complex Puiseux series** $\alpha(t) := c_0 t^{k_0} + c_1 t^{k_1} + \cdots$, with $k_0 < k_1 < \cdots \in \mathbb{R}$, and $c_0, c_1, \ldots \in \mathbb{C}$.
- val – function $\mathbb{K} \to \mathbb{R} \cup \{-\infty\}$, $0 \neq \alpha(t) \mapsto -k_0$.
- Trop : $\mathbb{K}^2 \to (\mathbb{R} \cup \{-\infty\})^2$, $z \mapsto (\text{val}(z_1),\ \text{val}(z_2))$
- $\text{Trop}(\mathbb{V}(P))$ – the **tropicalization** of $\mathbb{V}(P)$:
- is a rational piecewise-affine complex in $\mathbb{R}^2$ [Biery, Groves – 1984], and
- determines the dual fan $\mathcal{F}(\text{NP}(P))$ [Kapranov– 2000]

$P := 1 + 12t^{12} - 5t^2 \cdot z_1 + (7 \cdot t^{-2} + t^{-1}) \cdot z_2 + 2t^7 \cdot {z_1}^2$



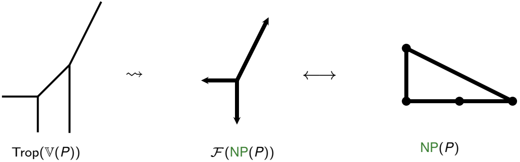
$\text{Trop}(\mathbb{V}(P))$ $\quad$ $\mathcal{F}(\text{NP}(P))$ $\quad$ $\text{NP}(P)$

Strategy for Newton polytope of $X_f \in \{D_f, S_f\}$**:**

A direction for computing the Newton polytope

- $\mathbb{K}$ – field of **generalized complex Puiseux series** $\alpha(t) := c_0 t^{k_0} + c_1 t^{k_1} + \cdots$, with $k_0 < k_1 < \cdots \in \mathbb{R}$, and $c_0, c_1, \ldots \in \mathbb{C}$.
- val – function $\mathbb{K} \to \mathbb{R} \cup \{-\infty\}$, $0 \neq \alpha(t) \mapsto -k_0$.
- Trop : $\mathbb{K}^2 \to (\mathbb{R} \cup \{-\infty\})^2$, $z \mapsto (\text{val}(z_1),\ \text{val}(z_2))$
- $\text{Trop}(\mathbb{V}(P))$ – the **tropicalization** of $\mathbb{V}(P)$:
- is a rational piecewise-affine complex in $\mathbb{R}^2$ [Biery, Groves – 1984], and
- determines the dual fan $\mathcal{F}(\text{NP}(P))$ [Kapranov– 2000]

$$P := 1 + 12t^{12} - 5t^2 \cdot z_1 + (7 \cdot t^{-2} + t^{-1}) \cdot z_2 + 2t^7 \cdot {z_1}^2$$



Strategy for Newton polytope of $X_f \in \{D_f, S_f\}$:

1. Generic maps $f \in \mathbb{C}^A$ and $g \in \mathbb{K}^A$ satisfy $\Delta := \text{NP}(X_f) = \text{NP}(X_g)$,
2. compute $\text{Trop}(\mathbb{V}(X_f)) \to \mathcal{F}(\Delta) \to \Delta$

A bit of tropical geometry

- For any $P : \mathbb{K}^2 \to \mathbb{K}$, $z \mapsto \sum_{a \in A} c_a z^a$

A bit of tropical geometry

- For any $P : \mathbb{K}^2 \to \mathbb{K}$, $z \mapsto \sum_{a \in A} c_a z^a$, we define the **tropical polynomial** $\mathcal{T}(P) : \mathbb{R}^2 \to \mathbb{R}$,

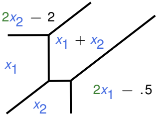
$$x \mapsto \max_{a \in A}(\langle x, a \rangle + \operatorname{val} c_a).$$

A bit of tropical geometry

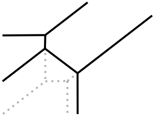
- For any $P : \mathbb{K}^2 \to \mathbb{K}$, $z \mapsto \sum_{a \in A} c_a z^a$, we define the **tropical polynomial** $\mathcal{T}(P) : \mathbb{R}^2 \to \mathbb{R}$,
$$x \mapsto \max_{a \in A}(\langle x, a \rangle + \operatorname{val} c_a).$$
- For any $y \in \mathbb{R} \cup \{-\infty\}$, define the **virtual preimage**

A bit of tropical geometry

- For any $P : \mathbb{K}^2 \to \mathbb{K}$, $z \mapsto \sum_{a \in A} c_a z^a$, we define the **tropical polynomial** $\mathcal{T}(P) : \mathbb{R}^2 \to \mathbb{R}$,
$$x \mapsto \max_{a \in A}(\langle x,\ a\rangle + \operatorname{val} c_a).$$
- For any $y \in \mathbb{R} \cup \{-\infty\}$, define the **virtual preimage** $\mathcal{V}_y(\mathcal{T}(P))$ – points $x \in \mathbb{R}^2$ where the below maximum is reached twice
$$\max_{a \in A}(y,\ \langle x,\ a\rangle + \operatorname{val} c_a)$$



$\mathcal{V}_{-\infty}(\mathcal{T}(P))$



$\mathcal{V}_{-1}(\mathcal{T}(P))$

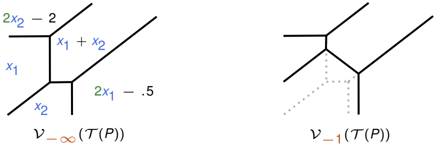
A bit of tropical geometry

- For any $P : \mathbb{K}^2 \to \mathbb{K}$, $z \mapsto \sum_{a \in A} c_a z^a$, we define the **tropical polynomial** $\mathcal{T}(P) : \mathbb{R}^2 \to \mathbb{R}$,

$$x \mapsto \max_{a \in A}(\langle x, a\rangle + \operatorname{val} c_a).$$

- For any $y \in \mathbb{R} \cup \{-\infty\}$, define the **virtual preimage** $\mathcal{V}_y(\mathcal{T}(P))$ – points $x \in \mathbb{R}^2$ where the below maximum is reached twice

$$\max_{a \in A}(y, \ \langle x, a\rangle + \operatorname{val} c_a)$$



Theorem (Kapranov's correspondence)

Let $P \in \mathbb{K}[z_1, z_2]$, $P(0,0) = 0$, $w \in \mathbb{K}$ and define $y := \operatorname{val}(w)$. Then, it holds

$$\operatorname{Trop}\left(P^{-1}(w) \cap (\mathbb{K}^*)^2\right) = \mathcal{V}_y(\mathcal{T}(P)).$$

(non-)Degenerate intersections

Let $f_1, f_2 \in \mathbb{K}[z_1, z_2]$, and let $X_1 := \operatorname{Trop} f_1$, $X_2 := \operatorname{Trop} f_2$ their tropical curves.

(non-)Degenerate intersections

Let $f_1, f_2 \in \mathbb{K}[z_1, z_2]$, and let $X_1 := \operatorname{Trop} f_1$, $X_2 := \operatorname{Trop} f_2$ their tropical curves.

Definition (Stable intersections)

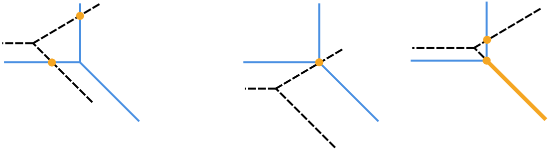
A point $p \in X_1 \cap X_2 \subset \mathbb{R}^2$ is **stable** if it is locally preserved under perturbations.

(non-)Degenerate intersections

Let $f_1, f_2 \in \mathbb{K}[z_1, z_2]$, and let $X_1 := \text{Trop } f_1$, $X_2 := \text{Trop } f_2$ their tropical curves.

Definition (Stable intersections)

A point $p \in X_1 \cap X_2 \subset \mathbb{R}^2$ is **stable** if it is locally preserved under perturbations.



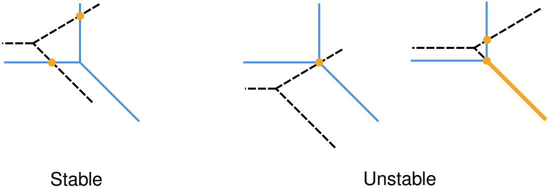
Stable Unstable

(non-)Degenerate intersections

Let $f_1, f_2 \in \mathbb{K}[z_1, z_2]$, and let $X_1 := \text{Trop } f_1$, $X_2 := \text{Trop } f_2$ their tropical curves.

Definition (Stable intersections)

A point $p \in X_1 \cap X_2 \subset \mathbb{R}^2$ is **stable** if it is locally preserved under perturbations.



Definition (Degenerate intersections)

An *un*stable intersection component C in $X_1 \cap X_2 \subset \mathbb{R}^2$ is **degenerate** if C satisfies one of the following conditions:

- is bounded,
- contains a half-line with direction (a, b) such that $a \cdot b < 0$, or
- contains a horizontal/vertical half-line with some extra properties.

A correspondence theorem

Tropical bifurcation set of a tropical polynomial map $F := (F_1,\ F_2) : \mathbb{R}^2 \to \mathbb{R}^2$,

A correspondence theorem

Tropical bifurcation set of a tropical polynomial map $F := (F_1,\ F_2) : \mathbb{R}^2 \to \mathbb{R}^2$,

$$TB(F) := \left\{ (y_1,\ y_2) \in \mathbb{R}^2 \;\middle|\; \mathcal{V}_{y_1}(F_1) \cap \mathcal{V}_{y_2}(F_2) \text{ has a degenerate component in } \mathbb{R}^2 \right\}$$

A correspondence theorem

Tropical bifurcation set of a tropical polynomial map $F := (F_1,\ F_2) : \mathbb{R}^2 \to \mathbb{R}^2$,

$$TB(F) := \left\{ (y_1,\ y_2) \in \mathbb{R}^2 \ \middle|\ \mathcal{V}_{y_1}(F_1) \cap \mathcal{V}_{y_2}(F_2) \text{ has a degenerate component in } \mathbb{R}^2 \right\}$$

Theorem (EH – 22)

*Let A_1 and A_2 be finite subsets of $\mathbb{N}^2 \setminus \{(0,0)\}$ and let $f := (f_1, f_2) : \mathbb{K}^2 \to (\mathbb{K}^*)^2$ be a **general** polynomial map with $A_i = \operatorname{supp} f_i$ and $F := (\mathcal{T}(f_1), \mathcal{T}(f_2))$. Then, it holds*

$$\operatorname{Trop}(B(f)) = TB(F).$$

A correspondence theorem

Tropical bifurcation set of a tropical polynomial map $F := (F_1,\ F_2) : \mathbb{R}^2 \to \mathbb{R}^2$,

$$TB(F) := \left\{ (y_1,\ y_2) \in \mathbb{R}^2 \ \middle|\ \mathcal{V}_{y_1}(F_1) \cap \mathcal{V}_{y_2}(F_2) \text{ has a degenerate component in } \mathbb{R}^2 \right\}$$

Theorem (EH – 22)

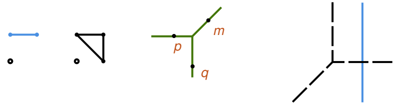
*Let A_1 and A_2 be finite subsets of $\mathbb{N}^2 \setminus \{(0,0)\}$ and let $f := (f_1, f_2) : \mathbb{K}^2 \to (\mathbb{K}^*)^2$ be a **general** polynomial map with $A_i = \operatorname{supp} f_i$ and $F := (\mathcal{T}(f_1), \mathcal{T}(f_2))$. Then, it holds*

$$\operatorname{Trop}(B(f)) = TB(F).$$

Example

Consider the polynomial map $(u,\ v) \to (v + t\,uv,\ u + v + uv)$. Its bifurcation set is $t^2\, s^2 - 2t\, r\, s - 4t\, r + r^2 + 2t\, s + 2r + 1 \in \mathbb{K}[r,\ s]$, and F is expressed as

$$(x_1,\ x_2) \to (\max(x_2,\ x_1 + x_2 - 1),\ \max(x_1,\ x_2,\ x_1 + x_2)).$$



A_1 A_2 $\operatorname{Trop}(B(f))$ $\mathcal{V}(F)$

A correspondence theorem

Tropical bifurcation set of a tropical polynomial map $F := (F_1,\ F_2) : \mathbb{R}^2 \to \mathbb{R}^2$,

$$TB(F) := \left\{ (y_1,\ y_2) \in \mathbb{R}^2 \ \middle|\ \mathcal{V}_{y_1}(F_1) \cap \mathcal{V}_{y_2}(F_2) \text{ has a degenerate component in } \mathbb{R}^2 \right\}$$

Theorem (EH – 22)

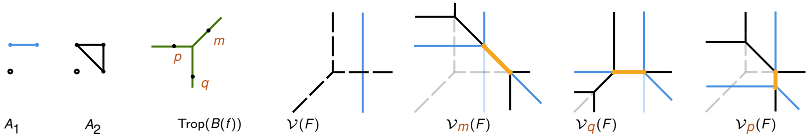
*Let A_1 and A_2 be finite subsets of $\mathbb{N}^2 \setminus \{(0,0)\}$ and let $f := (f_1, f_2) : \mathbb{K}^2 \to (\mathbb{K}^*)^2$ be a **general** polynomial map with $A_i = \operatorname{supp} f_i$ and $F := (\mathcal{T}(f_1), \mathcal{T}(f_2))$. Then, it holds*

$$\operatorname{Trop}(B(f)) = TB(F).$$

Example

Consider the polynomial map $(u,\ v) \to (v + t\,uv,\ u + v + uv)$. Its bifurcation set is $t^2\,s^2 - 2t\,r\,s - 4t\,r + r^2 + 2t\,s + 2r + 1 \in \mathbb{K}[r,\ s]$, and F is expressed as

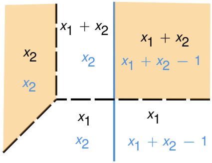
$$(x_1,\ x_2) \to (\max(x_2,\ x_1 + x_2 - 1),\ \max(x_1,\ x_2,\ x_1 + x_2)).$$



A recipe for the tropical bifurcation set

A tropical polynomial map $F : \mathbb{R}^2 \to \mathbb{R}^2$ induces a polyhedral subdivision Ξ of $\mathbb{R}^2$:

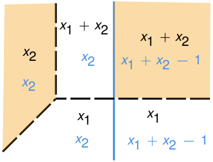
- Elements in Ξ are relative interiors of polyhedra.
- $F_{|\xi}$ is an affine map at each $\xi \in \Xi$.
- $F_{|\xi} = F_{|\xi'} \Rightarrow \xi = \xi'$ or $\overline{\xi}$ is a face of $\overline{\xi'}$.



A recipe for the tropical bifurcation set

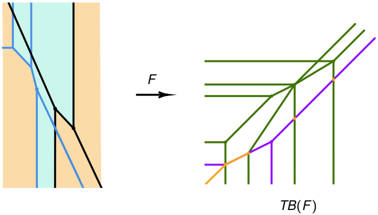
A tropical polynomial map $F : \mathbb{R}^2 \to \mathbb{R}^2$ induces a polyhedral subdivision Ξ of $\mathbb{R}^2$:

- Elements in Ξ are relative interiors of polyhedra.
- $F_{|\xi}$ is an affine map at each $\xi \in \Xi$.
- $F_{|\xi} = F_{|\xi'} \Rightarrow \xi = \xi'$ or $\overline{\xi}$ is a face of $\overline{\xi'}$.



Theorem (EH – 22)

The domains of linearity of a tropical polynomial map $F : \mathbb{R}^2 \to \mathbb{R}^2$ determine effectively the tropical bifurcation set of F.



Computing the Newton polytope of the discriminant

$A := (A_1, A_2)$ – pair of finite subsets in $\mathbb{N}^2$

Fact

There exists a polytope $\Delta \subset \mathbb{R}^2$, such that for any generic polynomial map $f := (f_1, f_2) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$ in $\mathbb{C}^A$, it holds

$$\mathrm{NP}(\mathrm{D}_f) = \Delta.$$

Computing the Newton polytope of the discriminant

$A := (A_1, A_2)$ – pair of finite subsets in $\mathbb{N}^2$

Fact

There exists a polytope $\Delta \subset \mathbb{R}^2$*, such that for any generic polynomial map* $f := (f_1, f_2) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$ *in* $\mathbb{C}^A$*, it holds*

$$\mathrm{NP}(\mathrm{D}_f) = \Delta.$$

Task

Compute Δ, up to translation, without using elimination.

Computing the Newton polytope of the discriminant

$A := (A_1, A_2)$ – pair of finite subsets in $\mathbb{N}^2$

Fact

There exists a polytope $\Delta \subset \mathbb{R}^2$, such that for any generic polynomial map $f := (f_1, f_2) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$ in $\mathbb{C}^A$, it holds

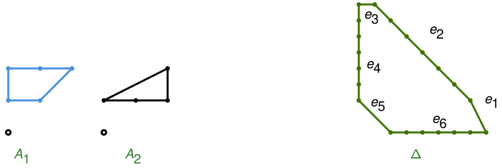
$$\mathrm{NP}(\mathrm{D}_f) = \Delta.$$

Task

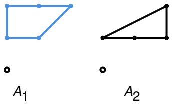
Compute Δ, up to translation, without using elimination.

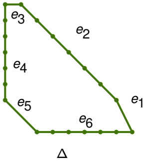
Example

Consider the polynomial map $(u, v) \to (v + v^2 + uv + uv^2 + u^2v^2,\ 2v + 3u^2v + 4u^2v^2)$, for which A_1, A_2 and Δ are illustrated below.



The Newton polytope of the discriminant





Task

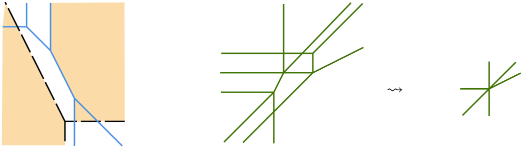
Given a pair $A := (A_1, A_2)$ of subsets in $\mathbb{N}^2$ and a generic map $f := (f_1, f_2) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$ in $\mathbb{C}^A$, compute $\Delta := \mathrm{NP}(\mathrm{D}_f)$.

The Newton polytope of the discriminant

A_1 A_2 Δ

Task

Given a pair $A := (A_1, A_2)$ of subsets in $\mathbb{N}^2$ and a generic map $f := (f_1, f_2) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$ in $\mathbb{C}^A$, compute $\Delta := \mathrm{NP}(\mathrm{D}_f)$.

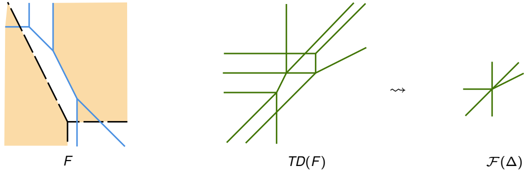


F $TD(F)$ $\mathcal{F}(\Delta)$

1. Compute the set $TD(F)$ for any "generic" $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that supp $F = A$.

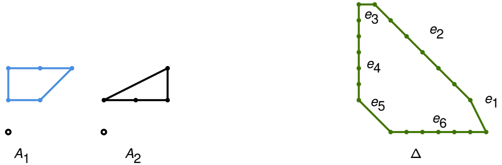
Task

Given a pair $A := (A_1, A_2)$ of subsets in $\mathbb{N}^2$ and a generic map $f := (f_1, f_2) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$ in $\mathbb{C}^A$, compute $\Delta := \mathrm{NP}(\mathrm{D}_f)$.



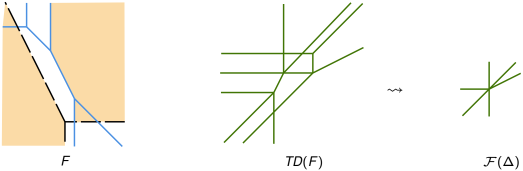
1. Compute the set $TD(F)$ for any "generic" $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that supp $F = A$.
2. We obtain the primitive vectors $\overrightarrow{e_1}, \dots, \overrightarrow{e_6}$ directing the respective edges.

The Newton polytope of the discriminant



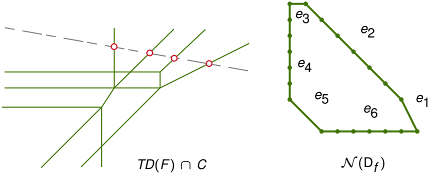
Task

Given a pair $A := (A_1, A_2)$ of subsets in $\mathbb{N}^2$ and a generic map $f := (f_1, f_2) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$ in $\mathbb{C}^A$, compute $\Delta := \mathrm{NP}(\mathrm{D}_f)$.



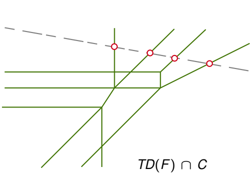
1. Compute the set $TD(F)$ for any "generic" $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{supp}\, F = A$.
2. We obtain the primitive vectors $\overrightarrow{e_1}, \ldots, \overrightarrow{e_6}$ directing the respective edges.
3. It is left to compute the lengths $\ell_1, \ldots, \ell_6$ of the corresponding edges.

The Newton polytope of the discriminant

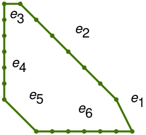


- $C - \mathrm{Trop}(\alpha\lambda^a + \beta\lambda^b) \subset \mathbb{R}^2$ intersecting $TD(F)$ transversally, and only at the unbounded edges of $TD(F)$, dual to $\{e_i\}_{i\in I}$ for some $I \subset \{1, \ldots, 6\}$.

The Newton polytope of the discriminant



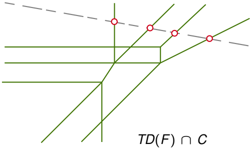
$TD(F) \cap C$



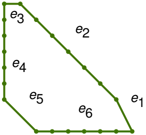
$\mathcal{N}(\mathrm{D}_f)$

- C – $\mathrm{Trop}(\alpha\lambda^a + \beta\lambda^b) \subset \mathbb{R}^2$ intersecting $TD(F)$ transversally, and only at the unbounded edges of $TD(F)$, dual to $\{e_i\}_{i\in I}$ for some $I \subset \{1, \dots, 6\}$.
- $MV(P,\ Q)$ – Mixed volume of two polytopes P, $Q \subset \mathbb{R}^2$

The Newton polytope of the discriminant



$TD(F) \cap C$



$\mathcal{N}(\mathrm{D}_f)$

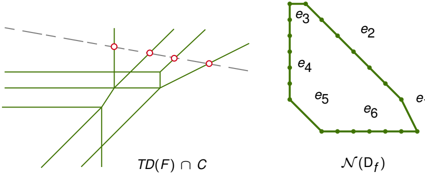
- C – $\mathrm{Trop}(\alpha\lambda^a + \beta\lambda^b) \subset \mathbb{R}^2$ intersecting $TD(F)$ transversally, and only at the unbounded edges of $TD(F)$, dual to $\{e_i\}_{i\in I}$ for some $I \subset \{1, \ldots, 6\}$.
- $MV(P,\ Q)$ – Mixed volume of two polytopes $P, Q \subset \mathbb{R}^2$

Claim

If the coefficients $\alpha, \beta \in \mathbb{K}$ are chosen to be generic enough, then it holds

$$\sum_{i\in I} \left|\det(a-b,\ \overrightarrow{e}_i)\right| \cdot \ell_i = \left|D(f) \cap \{\alpha\lambda^a + \beta\lambda^b = 0\}\right| = MV\big(\mathcal{N}(J_f),\ \mathcal{N}(\alpha f^a + \beta f^b)\big)$$

The Newton polytope of the discriminant



- C – $\mathrm{Trop}(\alpha\lambda^a + \beta\lambda^b) \subset \mathbb{R}^2$ intersecting $TD(F)$ transversally, and only at the unbounded edges of $TD(F)$, dual to $\{e_i\}_{i\in I}$ for some $I \subset \{1, \ldots, 6\}$.
- $MV(P,\ Q)$ – Mixed volume of two polytopes $P, Q \subset \mathbb{R}^2$

Claim

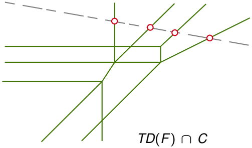
If the coefficients $\alpha, \beta \in \mathbb{K}$ are chosen to be generic enough, then it holds

$$\sum_{i\in I} \left|\det(a - b,\ \vec{e}_i)\right| \cdot \ell_i = \left|D(f) \cap \{\alpha\lambda^a + \beta\lambda^b = 0\}\right| = MV\big(\mathcal{N}(J_f),\ \mathcal{N}(\alpha f^a + \beta f^b)\big)$$

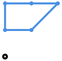
Proof:

- **Left equality:** BKK bound for $D(f) \cap X$
- **Right equality:** BKK bound for $C(f) \cap f^{-1}(X)$

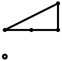
The Newton polytope of the discriminant



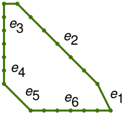
$TD(F) \cap C$



$\mathcal{N}(f_1)$



$\mathcal{N}(f_2)$



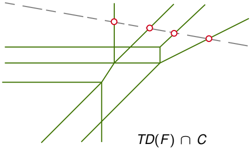
$\mathcal{N}(D(f))$

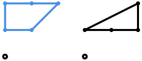
Claim

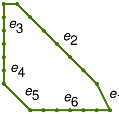
There exists $I \subset \{1, \ldots, 6\}$ *and a binomial curve* $\text{Val}(\{\alpha\lambda^a + \beta\lambda^b\}) \cap \mathbb{R}^2$ *satisfying*

$$\sum_{i \in I} \left| \det(a - b, \ \vec{e}_i) \right| \cdot \ell_i = MV\left(\mathcal{N}(J_f), \ \mathcal{N}(\alpha f^a + \beta f^b)\right)$$

The Newton polytope of the discriminant







Claim

There exists $I \subset \{1, \ldots, 6\}$ *and a binomial curve* $\mathrm{Val}(\{\alpha\lambda^a + \beta\lambda^b\}) \cap \mathbb{R}^2$ *satisfying*

$$\sum_{i \in I} \left|\det(a - b,\ \vec{e}_i)\right| \cdot \ell_i = MV\big(\mathcal{N}(J_f),\ \mathcal{N}(\alpha f^a + \beta f^b)\big)$$

We take enough binomial curves C as above to obtain the 6 equalities below

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \end{pmatrix} = \begin{pmatrix} 22 \\ 16 \\ 24 \\ 7 \\ 22 \\ 16 \end{pmatrix} \Rightarrow (\ell_1,\ \ell_2,\ \ell_3,\ \ell_4,\ \ell_5,\ \ell_6) = (1, 6, 1, 6, 2, 6)$$

Implementation

Algorithm: [EH, K. Rose – 2022]

Input: two subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$

Output: A polytope $\Delta := \mathrm{NP}(\mathrm{D}_f) \subset \mathbb{R}^2$ for any generic $f \in \mathbb{C}^A$.

1. Consider the tropical map $F := \mathcal{T}(f) : \mathbb{R}^2 \to \mathbb{R}^2$,
2. compute $TD(F) \subset \mathbb{R}^2$ and thus $\mathcal{F}(\Delta)$,
3. compute the edge lengths of Δ

Implementation

Algorithm: [EH, K. Rose – 2022]

Input: two subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$

Output: A polytope $\Delta := \mathrm{NP}(\mathrm{D}_f) \subset \mathbb{R}^2$ for any generic $f \in \mathbb{C}^A$.

OSCAR SYMBOLIC TOOLS

1. Consider the tropical map $F := \mathcal{T}(f) : \mathbb{R}^2 \to \mathbb{R}^2$,
2. compute $TD(F) \subset \mathbb{R}^2$ and thus $\mathcal{F}(\Delta)$,
3. compute the edge lengths of Δ

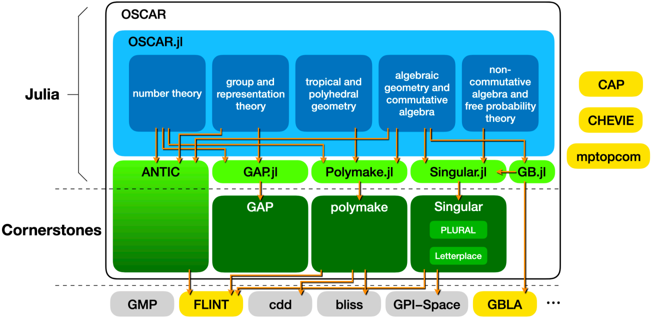
Implementation

Algorithm: [EH, K. Rose – 2022]

Input: two subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$

Output: A polytope $\Delta := \mathrm{NP}(\mathrm{D}_f) \subset \mathbb{R}^2$ for any generic $f \in \mathbb{C}^A$.

1. Consider the tropical map $F := \mathcal{T}(f) : \mathbb{R}^2 \to \mathbb{R}^2$,
2. compute $TD(F) \subset \mathbb{R}^2$ and thus $\mathcal{F}(\Delta)$,
3. compute the edge lengths of Δ



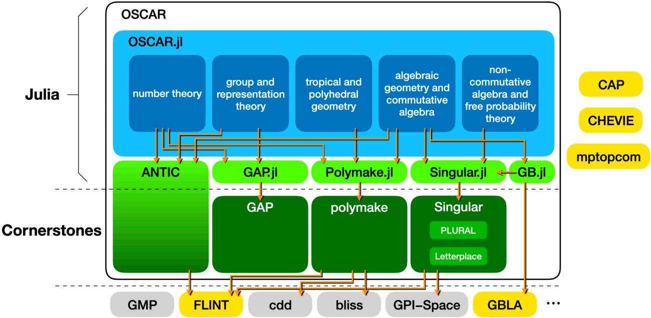
Implementation

Algorithm: [EH, K. Rose – 2022]

Input: two subsets $A_1, A_2 \subset \mathbb{N}^2 \setminus \{(0,0)\}$

Output: A polytope $\Delta := \mathrm{NP}(D_f) \subset \mathbb{R}^2$ for any generic $f \in \mathbb{C}^A$.

1. Consider the tropical map $F := \mathcal{T}(f) : \mathbb{R}^2 \to \mathbb{R}^2$,
2. compute $TD(F) \subset \mathbb{R}^2$ and thus $\mathcal{F}(\Delta)$,
3. compute the edge lengths of Δ



Thank you for your attention!

ArXiv: 2202.05052, 2207.00989