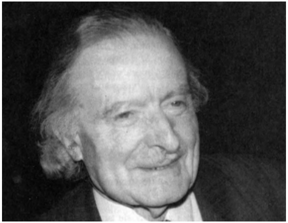
Poincaré-Hopf à la Lipschitz

inspired by Marie-Hélène Schwartz and Tadeusz Mostowski

Jean-Paul Brasselet [1], Tadeusz Mostowski [2], Thủy Nguyễn Thị Bích [3]

[1]CNRS - Université d'Aix-Marseille, France
[2]Uniwersytet Warszawski Polska
[3]IBILCE - UNESP - São José do Rio Preto, Brasil

Gdańsk - Kraków - Łódź - Warszawa Workshop in Singularity Theory
Celebrating Stanisław Łojasiewicz — Warszawa, Dec. 14, 2022



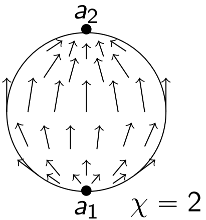
Theorem (Poincaré (1881), Hopf (1926))

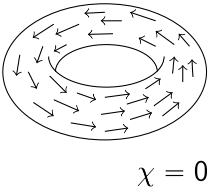
Let M be a compact differentiable manifold. Let v a vector field on M. with isolated zeroes a_i. If M has no boundary, the Euler-Poincaré characteristic of M is equal to :

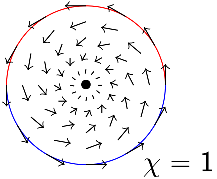
$$\chi(M) = \sum_{a_i} I(v, a_i).$$

If M has boundary ∂M, one has the same formula if v is pointing in the outward normal direction along the boundary. If v is pointing in the inward normal direction along the boundary. then one has

$$\chi(M) - \chi(\partial M) = \sum_{a_i} I(v, a_i).$$







.

We will consider real analytic varieties $X \subset \mathbb{R}^N$.
A stratification of a variety X, is a family of closed analytic subsets of X

$$\mathcal{S} \qquad \emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-2} \subset X^{n-1} \subset X = X^n$$

where each $\mathring{X}^\alpha = X^\alpha - X^{\alpha-1}$ is either empty or a smooth manifold of pure dimension α. The connected components S^α of $\mathring{X}^\alpha$ are the *strata*.

- Whitney-stratification, 1965 by Hassler Whitney,

Existence of stratifications satisfying conditions (a) and (b) for analytic varieties.

- Kuo-Verdier stratification, by Kuo 1971 and Verdier 1976.

Existence of stratifications satisfying condition (w) for analytic varieties.

- Lipschitz-stratification, 1985 by Tadeusz Mostowski,

Existence of stratifications satisfying Lipschitz L-conditions for complex analytic varieties – 1988 Existence for real analytic varieties ; Adam Parusiński.

Lipschitz $\Rightarrow$ (w) $\Rightarrow$ Whitney.

• X is a stratified variety with strata $\{\mathring{X}^\alpha\}$,
• v is a stratified vector field ($\mathcal{S}$-compatible in the terminology of Mostowski-Parusiński) i.e. for $x \in \mathring{X}^\alpha$, then $v(x) \in T_x(\mathring{X}^\alpha)$,
• v has isolated singular points a_i.

At each singular point a_i, the stratum $\mathring{X}^{\alpha(a_i)}$ containing the point a_i is smooth. The index $I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i)$ makes sense.

A first idea for a generalization of the Poincaré-Hopf Theorem, is to write

$$\chi(X) \overset{?}{=} \sum_{a_i} I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i),$$

where the points a_i are the isolated singular points of v.

- X is a stratified variety with strata $\{\mathring{X}^\alpha\}$,
- v is a stratified vector field (i.e. for $x \in \mathring{X}^\alpha$, then $v(x) \in T_x(\mathring{X}^\alpha)$),
- v has isolated singular points a_i.

At each singular point a_i, the stratum $\mathring{X}^{\alpha(a_i)}$ containing the point a_i is smooth. The index $I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i)$ makes sense.

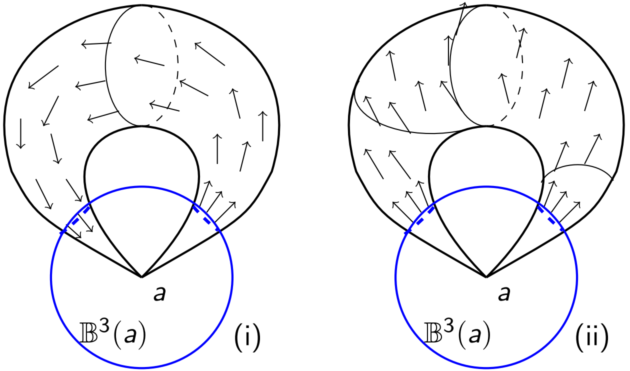
A first idea for a generalization of the Poincaré-Hopf Theorem, is to write

$$\chi(X) \overset{?}{=} \sum_{a_i} I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i),$$

where the points a_i are the isolated singular points of v.

The formula is not true. We provide two counter-examples.

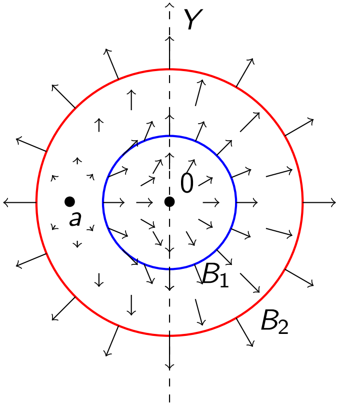
A first counter-example : The pinched torus (in $\mathbb{R}^3$)



$\chi(X) = 1 \neq 0 = I(v, a)$ $\qquad$ $\chi(X) = 1 = I(v, a)$

The picture in (ii) is the first example of M-H Schwartz radial vector field.

M.-H. Schwartz's counterexample :



$\chi(B_2) = +1$

$I(v,0) = 0,\ I(v,a) = +1$

$I(v|_Y, 0) = +1$

$B_1 = B(0;1) \quad B_2 = B(0;2)$
Inside B_1, vector field $v_1(x,y) = (|x|, y)$.
One has $v_1(0) = 0$, index $I(v_1, 0) = 0$.

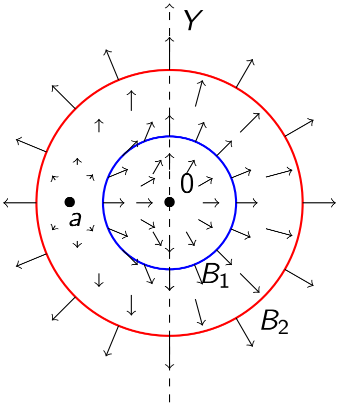
On ∂B_2, consider $v_2(x,y) = (x,y)$.

Consider a continuous vector field v :
$v = v_2$ along ∂B_2, $v = v_1$ inside B_1 and v tangent to the y-axis Y along Y.
For instance, on $B_2 \backslash B_1$, $v(x,y) = \left(2|x| - x + (x - |x|)\sqrt{x^2+y^2}, y\right)$.
The vector field v has another isolated singular point at $a = (-3/2, 0) \in B_2 \backslash B_1$.
By Poincaré-Hopf Theorem with boundary, we have

$$\chi(B_2) = +1 = I(v,0) + I(v,a),$$

that implies $I(v,a) = +1$.

M.-H. Schwartz's counterexample :

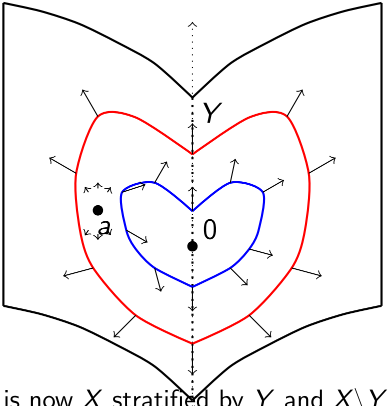


$$\chi(B_2) = +1$$

$$I(v,0) = 0,\ I(v,a) = +1$$

$$I(v|_Y,0) = +1$$

Singular surface $x^2 - z^3 = 0$

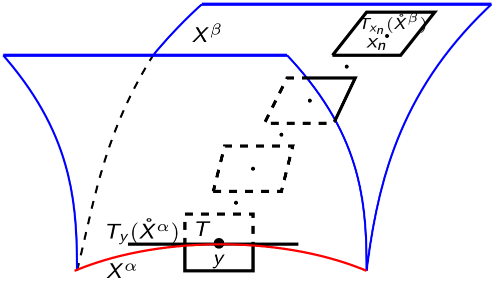


B_2 is now X stratified by Y and $X \setminus Y$. v becomes a stratified vector field and $\chi(X) = +1 \neq I(v,a) + I(v|_Y,0) = 2$.

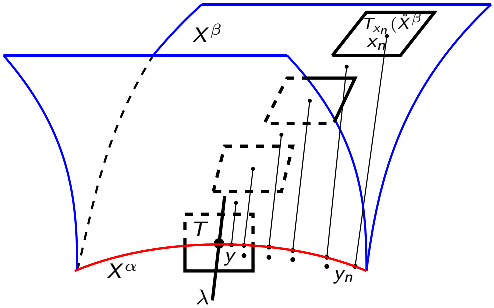
$$\chi(X) \neq \sum_{a_i} I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i),$$

a_i isolated singular points.

Whitney stratifications.

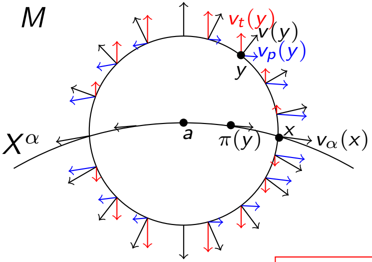


Whitney condition (a)



Whitney condition (b)

Local radial extension of a vector field.



v_α vector field along $\mathring{X}^\alpha$

v_p its local parallel extension

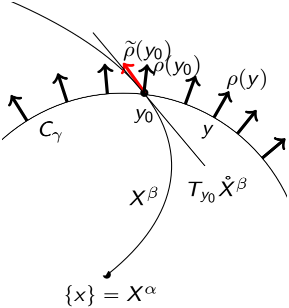
v_t the local transverse vector field

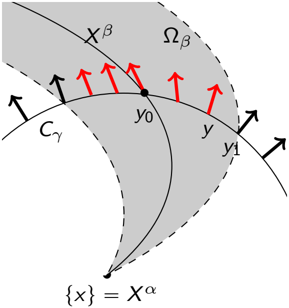
$$v = v_p + v_t$$

Main Property 1. $$I(v_\alpha, a; \mathring{X}^\alpha) = I(v, a; M)$$

Difficulty : the radial extension is not necessarily a stratified vector field.
Solution (MHS) : use the Whitney condition (a) for the parallel vector field v_p, and Whitney condition (b) for the transverse vector field v_t.

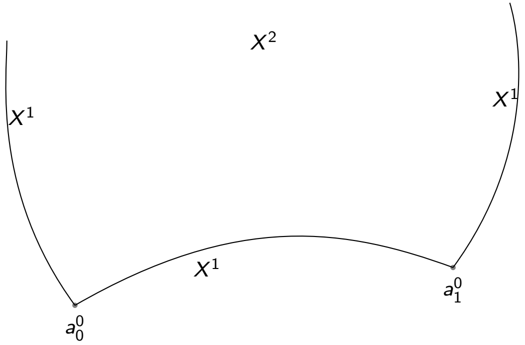
M.-H. Schwartz's radial extension - local construction



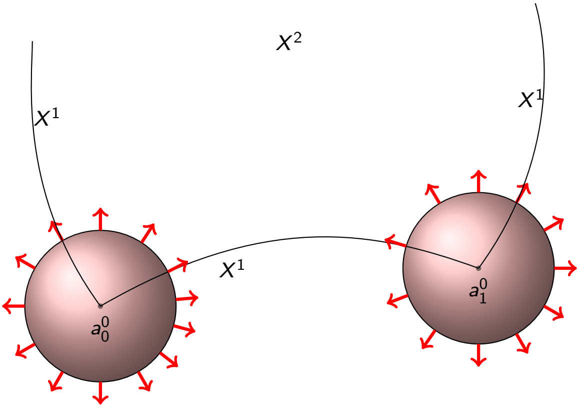


Difficulty : Even being stratified, the radial extension is not continuous.
Solution (MHS) : Use "tapered" neighbourhoods in which homotopy is performed. In the picture, homotopy is performed for the radial vector field v_p denoted ρ.

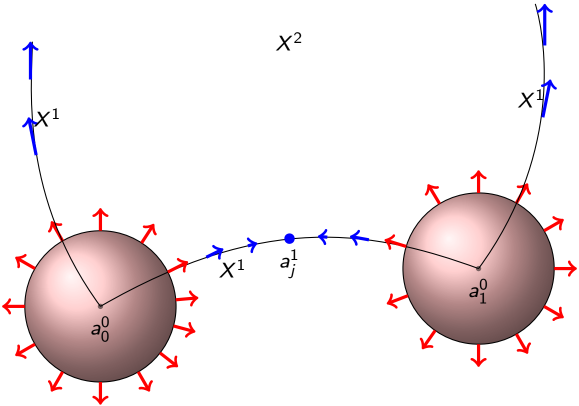
M.-H. Schwartz's radial extension - global construction - 1



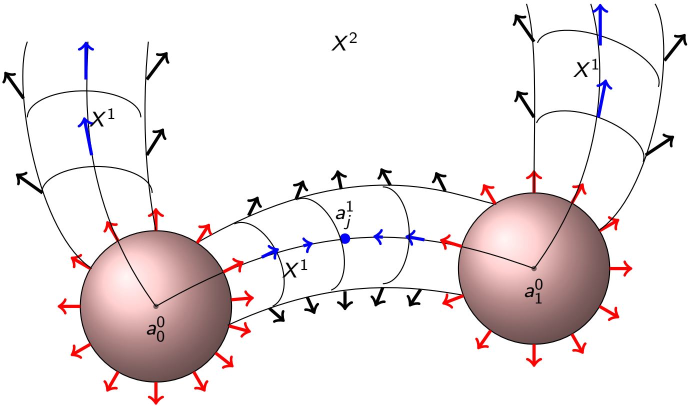
M.-H. Schwartz's radial extension - global construction - 2



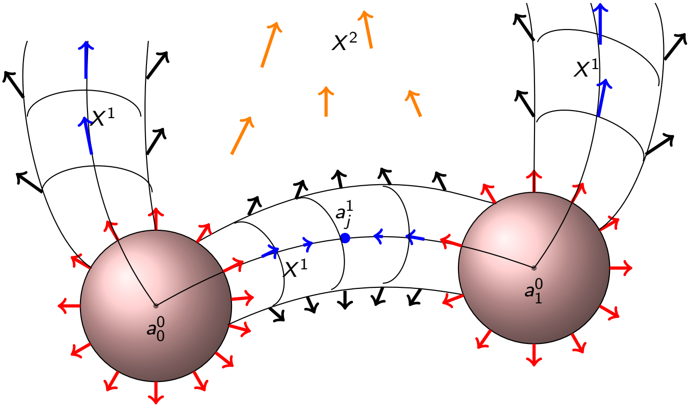
M.-H. Schwartz's radial extension - global construction - 3



M.-H. Schwartz's radial extension - global construction -4



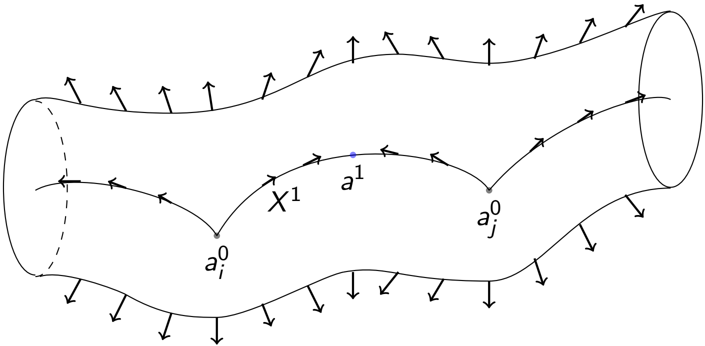
M.-H. Schwartz's radial extension - global construction -5



Observation : In order to obtain a continuous and stratified vector field, the process requires, at each step, the delicate and technical construction of systems of nested tubular neighbourhoods and tapered neighbourhoods in which the field is built by suitable homotopies.

This is a reason why the MHS construction has put off many mathematicians.

M.-H. Schwartz's radial extension - global construction



Important property 2 : **The radial vector field is pointing outwards the tubular neighbourhoods of strata.**

Proof of the Poincaré-Hopf Theorem

The proof is made by induction on the dimension of the strata.
+ For the 0-dimensional strata : $\chi(X^0) = \sum_{i=1}^{k} I(v, a_i^0; M)$ where $I(v, a_i^0; M) = 1$.
+ The obtained vector field v is pointing inwards the 1-dimensional strata along a neighbourhood of their boundary ∂X^1. By the classical Poincaré-Hopf Theorem for manifold with boundary and for a vector field pointing inwards the strata along the boundary :

$$\chi(X^1) = \sum_j I(v, a_j^1; M) + \chi(\partial X^1) = \sum_j I(v, a_j^1; M) + \sum_{i=1}^{k} I(v, a_i^0; M).$$

One continues till reaching the dimension of X and obtain :

$$\chi(X) = \sum_i I(v, a_i; M)$$

where a_i denotes all singularities of v in X and $I(v, a_i; M)$ denotes also the index of v at its singular point a_i computed in the stratum of the point a_i.

The M.-H. Schwartz method

Advantages

+ Clear use of Whitney conditions.

+ Clear use of radiality properties 1 and 2.

Note : Some authors tried to avoid radiality, but they use a different notion of index, which is to compensate the lack of radiality.

The M.-H. Schwartz method

Advantages

+ Clear use of Whitney conditions.
+ Clear use of radiality properties 1 and 2.

Note : Some authors tried to avoid radiality, but they use a different notion of index, which is to compensate the lack of radiality.

Inconvenients

+ The method is quite technical, it requires a lot of delicate constructions (of tubular neighbourhoods) which imply a lot of necessary attention.
+ It is unclear that the (Schwartz) radial extension of a Lipschitz vector field is Lipschitz.

The M.-H. Schwartz method

Advantages

+ Clear use of Whitney conditions.
+ Clear use of radiality properties 1 and 2.
Note : Some authors tried to avoid radiality, but they use a different notion of index, which is to compensate the lack of radiality.

Inconvenients

+ The method is quite technical, it requires a lot of delicate constructions (of tubular neighbourhoods) which imply a lot of necessary attention.
+ It is unclear that the (Schwartz) radial extension of a Lipschitz vector field is Lipschitz.

Idea

+ Keep the M.-H. Schwartz method (Main properties) BUT
+ Use the Lipschitz framework : Lipschitz stratifications and Lipschitz vector fields.

Lipschitz stratifications (Mostowski).

- c is a constant $c > 1$,
- $q = q_{j_1}$ is a point in a j_1-dimensional stratum $\mathring{X}^{j_1}$

A chain is a sequence of points $q_{j_s} \in \mathring{X}^{j_s}$, such that

$$X^{j_1} \supset X^{j_2} \supset \cdots \supset X^{j_s} \supset \cdots X^{j_r} = X^{\ell}$$

(with dimensions $j_1 > j_2 > \cdots > j_s > \cdots > j_r = \ell$)
and j_s is the bigger integer such that

$$d(q, X^{j_k}) \geqslant 2c^2\, d(q, X^{j_s}) \quad \text{for all } k \text{ for which } j_s > j_k \geqslant \ell$$

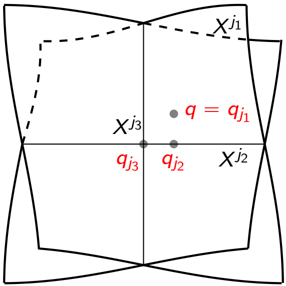
and

$$|q - q_{j_s}| \leqslant c\ d(q, X^{j_s}).$$

Here, $j_1 = 2, j_2 = j_s = 1, j_3 = 0$,

$d(q, X^{j_3}) \geqslant 2c^2\, d(q, X^{j_2})$

$|q - q_{j_2}| \leqslant c\ d(q, X^{j_2})$.



Lipschitz stratifications (Mostowski).

For $q \in \mathring{X}^j$,
let $P_q : \mathbb{R}^n \to T_q\mathring{X}^j$ be the orthogonal projection onto the tangent space and $P_q^\perp = Id - P_q$ the orthogonal projection onto the normal space $T_q^\perp \mathring{X}^j$.
The stratification is *L-stratification* if, for some constant $C > 0$ and every chain $q = q_{j_1}, q_{j_2}, \dots, q_{j_r}$ and every k, $1 \leqslant k \leqslant r$,

$$|P_{q_{j_1}}^\perp P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q_{j_1}, -q_{j_2}|/d(q_{j_1}, X^{j_k - 1})$$

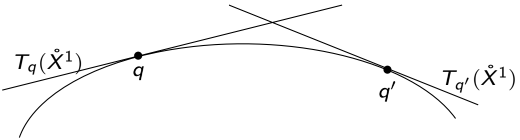
If, further, $q' \in \mathring{X}^{j_1}$ and $|q' - q| \leqslant (1/2c)\, d(q, X^{j_1 - 1})$, then

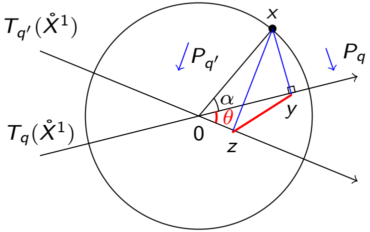
$$|(P_{q'} - P_q)P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q' - q|\, d(q, X^{j_k - 1}),$$

in particular, for q and q' in $\mathring{X}^{j_1}$,

$$|P_{q'} - P_q| \leqslant C|q' - q|\, d(q, X^{j_1 - 1}).$$

$\|P_{q'} - P_q\|$ geometrically is roughly the angle between $T_{q'}\mathring{X}^j$ and $T_q\mathring{X}^j$.





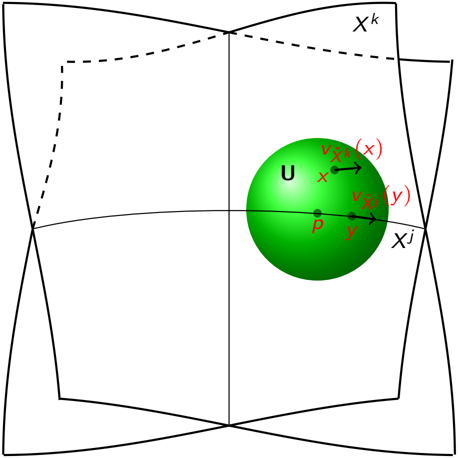
$$\|P_q - P_{q'}\| = \sup_{\|x\|=1} |(P_q - P_{q'})(x)| = \sin(\theta).$$

Let X stratified subset of the smooth manifold M.

A stratified vector field $v = \{v_{\mathring{X}^j} : \mathring{X}^j \text{ are strata}\}$ is said rugose if for each point $p \in \mathring{X}^j$, there is a constant $C > 0$ and a neighbourhood U of p in M such that for each point $y \in \mathring{X}^j \cap U$, and each point $x \in X \cap U$, if $\mathring{X}^k$ denotes the stratum of X containing x, then

$$\|v_{\mathring{X}^k}(x) - v_{\mathring{X}^j}(y)\| < C|x - y|.$$

For v to be Lipschitz one need to allow y not only to belong to $\mathring{X}^j$ but also to any stratum incident to $\mathring{X}^j$.



The "parallel" Lipschitz vector field

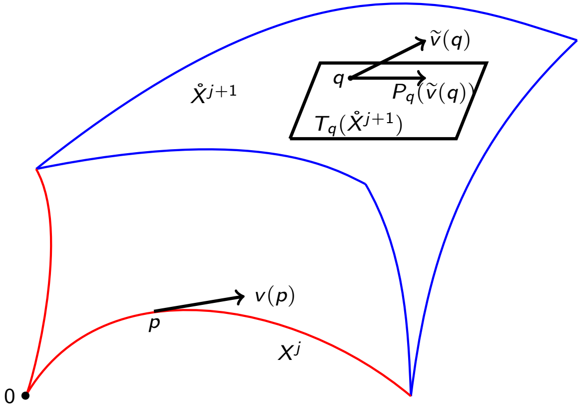
Proposition (Mostowski (1985) Prop. 2.1, Parusinski (1988) Sec 1.)

Let $\{\mathring{X}^j\}_{j=1}^m$ be a Lipschitz-stratification of X and let v be a Lipschitz stratified vector field on $\mathring{X}^j$ bounded on $\mathring{X}^\ell (\ell \leqslant j \leqslant m)$. Then v can be extended to a Lipschitz stratified vector field on $\mathring{X}^{j+1}$.

In a first step, one extends the Lipschitz vector field v, defined on $\mathring{X}^j$ in a Lipschitz vector field $\tilde{v}$ defined on M.
One defines a vector field w on X^{j+1} by

$$w(q) = \begin{cases} v(q) & \text{if } q \in X^j \\ P_q \tilde{v}(q) & \text{if } q \in \mathring{X}^{j+1} \end{cases}$$

where $P_q : \mathbb{R}^n \to T_q(\mathring{X}^{j+1})$. One shows that the obtained vector field is Lipschitz.



The "parallel" extension, à la Lipschitz : Let v a Lipschitz vector field tangent to a stratum $\mathring{X}^j$, then there is a neighbourhood of $\mathring{X}^j$ such that the Lipschitz extension of v provides a stratified vector field "parallel" to v.

The "transverse" Lipschitz vector field

Lemma

For each stratum $\mathring{X}^{j_0}$, there is a neighbourhood of $\mathring{X}^{j_0}$ and a "transverse" vector field which is a stratified Lipschitz vector field.

We work in a neighbourhood of some point in $\mathring{X}^{j_0}$ and assume that $X^i = \emptyset$ for $i < j_0$ and that $\mathring{X}^{j_0}$ is a linear subspace : $x_\mu = 0, (\mu > j_0)$. Put $r_0 = \sum_{\mu > j_0} x_\mu \frac{\partial}{\partial x_\mu}$.

For every $\varepsilon > 0$, there is a tubular neighbourhood $\mathcal{N}$ of $\mathring{X}^{j_0}$ in $\mathbb{R}^n$ and a vector field r in it, s.t. :

1. r is Lipschitz, tangent to the strata,
2. $\measuredangle(r, r_0) < \varepsilon$.

By increasing dimension on j, we construct a Lipschitz vector field r_j on $\mathcal{N}_j \cap X^j$ (for some tubular neighbourhood $\mathcal{N}_j$ of X^j) with the same properties.
Suppose r_j is defined on $\mathcal{N}_j \cap X^j$. Let v be its Lipschitz extension to $\mathcal{N}_j \cap X^{j+1}$, tangent to $\mathring{X}^{j+1}$; let C be the Lipschitz constant for v. For every $\eta > 0$, put

$$C_\eta(X^j) = \{x \in \mathbb{C}^n : d_j(x) \leqslant \eta\, d_{j_0}(x)\}.$$

For η small enough $\measuredangle(v, r_0) < 2\varepsilon$ in $C_\eta(X^j)$.
On $X^{j+1} \setminus C_{\eta/2}(X^j)$ we put $w(x) = P_x r_0(x)$. Then :

1. w is a Lipschitz vector field,
2. $|w(x) - r_0(x)| \leqslant (d_{j_0}(x))^{1+j_0}$ for some $j_0 > 0$ and $d_{j_0}(x)$ sufficiently small ; this implies $|w(x) - r_0(x)| \leqslant |r_0(x)|^{1+j_0} \leqslant \varepsilon |r_0(x)|$ and so $\measuredangle(w(x), r_0(x)) < \varepsilon$

Finally glue v and w using standard arguments (partition of unity).

Conclusion

The sum of the parallel and transverse Lipschitz vector fields provides a Lipschitz vector field v such that the main properties are fulfilled :
If a is a singularity of v in a stratum $\mathring{X}^\alpha$, then

$$\boxed{I(v_\alpha, a; \mathring{X}^\alpha) = I(v, a; M)}$$

The obtained vector field is pointing outwards the tubular neighbourhoods of the strata.

One concludes by the same argument than M.-H. Schwartz.

Thanks a lot for your attention

Dziękuję za uwagę

Merci pour votre attention