The extra-nice dimensions

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Special session dedicated to the memory of Stanislaw Łojasiewicz

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Introduction

The extra-nice dimensions, with Raul Oset-Sinha & Roberta Wik Atique, Math. Ann. 2022.

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- Stable families of mappings $F : N \times [0, 1] \to P$

Let $C^\infty_{pr}(N, P) = \{f : N \to P,\ f \in C^\infty\}$ with the Whitney topology.

$\mathcal{A}$-equivalence:

$f \sim_{\mathcal{A}} g$

$$\begin{array}{ccc} N & \xrightarrow{f} & P \\ h\downarrow & \circlearrowleft & \downarrow k \\ N & \xrightarrow{g} & P \end{array},$$

h, k C^∞ diffeomorphisms, $g = k \circ f \circ h^{-1}$.

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$f \in C^{\infty}(N, P)$ is stable if there exists a neighbourhood U of f such that for every $g \in U$, it follows that $f \sim_{\mathcal{A}} g$.

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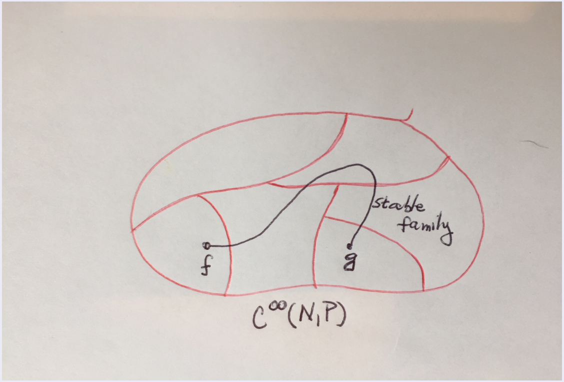
$f \in C^\infty(N, P)$ is stable if there exists a neighbourhood U of f such that for every $g \in U$, it follows that $f \sim_{\mathcal{A}} g$.

A family $F : N \times [0, 1] \to P$ is a stable one parameter family if F_t is stable for all $t \in [0, 1] \setminus \{t_1, \dots, t_k\}$ and at each point t_i, the family F is transversal to the orbits in jet space.

E.Chíncaro (n,2), J. Rieger (2,2), Goryunov, Mond and Marar (2,3). Cerf, Igusa-Pseudo-isotopies



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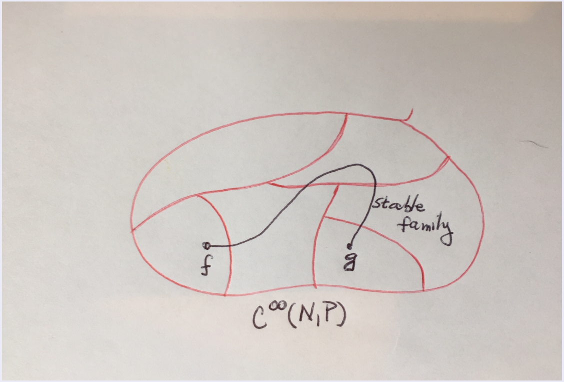

Goals: The set of stable families is a residual set in $C^\infty(N^n \times [0,1], P^p)$ if and only the pair (n, p) is in the extra-nice dimensions.

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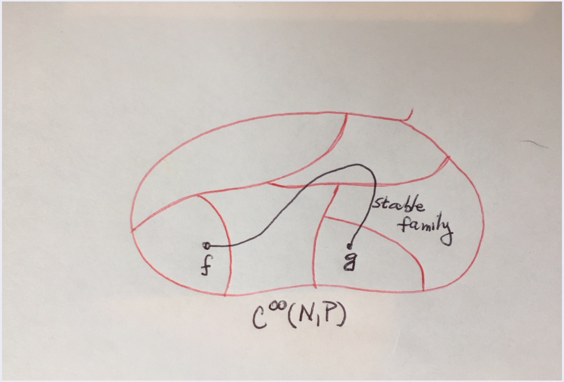

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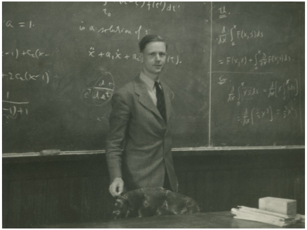
Singularity theory of smooth mappings began with the work of Hassler Whitney in the decades of 40 and 50's of century XX.



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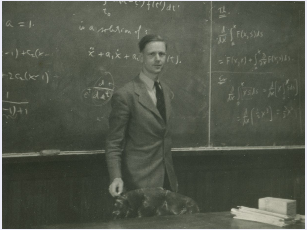

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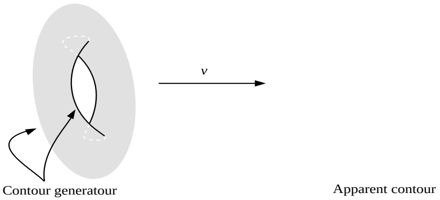

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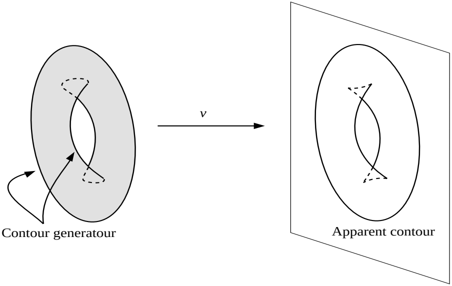

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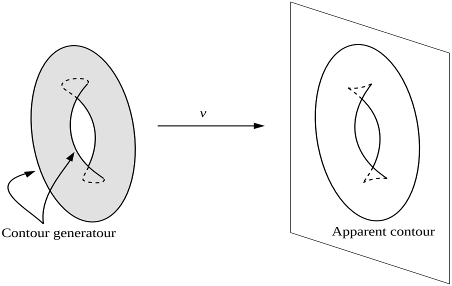
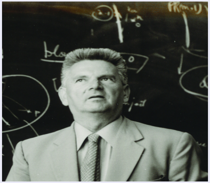


Whitney conjecture (1958)

The set of stable mappings is a dense set for all pairs (n, p). ..

René Thom



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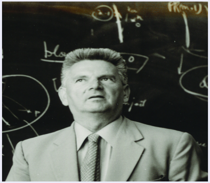

R. Thom, in his 1959 lectures *Singularities of differentiable mappings, I,* Bonn, 1959, sketched a proof that stable maps are not dense when $(n, p) = (9, 9)$, and formulated conjectures on density of C^0-stable maps that led to great developments in singularity theory.

John Mather, Stable mappings, 1968 to 1971

Mather's theorem

Stable mappins are dense $\Leftrightarrow$ (n, p) is in the nice dimensions.

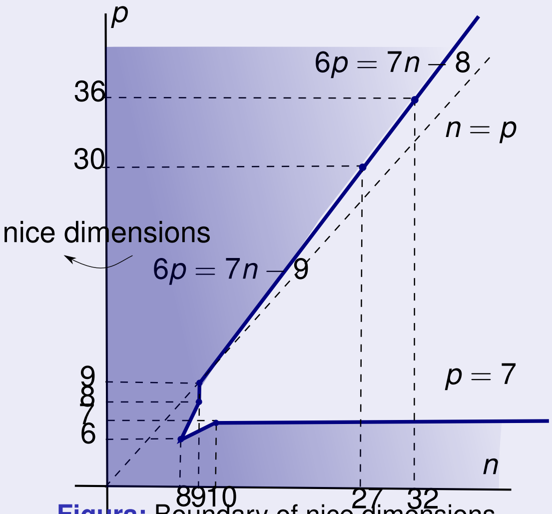

Figura: Boundary of nice dimensions

Definition

$$\mathcal{A}_e - cod(f) = dim_{\mathbb{K}} \frac{\Theta_f}{tf(\Theta_n) + wf(\Theta_p)}$$

- $\mathcal{A}_e -$ codimension $f_t = 1$
- If $t \neq t'$, then $f_t \not\sim_{\mathcal{A}} f'_t$ (f_t are not simple.)

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Thom's example

There is a one-parameter family of germs
$f_t : (\mathbb{K}^9, 0) \to (\mathbb{K}^9, 0), \ \mathbb{K} = \mathbb{R}, \mathbb{C}$ corank$f_t = 3$ such that

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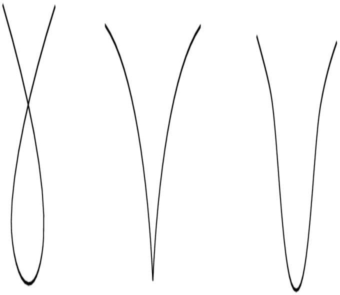
A germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is simple if $\exists k_0$ such that for all $k \geq k_0$, there is a finite number of orbits in a neighbourhood of $j^k f(0)$ in $J^k(n, p)$.

$\mathcal{A}_e - \text{cod}(f) = \#$ parameters of a miniversal unfolding of f.



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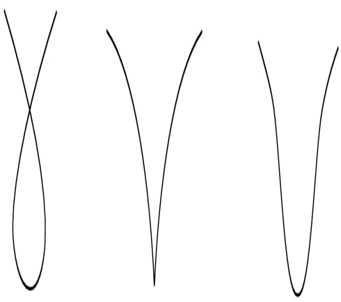

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(n, p) is in nice-dimensions $\Longleftrightarrow$ all $\mathcal{A}_e$-codimension 1 singularities $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ are simple.

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T. Cooper,D. Mond and R. Wik Atique, *Compositio Math.* 131 (2002), no. 2, 121–160.

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Let $(n+1, p+1)$ be a nice pair of dimensions, $F : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{p+1}, 0)$ a minimal stable map-germ and $\Delta(F)$ its discriminant.

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$$\begin{array}{ccc} (\mathbb{C}^{n+1},0) & \xrightarrow{F} & \Delta(F) \subset (\mathbb{C}^{p+1},0) \\ \uparrow & & \uparrow i \\ (\mathbb{C}^{n},0) & \xrightarrow{f} & (\mathbb{C}^{p},0) \end{array}$$

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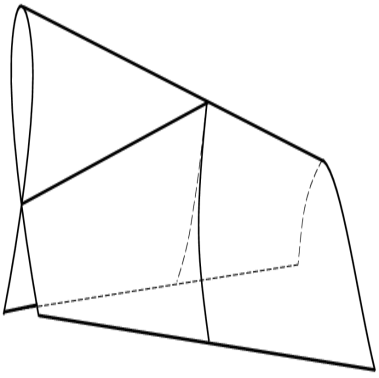
A hyperplane section $H = i(\mathbb{C}^p)$ transversal to the discriminant $V = \Delta(F) \subset \mathbb{C}^{p+1}$ away from the origin pulls back by F to an $\mathcal{A}$-finite map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

The group $\mathcal{K}_V$ acts on the space of embeddings
$i : (\mathbb{K}^p, 0) \to (\mathbb{K}^{p+1}, 0)$, and $\mathcal{K}_{V_e} - cod(i) = \mathcal{A}_e - cod(f)$.

The cross-cap



$$F : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0), \ F(x, y) = (x, y^2, xy + y^3),$$

Proposition: (Oset-Sinha, R., Wik Atique, 2022).

Let $(n+1, p+1)$ be a nice dimension. Then, all corank 1 $\mathcal{A}_e$-cod 2 germs (n, p) are simple.

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Proposition

Let $(n+1, p+1)$ be nice dimensions. If all stable germs $F : (\mathbb{K}^{n+1}, 0) \longrightarrow (\mathbb{K}^{p+1}, 0)$ admit a $\mathcal{A}_e$-cod 1 hyperplane section, then all $\mathcal{A}_e$- cod 2 germs $f : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^p, 0)$ are simple.

A non simple germ of $\mathcal{A}_e$-codimension 2.

Example: algebra $B_{3,3} : (x^3 + y^3, xy)$

The stable corank 2 germ $F : (\mathbb{K}^6, 0) \to (\mathbb{K}^6, 0)$ given by

$$F(x, y, u) = (x^3 + y^3 + u_1 x + u_2 y + u_3 x^2 + u_4 y^2, xy, u) =$$

where $u = (u_1, u_2, u_3, u_4)$.

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It does not admit a $\mathcal{A}_e$-codimension 1 hyperplane section, but it admits a section of codimension 2, $U_3 + \lambda U_4 + U_4^2 = 0$. We use capital letters for target coordinates.

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with $\lambda \neq 0, -1$, has $\mathcal{A}_e$-codimension 2 and is not simple.

The extra-nice dimensions

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Definition

(n, p) is in the extra-nice dimensions if for large enough k, there exists a Zariski closed subset $\Lambda \subset J^k(n, p)$, $\mathcal{A}$-invariant, cod $(\Lambda) \geq n + 2$ such that its complement $J^k(n, p) \setminus \Lambda$ is a finite union of $\mathcal{A}$-orbits.

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Extra-nice $\Rightarrow$ nice.

Theorem:

The pair (n, p) is in the extra-nice dimensions if and only if $(n+1, p+1)$ is in the nice dimensions and every stable germ $F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$ admits a hyperplane $\mathcal{A}_e$-codimension 1 section $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$.

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Corollary

(n, p) *extra-nice* $\Longrightarrow$ $\mathcal{A}_e$*-codimension* 2 *germs are simple.*

Density of stable families

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The subset of 1-parameter locally stable families in $C^{\infty}_{pr}(N^n \times [0,1], P^p)$, is dense $\Longleftrightarrow$ (n,p) is in the extra-nice dimensions.

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Let $\Lambda^k(n,p) = \{\sigma \in J^k(n,p) | \mathcal{A}^k - \text{cod}(\sigma) \geq n+2\}$.

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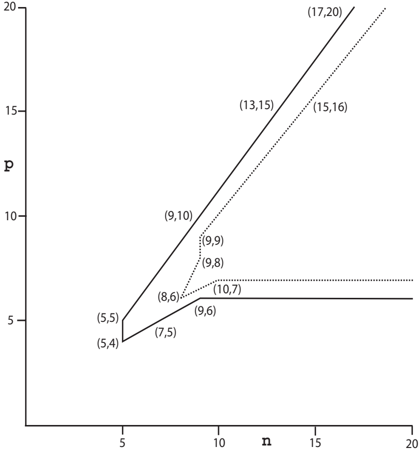
The result follows from the Transversality theorem for families of mappings.

The extra-nice dimensions



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Let $F : (\mathbb{K}^{n+1}, 0) \rightarrow (\mathbb{K}^{p+1}, 0)$, be stable and $\Delta(F)$ the discriminant of F.

Let $Derlog(F)$ be the $\mathcal{O}_{p+1}$-submodule of θ_{p+1}, the module of vector fields in $(\mathbb{K}^{p+1}, 0)$, defined by

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Proposition

$\exists\, \mathcal{A}_e$-cod 1 hyperplane section of $\Delta(F) \Longleftrightarrow \exists\, L : \mathbb{K}^{p+1} \to \mathbb{K}$, such that $\mathcal{M}_{p+1} \subset DL(Derlog(\Delta(F)))$

$$F_{3,3} := (x^3 + y^3 + u_1 x + u_2 y + u_3 x^2 + u_4 y^2, xy, u) = (X, Y, U_1, ..., U_4)$$

Lema

Derlog$(\Delta(V)) = $ *Lift*$(F_{3,3})$, *where*
Lift$(F) = \{\eta \in \theta_{p+1} \mid \exists \xi \in \theta_{n+1}$, *such that* $DF(\xi) = \eta \circ F\}$

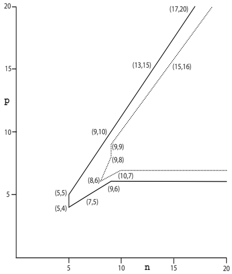
$$\eta_{1,2,3} = \begin{pmatrix} 3X \\ 2Y \\ 2U_1 \\ 2U_2 \\ U_3 \\ U_4 \end{pmatrix}, \begin{pmatrix} 2U_1U_2 + 6Y^2 + 4U_3U_4Y \\ X \\ -3U_2U_3 - 5U_4Y \\ -3U_1U_4 - 5U_3Y \\ -4U_2 \\ -4U_1 \end{pmatrix}, \begin{pmatrix} \frac{4}{3}U_2Y \\ -\frac{1}{9}U_3Y \\ X + \frac{1}{9}U_1U_3 \\ -\frac{5}{3}U_4Y \\ -\frac{2}{3}U_1 + \frac{2}{9}U_3^2 \\ -2Y \end{pmatrix}$$

$$\eta_{4,5,6} = \begin{pmatrix} \frac{4}{3}U_1Y \\ -\frac{1}{9}U_4Y \\ -\frac{5}{3}U_3Y \\ X + \frac{1}{9}U_2U_4 \\ -2Y \\ -\frac{2}{3}U_2 + \frac{2}{9}U_4^2 \end{pmatrix}, \begin{pmatrix} \frac{5}{3}U_4Y^2 + \frac{1}{9}U_2U_3Y \\ (-\frac{2}{9}U_1 + \frac{2}{27}U_3^2)Y \\ -\frac{4}{3}U_2Y + \frac{2}{9}U_1^2 - \frac{2}{27}U_1U_3^2 \\ -2Y^2 - \frac{2}{9}U_3U_4Y \\ X + \frac{5}{9}U_1U_3 - \frac{4}{27}U_3^3 \\ -\frac{1}{3}U_3Y \end{pmatrix}, \begin{pmatrix} \frac{5}{3}U_3Y^2 + \frac{1}{9}U_1U_4Y \\ -\frac{2}{9}U_2Y + \frac{2}{27}U_4^2Y \\ -2Y^2 - \frac{2}{9}U_3U_4Y \\ -\frac{4}{3}U_1Y + \frac{2}{9}U_2^2 - \frac{2}{27}U_4^2 U \\ -\frac{1}{3}U_4Y \\ X + \frac{5}{9}U_2U_4 - \frac{4}{27}U_4^3 \end{pmatrix}$$

$\Delta(F)$ does not admit a $\mathcal{A}_e$-codimension 1 hyperplane section. A generic section (with respect to analytic equivalence) is not linear. It is a section of $\mathcal{A}_e$-codimension 2, given by

$$G(X, Y, U_1, \ldots, U_3, U_4) = U_3 + \lambda U_4 + U_4^2,$$

with $\lambda \neq 0, 1$.



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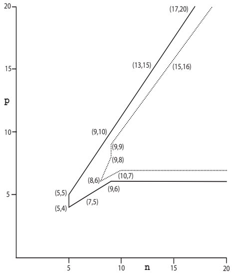

Properties of Stable Discriminants in the nice dimensions.

Let $\Delta(F) \subset (\mathbb{K}^{p+1}, 0)$ be the discriminant of a stable map-germ $F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$, where $(n+1, p+1)$ is in the nice dimensions.

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- For any pair (n, p) in the nice dimensions, the topological type of a generic hyperplane section is constant. (Damon)

Conjecture: rigidity of bi-Lipschitz triviality outside the extra-nice dimensions

Suppose the pair $(n+1, p+1)$ is in the nice dimensions, and let $G_t : (\mathbb{K}^{p+1}, 0) \to (\mathbb{K}, 0)$ be a family of generic sections to the discriminant $V = \Delta(F)$ of a stable map F.

We can use the $\mathbb{K}^*$-action to find a control function ρ and a polynomial vector field $\eta \in Derlog(V)$, $\eta = \Sigma_{i=1}^{6} \alpha_i \eta_i$, such that the vector field

$$\mathbf{w} = \frac{\partial}{\partial \lambda} + \frac{\eta}{\rho}$$

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The discriminant $V = \Delta(F)$ is weighted homogeneous of type $(3, 2, 2, 2, 1, 1; 18)$.

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Thank you !