Regularity in analysis & geometry - influence of Łojasiewicz.

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Singularity Theory - in memory of Łojasiewicz

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Topic:
Central question in geometry & PDEs:
Regularity of equations that describe the world around us.

Approach:
Many problems in geometry & analysis can be thought of
as questions about gradient flows on ∞-dimensional spaces.

Regularity in analysis and geometry

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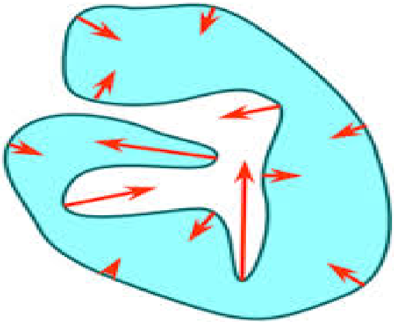
Many problems in geometry & analysis can be thought of as questions about gradient flows on ∞-dimensional spaces.

Mean curvature flow

A surface evolves in time
by each point moving normal $\vec{\mathbf{n}}$ to the surface with speed $\mathbf{H}$:

$$x_t = -\mathbf{H}\,\vec{\mathbf{n}}\,.$$



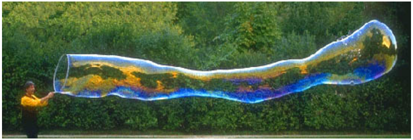
Convex points move inward
– concave points move out.

Mathematics of surface tension

Mean curvature flow is a nonlinear heat equation.

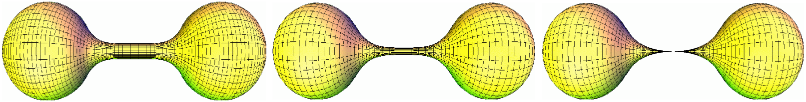
It is the (negative) gradient flow for area
on the infinite dimensional space of surfaces:

The flow makes the area shrink as fast as possible.

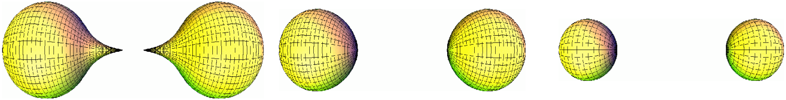


Mean curvature flow goes back more than 100 years
in mathematics & material science.

Example: Evolution of Grayson's dumbbell



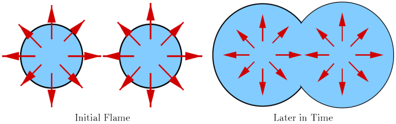
Initial dumbbell, shrinking neck, & neck pinch singularity.



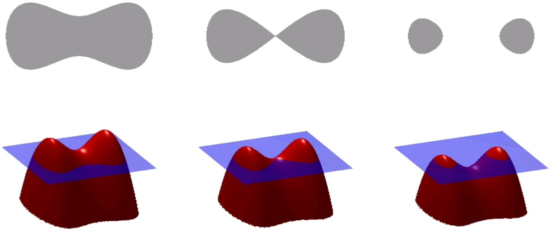
Cusps retract & each piece becomes round.

Level Set Method from applied mathematics

Tracking moving front = Level Set Method.



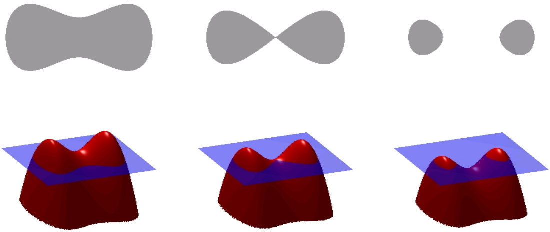
Idea: Represent evolving front as level sets of a function:



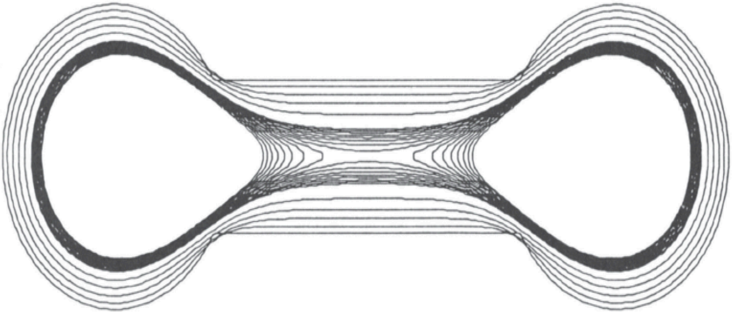
Which PDE does the function satisfy?

The Level Set Method allows for:

Singularities & topological changes.



Monotone front: Flow that moves inward.



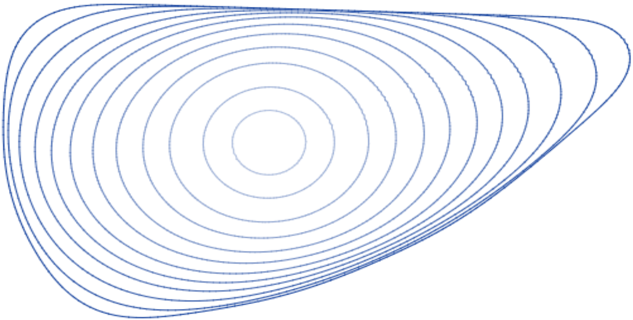
Arrival time function u: $u^{-1}(t)$ are the fronts.

$u(x)$ the time when the front arrives at x.

u defined on domain that initial front bounds.

Arrival time equation

Evolving curves are level sets of u:



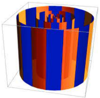
$$-1 = |\nabla u| \operatorname{div} \frac{\nabla u}{|\nabla u|}\,.$$

Degenerate elliptic equation.

Arrival time functions on $\mathbf{R}^3$:

For spheres becoming extinct at the origin at time 0:

$$-\frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$$



For cylinders becoming extinct in the line $x_2 = x_3 = 0$ at time 0:

$$-\left(x_2^2 + x_3^2\right) .$$

Existence & regularity

Evans-Spruck, Chen-Giga-Goto:
Viscosity solutions exist & are Lipschitz.

Fundamental question:
How smooth are solutions?

Examples of Ilmanen: NOT C^2 in general;
cf. Huisken, Kohn-Serfaty, Sesum.

Weak solutions in PDEs: Viscosity

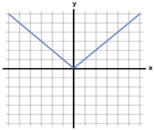
A continuous function $u : \mathbf{R}^n \to \mathbf{R}$ has $\Delta\, u \geq 0$ at $x = 0$
in the viscosity sense if

$\exists$ a smooth barrier function v with

- $v(0) = u(0)$,
- $u \geq v$,
- $\Delta\, v \geq 0$ at $x = 0$.

Maximum principle holds for such u.

E.g., $u(x) = |x|$ has $\Delta\, |x| \geq 0$ at $x = 0$.



Optimal regularity & geometry

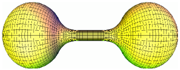
Thm (**CM**): The arrival time is twice differentiable everywhere.

Not always C^2!

Being C^2 has geometric meaning:

Thm (**CM**): The arrival time is C^2 iff the entire evolving front becomes singular at the same time & then extinct.

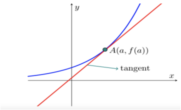
C^2 only if it is like a marriage ring or sphere,

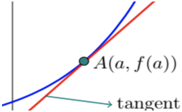


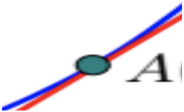
dumbbell NOT C^2.

Differentiability & uniqueness

A function is differentiable if it looks like
the **same** linear function on all sufficiently small scales.



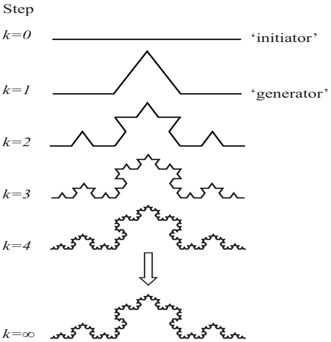


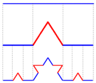


Derivatives as unique limit of rescalings.

Koch curve – uniqueness fails

Koch curve is a fractal; it is an iterative limit of broken lines.





When each broken line is almost flat:

On each scale the curve looks roughly like a line.

Yet under larger magnifications looks like a different line.

Rescaling of arrival time functions

The flow & u are both smooth away from critical points, i.e., away from points where $\nabla u = 0$.

If 0 is critical point define rescalings

$$v_\lambda(x) = \lambda^{-2}\, u(\lambda\, x)\,.$$

v_λ satisfies same equation.

Homogeneous quadratic polynomials are preserved.

Two examples: cylinders & spheres:

- Both have quadratic polynomials as arrival time.
- For both v_λ is independent of λ.

Rescaling & regularity

Twice differentiable means:

There is a 2nd order Taylor expansion.

Thus must show that $v_\lambda(x)$ has a limit as $\lambda \to 0$.

A priori – no reason to expect any limit!

– Even for a subsequence.

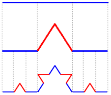
Rescaling of the flow

Huisken-Ilmanen-White get geometric blowups
– but depend on choice of subsequence.

For the flow:
Blowups $=$ dilation-invariant solutions called shrinkers.
Most blowups are non-compact.

Might be like Koch?

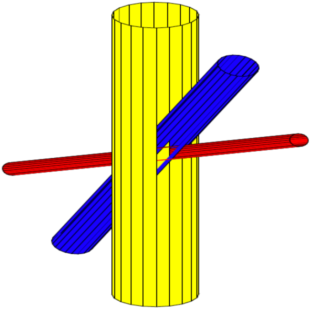
For Koch: One sequence of rescalings gives one blowup,
another gives a different blowup.



Uniqueness

Are limits unique?

Or, do different subsequences give different limits?



Thm (**CM**): Uniqueness of blowups for all monotone flows.

Questions of uniqueness
have long been recognized in geometry
as fundamental.

Fundamental work of Allard-Almgren, Simon
on uniqueness questions for minimal varieties.

Uniqueness & twice differentiabilty

Thm (**CM**): Uniqueness of blowups implies:

u looks like the same quadratic polynomial at all small scales.

This gives the 2nd order Taylor expansion
& twice differentiability.

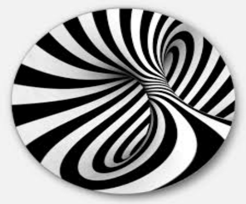
Dynamics

Uniqueness can be understood dynamically for rescaled flow.

Rescaled flow $=$ magnifying continuously along the flow.

Uniqueness $\Leftrightarrow$ solution of rescaled flow with a limit point has a unique limit.

This contrasts with wandering points in dynamics:



Finite dimensional model problem

Function $f : \mathbf{R}^n \to \mathbf{R}$, negative gradient flow line

$$x_t = -\nabla f(x(t))$$

has sequence $t_i \to \infty$ with $x(t_i) \to x_\infty$.

Do we get uniqueness $\lim_{t\to\infty} x(t) = x_\infty$?

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Model problem

Model: $x_t = -\nabla f(x(t))$, sequence $t_i \to \infty$ with $x(t_i) \to x_\infty$.

Uniqueness would follow from finite length

$$\int_0^\infty |\nabla f|(x(t)) < \infty \,.$$

We don't necessarily have this. We have $f(x(t)) \to f(x_\infty)$ and

$$\int_0^\infty |\nabla f|^2(x(t)) < \infty \,.$$

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Łojasiewicz's theorem

If $f : \mathbf{R}^n \to \mathbf{R}$ is analytic and $x(t_i) \to x_\infty$, then length is finite and

$$\lim_{t\to\infty} x(t) = x_\infty \, .$$

x_∞ is a critical point for f.

Even in $\mathbf{R}^2$, it is easy to construct **smooth** (but not analytic!) functions where uniqueness fails.

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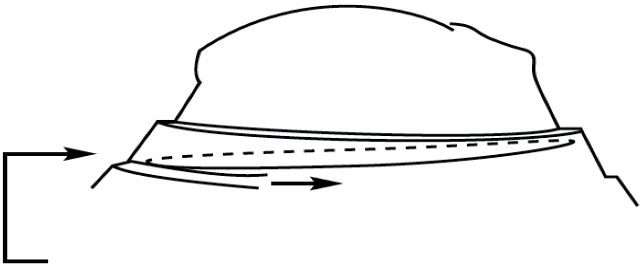
Goat tracks



Narrow ledge on steep hillside, becoming more and more narrow and sloping less and less as it spirals infinitely down to the base.

Proof of Łojasiewicz's theorem:

Uses the Gradient Łojasiewicz inequality (1962–1963):

Analytic f: There is a nbhd and constants $\beta \in (0, 1)$ and C so

$$|f(x) - f(x_\infty)|^\beta \leq C\, |\nabla_x f| \,.$$

Our main tools for proving uniqueness:

Two ∞ dimensional Łojasiewicz type inequalities
on non-compact spaces.

Approaching uniqueness in ∞ dimensional by using Łojasiewicz' original inequalities was pioneered by Leon Simon in 1983.

Simon used reduction to finite dimensions and then appealed to Łojasiewicz.

Crucial for Simon's approach that blowup is compact.

Schulze used Simon's approach to prove uniqueness for compact shrinkers.

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Key difficulties - new ideas

The functional that plays the role of the analytical function in Łojasiewicz thm:

Gaussian surface area:

$$F(\Sigma) = \int_{\Sigma} e^{-\frac{|x|^2}{4}} .$$

F defined on ∞ dimensional space of hypersurfaces Σ.

The uniqueness for cylinders had been open for decades.

Key difficulties - new ideas

The key difficulty is that singularities are non-compact

- entirely new techniques were needed.

The parallel to the Łojasiewicz inequalities is a vague guiding principle

- nothing from the Łojasiewicz inequalities can be used.

Instead, we discovered **The Shrinker Principle**:

- information travels out from a compact set.